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ON NONEXISTENCE OF GLOBAL IN TIME SOLUTION FOR A MIXED PROBLEM FOR A NONLINEAR EVOLUTION EQUATION WITH MEMORY GENERALIZING THE VOIGT–KELVIN RHEOLOGICAL MODEL

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Abstract. The paper deals with investigating of the first mixed problem for a fifth-order nonlinear evolutional equation which generalizes well known equation of the vibrations theory. We obtain sufficient conditions of nonexistence of a global solution in time variable.

Keywords: boundary value problem, beam vibrations, nonlinear evolution equation, Voigt-Kelvin model, memory, blowup.

Mathematics Subject Classification: 35G20, 35G31.

1. INTRODUCTION

We study the nonlinear evolution fifth-order equation with second-order temporal derivative which is a multidimensional nonlinear generalization of the well known one-dimensional linear equation of beam vibrations in the Timoshenko model [7]. Equations of such a type describe propagation of perturbations in a viscoelastic material under action of external ultrasonic aerodynamical forces [8]. Investigation of mixed problems for these equations and systems can be explained by the worn-out contact surfaces [7]. In paper [7] there is investigated the existence of weak solutions for the mixed problems in the bounded domain D for a system of two linear evolution equations with partial one- and second-order temporal derivatives, where one of unknown functions describes a vertical displacement of a beam.

General mathematical models of contact dynamics for the elastic structures, described by such equations and systems, have been studied recently in many papers [2, 13, 22]. In paper [22] there was formulated a mathematical problem for dynamical viscoelastic friction with worn-out. Dynamical contact between the beam and movable surface was investigated in [2], thermoelastic contact was analyzed in [13].

A general equation that has finite speed of propagation compatible with Einstein's theory of special relativity is investigated in the paper [5]. Both stationary and evolutionary problems are considered.

Boundary value problems for the differential equations of such a type with odd order partial derivatives were also a topic of modern research [1, 3, 4, 6, 9, 11, 14, 17–19, 24]. The mixed problem for a strongly nonlinear equation of beam vibrations in a bounded domain was in detail studied in [17]. The case of a weakly nonlinear equation in an unbounded space domain was, in particular, considered in [18, 19]. The question of existence of the unique generalized solution to the mixed problem for a strongly nonlinear beam vibrations type equation in the domain $\Omega \times [0, +\infty)$ (Ω is a bounded domain) and a behavior of this solution as $t \to \infty$ were analyzed in [4]. The mixed problem for the nonlinear third-order equation was also investigated in the same domain in [6]. The existence of a unique classical solution, stable under perturbations of the initial data, was there proved, as well as the behavior of this solution as $t \to \infty$ was described. The conditions for existence of local and global solutions to the mixed problem in Sobolev spaces were formulated in [1]. The case, where the degree of nonlinearity in the main part is a function of space variables was studied in [3].

The phenomena of nonexistence of solutions global in time (also known as blowup) was considered in [14, 24], in particular, for the hyperbolic fourth-order equation it was studied in [11]. In [9] the sufficient conditions for existence of local and nonexistence of global in time solutions to a mixed problem for a hyperbolic third-order equation with the integral term were discussed. This integral term simulates the well-known phenomena of "memory" in oscillation processes. The description of mathematical model of propagating longitudinal waves in the inhomogenous rod one can consult [23]. The mixed problem for some nonlinear fifth-order equation similar to the previous view without integral term was proposed in [21].

Important questions of existence and stability as $t \to +\infty$ of solutions to nonlinear Hamilton-Jacobi equation in suitable functional spaces were studied in [15, 16], where there were devised effective tools for investigating nonlinear evolution problems based on the fixed point approach stemming from [10]. A related general method for studying the solution existence, based on the Leray-Schauder fixed point approach within the Calogero type projection-algebraic scheme of discrete approximations, was suggested for linear and nonlinear differential-operator equations in Banach spaces in [12].

The main aim of our paper is to establish sufficient conditions for the nonexistence of global solution to a mixed problem for some fifth-order partial differential equation with a fourth order spatial derivative. As a main tool, the method of estimating the energy functional for the mechanical oscillation system will be used.

2. PROBLEM STATEMENT. EXISTENCE OF LOCAL SOLUTION

Let $T > 0$ be an arbitrary number, $\Omega \subset R^n$ $(n \geq 1)$ be a bounded domain with the smooth bound $\partial\Omega$ of class C^1. Denote $Q_\tau = \Omega \times (0, \tau)$, $S_\tau = \partial\Omega \times (0, \tau)$, $\Omega_\tau = \{(x,t) : x \in \Omega,\ t = \tau\}$, $\tau \in [0, T]$.

We will consider the following nonlinear evolution fifth-order equation in the domain Q_T:

$$\begin{aligned} u_{tt} + &\sum_{|\alpha|=|\beta|=2} D^\beta \left(a_{\alpha\beta}(x) D^\alpha u_t\right) + \sum_{|\alpha|=|\beta|=2} D^\beta \left(b_{\alpha\beta}(x) D^\alpha u\right) \\ &+ \sum_{|\alpha|=2} D^\alpha \left(b_\alpha(x) |D^\alpha u|^{q-2} D^\alpha u\right) \\ &- \int_0^t g(t-\theta) \sum_{|\alpha|=2} D^\alpha \left(d_\alpha(x) D^\alpha u(x,\theta)\right) d\theta = c(x)|u|^{p-2}u, \end{aligned} \tag{2.1}$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1{}^{\alpha_1} \ldots \partial x_n{}^{\alpha_n}}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in N \cup \{0\}$, $i = 1, \ldots, n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, with initial conditions

$$u|_{t=0} = u_0(x), \tag{2.2}$$

$$u_t|_{t=0} = u_1(x) \tag{2.3}$$

and boundary conditions

$$u|_S = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_S = 0, \tag{2.4}$$

ν is the external normal unit vector of the surface $\partial\Omega$.

Problem (2.1)–(2.4) is multidimensional generalization of rheological nonlinear Voigt-Kelvin model. An influence of the internal friction as a result of waves dispersion on the accidental inhomogeneous material is investigated in this model [23].

Assume the next conditions are satisfied.

(A) $a_{\alpha\beta} \in L^\infty(\Omega)$, $|\alpha| = |\beta| = 2$,

$$\sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq A_2 \sum_{|\alpha|=2} |\xi_\alpha|^2, \quad A_2 > 0,$$

for arbitary real numbers ξ_α, $|\alpha| = 2$, and almost all $x \in \Omega$.

(B) $b_{\alpha\beta} \in L^\infty(\Omega)$, $|\alpha| = |\beta| = 2$,

$$\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq B_2 \sum_{|\alpha|=2} |\xi_\alpha|^2, \quad B_2 > 0,$$

for arbitrary real numbers ξ_α, $|\alpha| = 2$, and for almost all $x \in \Omega$; $b_{\alpha\beta}(x) = b_{\beta\alpha}(x)$ for almost all $x \in \Omega$.

(B1) $b_\alpha \in L^\infty(\Omega)$, $b_\alpha(x) \geq b_2 > 0$ for almost all $x \in \Omega$ and for all α, $|\alpha| = 2$.

(C) $c \in L^\infty(\Omega)$, $c(x) \geq c_0 > 0$ for almost all $x \in \Omega$.

(D) $d_\alpha \in L^\infty(\Omega)$, $d_\alpha(x) \geq d_2 \geq 0$ for almost all $x \in \Omega$ and for all α, $|\alpha| = 2$, $d_3 = \operatorname{ess\,sup}_{\Omega} \sum_{|\alpha|=2} d_\alpha^2(x)$.

(G) $g(t) \geq 0$, $g'(t) \leq 0$ for all $t \in [0, +\infty)$,

$$G(t) := \int_0^t g(\tau)\,d\tau \geq 0, \quad G(+\infty) = G < \frac{2p-4}{2p-3} B_2, \quad l(t) := B_2 - G(t)d_3 > 0,$$

$$l(+\infty) = B_2 - Gd_3 \equiv l > 0.$$

(PQ) $p > q > 2$.

(U) $u_0, u_1 \in W_0^{2,q}(\Omega)$.

Definition 2.1. Function $u : \Omega \times [0, T) \to \mathbb{R}$ (T is a positive number or $+\infty$) such that

$$u \in C([0, T_0]; W_0^{2,q}(\Omega)) \cap L^p((0, T_0); L^p(\Omega)), \quad u_t \in C([0, T_0]; W_0^{2,q}(\Omega)),$$

$$u_{tt} \in L^\infty((0, T_0); L^2(\Omega))$$

for arbitrary number T_0 from $(0, T)$ denote generalized solution of the problem (2.1)–(2.4) in the domain Q_T, if it satisfies the initial conditions (2.2), (2.3) and an integral identity

$$\begin{aligned} \int_{\Omega_t} \Big[& u_{tt} v + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta v \\ & + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta v + \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^{q-2} D^\alpha u D^\alpha v \\ & - \int_0^t g(t-\theta) \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x,\theta) D^\alpha v\,d\theta - c_0(x)|u|^{p-2}uv \Big] dx = 0 \end{aligned} \tag{2.5}$$

for almost all $t \in (0, T)$ and for all $v \in W_0^{2,q}(\Omega) \cap L^p(\Omega)$. In case $T = +\infty$, solution is called global.

Remark 2.2. If $T < +\infty$, then solution is called local. Under some conditions on the coefficients, right part of equation and initial data it is possible to find a finite time moment T (depending on the coefficients, right part of equation and initial data), such that the local solution u of the problem (2.1)–(2.4) exists in the domain Q_T. Sufficient conditions of local in time solution existence of the previous problem are proved via Faedo-Galerkin method [10] in [20] (see Theorem 1 therein).

3. THE MAIN RESULT. SUFFICIENT CONDITIONS OF BLOWUP

Further we will use the following notation:

$$\|v\|_r := \|v\|_{L^r(\Omega)} = \Big(\int_\Omega |v|^r\,dx\Big)^{1/r},\ r>1;\ \ \|D^2 v\|_2 := \Big(\int_\Omega \sum_{|\alpha|=2} |D^\alpha v|^2\,dx\Big)^{1/2}.$$

Taking into account $p \le \frac{2n}{n-4}$, $n > 4$, via Sobolev imbedding theorem the next is true $H^2(\Omega) \subset L^p(\Omega)$, i.e.

$$\|u\|_p \le B\|D^2 u\|_2, \quad B > 0.$$

Denote

$$B_1 = Bl^{-1/2},\ \ C = \operatorname*{ess\,sup}_\Omega c(x),\ \ A = \operatorname*{ess\,sup}_\Omega \sum_{|\alpha|=|\beta|=2} a^2_{\alpha\beta}(x).$$

Let us consider the functional (energy functional) of the form

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_{\Omega_t}\Big[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u\Big]dx \\ & + \frac{1}{q}\int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx - \frac{1}{p}\int_{\Omega_t} c(x)|u|^p\,dx \\ & + \frac{1}{2}\int_0^t g(t-\theta)\int_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta \\ & - \frac{1}{2}G(t)\int_{\Omega_t}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,t)|^2\,dx, \quad t\in[0,T). \end{aligned} \tag{3.1}$$

Let us denote

$$\begin{aligned} E_0 := {} & \frac{1}{2}\int_\Omega \Big[u_1^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0\Big]dx \\ & + \frac{1}{q}\int_\Omega \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u_0|^q\,dx - \frac{1}{p}\int_\Omega c(x)|u_0|^p\,dx, \\ E_1 := {} & \frac{p-2}{2p} C^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}}. \end{aligned}$$

Theorem 3.1. *Suppose that the conditions indicated above are satisfied and, furthermore,* $p \le \frac{2n}{n-4}$ *if* $n > 4$; $2 < q < \frac{p+2}{2}$; $E_0 < E_1$, $\|D^2 u_0\|_2 > \sqrt{\frac{E_1}{B_2}\cdot\frac{2p}{p-2}}$. *Then a global solution of problem* (2.1)–(2.4) *does not exist.*

4. THE MAIN RESULT PROOF

Assume the contrary, i.e., assume that global solutions of problem (2.1)–(2.4) exist. It follows from the definition of generalized solution that the function $E(t)$, $t \in [0,+\infty)$, is continuous and its restriction to an arbitrary segment $[0,\tau)$, $\tau > 0$, is an absolutely continuous function. Moreover, it is obvious that $E(0) = E_0$. Let function u is the global solution of the problem (2.1)–(2.4). Then it satisfies (2.5). Considering $v = u_t$ in (2.5), one can obtain identity

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int\limits_{\Omega_t}\left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u\right]dx + \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\right]\\
&-\frac{d}{dt}\left[\frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p\,dx\right] + \int\limits_{\Omega_t}\sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)D^\alpha u_t D^\beta u_t\,dx\\
&-\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)D^\alpha u(x,\theta)D^\alpha u_t(x,t)\,dx\,d\theta = 0
\end{aligned} \tag{4.1}$$

for almost all $t \in [0,+\infty)$. We will transform integral

$$\begin{aligned}
&\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)D^\alpha u(x,\theta)D^\alpha u_t(x,t)\,dx\,d\theta\\
&=\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)D^\alpha u_t(x,t)[D^\alpha u(x,\theta) - D^\alpha u(x,t)]\,dx\,d\theta\\
&\quad+\int\limits_0^t g(t-\theta)\,dt\int\limits_{\Omega_t}\sum_{|\alpha|=2} d_\alpha(x)D^\alpha u(x,t)D^\alpha u_t(x,t)\,dx\\
&=-\frac{1}{2}\int\limits_0^t g(t-\theta)\frac{d}{dt}\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\\
&\quad+\frac{1}{2}\int\limits_0^t g(\theta)\,d\theta\frac{d}{dt}\int\limits_{\Omega_t}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,t)|^2\,dx\\
&=-\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\right]\\
&\quad+\frac{1}{2}\int\limits_0^t g'(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\\
&\quad+\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(\theta)\,d\theta\int\limits_{\Omega_t}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,t)|^2\,dx\right] - \frac{1}{2}g(t)\int\limits_{\Omega_t}\sum_{i,j=1}^n d_\alpha(x)|D^\alpha u(x,t)|^2\,dx.
\end{aligned}$$

Based on the last equality it follows from (4.1) that

$$\frac{d}{dt}\left[\frac{1}{2}\int\limits_{\Omega_t}\left[u_t^2+\sum_{|\alpha|=|\beta|=2}b_{\alpha\beta}(x)D^\alpha u D^\beta u\right]dx+\frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2}b_\alpha(x)|D^\alpha u|^q\,dx\right]$$

$$-\frac{d}{dt}\left[\frac{1}{p}\int\limits_{\Omega_t}c(x)|u|^p\,dx\right]$$

$$=-\int\limits_{\Omega_t}\sum_{|\alpha|=|\beta|=2}a_{\alpha\beta}(x)D^\alpha u_t D^\beta u_t\,dx$$

$$-\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta\right]$$

$$+\frac{1}{2}\int\limits_0^t g'(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta$$

$$+\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(\theta)d\theta\int\limits_{\Omega_t}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,t)|^2\,dx\right]$$

$$-\frac{g(t)}{2}\int\limits_{\Omega_t}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,t)|^2dx$$

$$\le-\int\limits_{\Omega_t}\sum_{|\alpha|=|\beta|=2}a_{\alpha\beta}(x)D^\alpha u_t D^\beta u_t\,dx$$

$$-\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta\right]$$

$$+\frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(\theta)\,d\theta\int\limits_{\Omega_t}\sum_{|\alpha|=2}d_\alpha(x)|D^\alpha u(x,t)|^2\,dx\right].$$

One can obtain

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\left[\frac{1}{2}\int\limits_{\Omega_t}\left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u\right]dx + \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx \right.\\
&\quad \left. - \frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p\,dx\right]\\
&\quad + \frac{1}{2}\frac{d}{dt}\left[\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta \right.\\
&\quad \left. - \int\limits_0^t g(\theta)\,d\theta \int\limits_{\Omega_t}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,t)|^2\,dx\right]\\
&\le -\int\limits_{\Omega_t}\sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)D^\alpha u_t D^\beta u_t\,dx < 0
\end{aligned}$$

for almost all $t \in [0,+\infty)$. From (3.1) it follows the conclusion

$$\begin{aligned}
E(t) &\ge \frac{1}{2}l(t)\|D^2u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\
&\quad - \frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p\,dx + \frac{1}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\\
&\ge \frac{1}{2}l(t)\|D^2u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\
&\quad + \frac{1}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta - \frac{C}{p}B_1^p l^{\frac{p}{2}}\|D^2u\|_2^p\\
&\ge \frac{1}{2}l(t)\|D^2u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\
&\quad + \frac{1}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\\
&\quad - \frac{CB_1^p}{p}\left[l(t)\|D^2u\|_2^2 + \frac{2}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx \right.\\
&\quad \left. + \|u_t\|_2^2 + \int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\right]^{\frac{p}{2}}.
\end{aligned} \tag{4.2}$$

Therefore,

$$E(t) \geq h(\xi(t)),$$

where $h(y) = \frac{1}{2}y^2 - \frac{CB_1^p}{p}y^p$ and

$$\xi(t) = \Bigg[l(t)\|D^2u\|_2^2 + \|u_t\|_2^2 + \frac{2}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx + \int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\Bigg]^{\frac{1}{2}}.$$

Obviously, $y_0 = C^{-\frac{1}{p-2}}B_1^{-\frac{p}{p-2}}$ is the maximum of the function h, because of $h'(y) = y - CB_1^p y^{p-1}$. Accordingly,

$$h(y_0) = \left(\frac{1}{2} - \frac{1}{p}\right)C^{-\frac{2}{p-2}}B_1^{-\frac{2p}{p-2}} = E_1 > E_0.$$

So there exists such $\beta > y_0$, that $h(\beta) = E_0$.

If

$$\beta_0 = \left[\int\limits_\Omega \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u_0 D^\beta u_0\,dx\right]^{\frac{1}{2}},$$

then

$$\begin{aligned}
h(\beta_0) &\leq \frac{1}{2}\int\limits_\Omega \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u_0 D^\beta u_0\,dx - \frac{CB_1^pB_2^{\frac{p}{2}}}{p}\left[\left(\int\limits_\Omega \sum_{|\alpha|=2}|D^\alpha u_0|^2\,dx\right)^{\frac{1}{2}}\right]^p \\
&\leq \frac{1}{2}\int\limits_\Omega \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u_0 D^\beta u_0\,dx - \frac{CB_1^pB_2^{\frac{p}{2}}}{pB^p}\|u_0\|_p^p \\
&\leq \frac{1}{2}\int\limits_\Omega \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u_0 D^\beta u_0 + |u_1|^2\right]dx + \frac{1}{q}\int\limits_\Omega \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u_0|^q\,dx \\
&\quad - \frac{1}{p}\int\limits_{\Omega_t} c(x)|u_0|^p\,dx + \frac{1}{p}\int\limits_{\Omega_t} c(x)|u_0|^p\,dx - \frac{CB_1^pB_2^{\frac{p}{2}}}{pB^p}\|u_0\|_p^p.
\end{aligned}$$

Since $B_2 > l$, then $h(\beta_0) \leq E_0 = h(\beta)$. Obviously, $\beta_0 \geq \sqrt{B_2}\|D^2u_0\|_2$.

According to the theorem

$$\|D^2u_0\|_2 > \sqrt{\frac{E_1}{B_2}\cdot\frac{2p}{p-2}},$$

consequently

$$\beta_0 > \sqrt{E_1 \cdot \frac{2p}{p-2}} \geq \sqrt{C^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}}} = y_0.$$

As function $h(y)$ is monotonically decreasing while $\beta > y_0$, taking into account the last estimations $h(\beta_0) \leq h(\beta)$, $\beta_0 > y_0$, one can get $\beta_0 > \beta > y_0$.

Further we will assume existence of $t_0 \in [0, +\infty)$, such that $\xi(t_0) < \beta$. Since ξ is continuous function, t_0 can be choosed as following $\xi(t_0) > y_0$, then $y_0 < \beta < \beta_0$. Hence, $E(t_0) \geq h(y_0) > h(\xi(t_0)) > h(\beta) = E_0$. That is impossible, because $E(t) \leq E_0$ for all $t \in [0, T)$ by the reason of strongly monotonically decreasing function $E(t)$.

So it is proved, that in case

$$E_0 < E_1, \quad \|D^2 u_0\|_2 > \sqrt{\frac{E_1}{B_2} \cdot \frac{2p}{p-2}}$$

exists $\beta \geq y_0$, such that

$$\xi(t) = \Bigg[l(t)\|D^2 u\|_2^2 + \|u_t\|_2^2 + \frac{2}{q} \int\limits_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx + \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta \Bigg]^{\frac{1}{2}} \geq \beta, \quad t \in [0, +\infty). \tag{4.3}$$

Moreover, since $E(t) < E_0$ on $(0, +\infty)$, based on (4.2) we obtain

$$\frac{1}{2} l(t)\|D^2 u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{q} \int\limits_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx + \frac{1}{2} \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta < E_0 + \frac{C}{p} \int\limits_{\Omega_t} |u|^p \, dx. \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\frac{C}{p}\|u\|_p^p \geq \frac{1}{2}\beta^2 - E_0 \geq \frac{1}{2}\beta^2 - h(\beta) = \frac{B_1^p C}{p} \beta^p \quad \text{or} \quad \|u\|_p \geq B_1 \beta.$$

If $\|u\|_p \leq 1$, then $\|u\|_p^s \leq \|u\|_p^2 \leq B\|D^2 u\|_2^2$ as $2 \leq s \leq p$. If $\|u\|_p > 1$, then $\|u\|_p^s \leq \|u\|_p^p$ as $2 \leq s \leq p$. So there is an obvious estimation

$$\|u\|_p^s \leq \kappa_1 (\|D^2 u\|_2^2 + \|u\|_p^p), \quad \kappa_1 = \max\{B, 1\}, \quad s \in [2, p]. \tag{4.5}$$

Let $H(t) = E_1 - E(t)$. Using (4.2) one can get

$$\begin{aligned}\frac{1}{2}\bigg(B_2 - l\bigg)\|D^2u\|_2^2 &\le \frac{1}{2}l(t)\|D^2u\|_2^2\\ &\le E(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\ &\quad - \frac{1}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta \qquad (4.6)\\ &\quad + \frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p\,dx.\end{aligned}$$

Additionally

$$\begin{aligned}\|u\|_p^p \ge B_1^p\beta^p > B_1^p y_0^p &= B_1^p B_1^{-\frac{p^2}{p-2}} C^{-\frac{p}{p-2}}\\ = B_1^{-\frac{2p}{p-2}} C^{-\frac{p}{p-2}} &= B_1^{-\frac{2p}{p-2}} C^{-\frac{2}{p-2}} C^{-1} = \frac{2p}{(p-2)C}E_1,\end{aligned}$$

therefore $E_1 \le \frac{(p-2)C}{2p}\|u\|_p^p$ and taking into consideration (4.6),

$$\begin{aligned}\frac{1}{2}\bigg(B_2 - l\bigg)\|D^2u\|_2^2 &\le -E_1 + E_1 + E(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\ &- \frac{1}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\\ &+ \frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p\,dx\\ &\le -H(t) - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{q}\int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx - \frac{1}{2}\int\limits_0^t g(t-\theta) \qquad (4.7)\\ &\times \int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta + \bigg(\frac{1}{p} + \frac{p-2}{2p}\bigg)C\|u\|_p^p\\ &\le \frac{1}{q}\bigg[-H(t) - \|u_t\|_2^2 - \int\limits_{\Omega_t}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\,dx\\ &- \int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta + \frac{q}{2}\|u\|_p^p\bigg].\end{aligned}$$

From inequalities (4.5) and (4.7) it follows that

$$\|u\|_p^s \le \kappa_2 \Big[- H(t) - \|u_t\|_2^2 - \int\limits_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx \\ - \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta \\ + \frac{q}{2} \|u\|_p^p \Big], \quad \kappa_2 > 0, s \in [2,p].$$

Since $H'(t) > 0$ almost everywhere on $[0, +\infty)$, then $H(t) \ge H(0) = E_1 - E_0 > 0$. From (4.4) additionally obtain the following

$$H(t) \le E_1 - \frac{1}{2} \Big[l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int\limits_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx \\ + \frac{1}{2} \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta \Big] + \frac{C}{p} \|u\|_p^p \\ \le E_1 - \frac{1}{2} \beta^2 + \frac{C}{p} \|u\|_p^p \\ \le E_1 - \frac{1}{2} \xi_0^2 + \frac{C}{p} \|u\|_p^p \\ \le \frac{p-2}{2p} \xi_0^2 - \frac{1}{2} \xi_0^2 + \frac{C}{p} \|u\|_p^p \\ \le \frac{C}{p} \|u\|_p^p.$$

Accordingly,

$$0 < H(0) \le H(t) \le \frac{C}{p} \|u\|_p^p, \quad t \in [0,T). \tag{4.8}$$

Hereafter, let

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int\limits_{\Omega_t} u u_t \, dx,$$

where $\varepsilon > 0$ and $\alpha \in (0,1)$ are arbitrary numbers. Then

$$
\begin{aligned}
L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int\limits_{\Omega_t} [u_t^2 + u u_{tt}]\,dx = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int\limits_{\Omega_t} u_t^2\,dx \\
&\quad + \varepsilon \int\limits_{\Omega_t} \Bigg[c(x)|u|^p - \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u - \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \Bigg]\,dx \\
&\quad - \varepsilon \int\limits_{\Omega_t} \Bigg[\sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \Bigg]\,dx \\
&\quad + \varepsilon \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x,\theta) D^\alpha u(x,t)\,dx\,d\theta \\
&= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int\limits_{\Omega_t} u_t^2\,dx + \varepsilon \int\limits_{\Omega_t} \Bigg[c(x)|u|^p \\
&\quad - \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u - \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u - \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \Bigg]\,dx \\
&\quad + \varepsilon \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) \Big[D^\alpha u(x,\theta) - D^\alpha u(x,t) \Big] D^\alpha u(x,t)\,dx\,d\theta \\
&\quad + \varepsilon \int\limits_0^t g(\theta)\,d\theta \int\limits_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x,t)|^2\,dx \geq (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int\limits_{\Omega_t} u_t^2\,dx \\
&\quad + \varepsilon \int\limits_{\Omega_t} c(x)|u|^p\,dx - \frac{\varepsilon\delta_1}{2} \int\limits_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u|^2\,dx - \frac{\varepsilon C_1}{2\delta_1} \int\limits_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u_t|^2\,dx \\
&\quad - \varepsilon \int\limits_{\Omega_t} \Bigg[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \Bigg]\,dx \\
&\quad - \frac{\delta_2 C_2 \varepsilon}{2} \int\limits_0^t g(t-\theta) \int\limits_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta \\
&\quad - \frac{\varepsilon C_2}{2\delta_2} \int\limits_0^t g(\theta)\,d\theta \int\limits_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u(x,t)|^2\,dx + \varepsilon C_3 \int\limits_0^t g(\theta)\,d\theta \int\limits_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u(x,t)|^2\,dx,
\end{aligned}
$$

where δ_1, δ_2 are arbitrary positive constants, positive constant C_1 depends on A, positive constant C_2 depends on $\operatorname{ess\,sup}_\Omega \sum_{|\alpha|=2} d_\alpha^2(x)$, positive constant C_3 depends on d_2.

Therefore,

$$L'(t) \geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1}\right]\|D^2u_t\|_2^2 + \varepsilon\int\limits_{\Omega_t} u_t^2\,dx + \varepsilon\int\limits_{\Omega_t} c(x)|u|^p\,dx$$

$$-\,\varepsilon\int\limits_{\Omega_t}\left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\right]dx\Bigg]$$

$$+\,\varepsilon\left[\left(1-\frac{C_2}{2\delta_2}\right)\int\limits_0^t g(\theta)\,d\theta - \frac{\delta_1}{2}\right]\|D^2u\|_2^2$$

$$-\,\frac{\varepsilon\delta_2 C_2}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta$$

$$\geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1}\right]\|D^2u_t\|_2^2 + \varepsilon\int\limits_{\Omega_t} u_t^2 dx + \varepsilon p\Big[-H(t) + H(t) + \frac{1}{p}\int\limits_{\Omega_t} c(x)|u|^p dx\Big]$$

$$-\,\varepsilon\int\limits_{\Omega_t}\left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\right]dx + \varepsilon\left[\left(1-\frac{C_2}{2\delta_2}\right)\right.$$

$$\times\left.\int\limits_0^t g(\theta)\,d\theta - \frac{\delta_1}{2}\right]\|D^2u\|_2^2 - \frac{\varepsilon\delta_2 C_2}{2}\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta$$

$$\geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1}\right]\|D^2u_t\|_2^2 + \varepsilon\|u_t\|^2 + \varepsilon p\Big[-H(t)$$

$$+\,\frac{\delta_3}{2}\int\limits_{\Omega_t}\left[|u_t|^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \frac{2}{q}\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\right]dx + \frac{\delta_3\varepsilon p}{2}$$

$$\times\int\limits_0^t g(t-\theta)\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta$$

$$-\,\frac{\delta_3\varepsilon p}{2}\int\limits_0^t g(\theta)\,d\theta\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u|^2\,dx$$

$$+\,\varepsilon(1-\delta_3)\int\limits_{\Omega_t} c(x)|u|^p\,dx - \varepsilon\int\limits_{\Omega_t}\left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q\right]dx$$

$$+\,\varepsilon\left[\left(1-\frac{C_2}{2\delta_2}\right)\int\limits_0^t g(\theta)\,d\theta - \frac{\delta_1}{2}\right]\|D^2u\|_2^2 - \frac{\varepsilon\delta_2 C_2}{2}\int\limits_0^t g(t-\theta)$$

$$\times\int\limits_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta)-D^\alpha u(x,t)|^2\,dx\,d\theta = \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1}\right]$$

$$\times\,\|D^2u_t\|_2^2 + \varepsilon\left(1+\frac{\delta_3 p}{2}\right)\|u_t\|^2\,dx - \varepsilon pH(t) + \varepsilon\left(\frac{\delta_3 p}{2} - 1\right)$$

$$\times \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \, dx + \varepsilon\Big(\frac{\delta_3 p}{q} - 1\Big) \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx + \varepsilon(1-\delta_3)$$
$$\times \int_0^t c(x)|u|^p \, dx + \varepsilon\Big(\frac{\delta_3 p}{2} - \frac{\delta_2}{2}\Big) \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta$$
$$- \varepsilon\Big(\Big(\frac{\delta_3 p}{2} - 1 + \frac{C_2}{2\delta_2}\Big) \int_0^t g(\theta)\, d\theta + \frac{\delta_1}{2}\Big) \|D^2 u\|_2^2, \tag{4.9}$$

and $0 < \delta_3 < 1$.

Choosing $\delta_1 = H^\alpha(t)\delta_4$ in (4.9) one can get

$$L'(t) \geq \Big[(1-\alpha)A_2 - \frac{\varepsilon C_1}{2\delta_4}\Big] H^{-\alpha}(t) \|D^2 u_t\|_2^2 + \varepsilon\Big(1 + \frac{\delta_3 p}{2}\Big) \|u_t\|^2 \, dx - \varepsilon p H(t) + \varepsilon$$
$$\times \Big(\frac{\delta_3 p}{2} - 1\Big) \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \, dx$$
$$+ \varepsilon\Big(\frac{\delta_3 p}{q} - 1\Big) \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx$$
$$+ \varepsilon(1-\delta_3) \int_0^t c(x)|u|^p \, dx + \varepsilon\Big(\frac{\delta_3 p}{2} - \frac{\delta_2}{2}\Big) \int_0^t g(t-\theta)$$
$$\times \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 \, dx \, d\theta$$
$$- \varepsilon\Big(\frac{\delta_3 p}{2} - 1 + \frac{C_2}{2\delta_2}\Big) \int_0^t g(\theta)\, d\theta \|D^2 u\|_2^2 - \frac{\varepsilon \delta_4}{2} H^\alpha(t) \|D^2 u\|_2^2. \tag{4.10}$$

Let us set $\alpha = \frac{q-2}{p}$. By (4.8) and the spaces embeddings $L^{2+p\alpha}(\Omega) \subset L^2(\Omega)$, we get

$$H^\alpha(t)\|D^2 u\|_2^2 \leq \Big(\frac{C}{p}\Big)^\alpha \|u\|_p^{p\alpha} \|D^2 u\|_2^2 \leq \Big(\frac{C}{p}\Big)^\alpha B^{p\alpha} \|D^2 u\|_2^{p\alpha} \|D^2 u\|_2^2$$
$$\leq \Big(\frac{CB^p}{p}\Big)^\alpha \|D^2 u\|_2^{p\alpha+2} \leq \Big(\frac{CB^p}{p}\Big)^\alpha C_3 \|D^2 u\|_q^{p\alpha+2} = C_4 \|D^2 u\|_q^q,$$

$$C_4 = \Big(\frac{C}{p}\Big)^\alpha B^{p\alpha} C_3, \quad C_3 > 0.$$

Inequality (4.10) can be rewritten as follows:

$$\begin{aligned} L'(t) \geq & \left[(1-\alpha)A_2 - \frac{\varepsilon C_1}{2\delta_4}\right] H^{-\alpha}(t)\|D^2 u_t\|_2^2 + \varepsilon\left(1 + \frac{\delta_3 p}{2}\right)\|u_t\|^2\,dx - \varepsilon p H(t) \\ & + \varepsilon\left[\left(\frac{\delta_3 p}{2} - 1\right)B_2 - \left(\frac{\delta_3 p}{q} - 1 + \frac{C_2}{2\delta_2}\right)\left(B_2 - l\right)\right]\|D^2 u\|_2^2 \\ & + \varepsilon\left[\left(\frac{\delta_3 p}{2} - 1\right)b_2 - \frac{C_4\delta_4}{2}\right]\|D^2 u\|_q^q + \varepsilon(1-\delta_3)c_0\|u\|_p^p \\ & + \varepsilon\left(\frac{\delta_3 p}{2} - \frac{\delta_2}{2}\right)\int_0^t g(t-\theta)\int_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta. \end{aligned} \tag{4.11}$$

Due to the conditions of the theorem we can choose the parameters δ_i, $i = 2,3,4$, so that inequality (4.11) yields

$$\begin{aligned} L'(t) \geq C_5 \Bigg[& -H(t) + \|D^2 u\|_2^2 + \|D^2 u\|_q^q + \|u_t\|\|_2^2 + \|u\|_p^p \\ & + \int_0^t g(t-\theta)\int_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\Bigg], \quad C_5 > 0. \end{aligned} \tag{4.12}$$

The next point under consideration

$$\begin{aligned} \Big[L(t)\Big]^{\frac{1}{1-\alpha}} & = \left[H^\alpha(t) + \varepsilon\int_{\Omega_t} u u_t\,dx\right]^{\frac{1}{1-\alpha}} \leq C_6\left[H(t) + \|u_t\|_2^{\frac{1}{1-\alpha}}\,\|u\|_2^{\frac{1}{1-\alpha}}\right] \\ & \leq C_7\left[H(t) + \|u_t\|_2^2 + \|u\|_2^{\frac{2}{1-2\alpha}}\right], \quad C_6 > 0, \quad C_7 > 0. \end{aligned} \tag{4.13}$$

Since $q < \dfrac{p+2}{2}$ and $\alpha = \dfrac{q-2}{p}$, then $\alpha < \dfrac{1}{2}$ and $2 \leq \dfrac{2}{1-2\alpha} \leq p$. Hence

$$\|u\|_2^{\frac{2}{1-2\alpha}} \leq C_8\left(\|u\|_2^2 + \|u\|_p^p\right), \quad \|u\|_2^2 \leq C_9\|D^2 u\|_2^2 \leq C_9\Big[-H(t) + C_{10}\|u\|_p^p\Big],$$

$C_8 > 0$, $C_9 > 0$, $C_{10} > 0$. Thereby from (4.13)

$$\begin{aligned} L'(t) \leq C_{11} \Bigg[& -H(t) + \|D^2 u\|_2^2 + \|D^2 u\|_q^q + \|u_t\|\|_2^2 + \|u\|_p^p \\ & + \int_0^t g(t-\theta)\int_{\Omega_\theta}\sum_{|\alpha|=2}|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2\,dx\,d\theta\Bigg], \quad C_{11} > 0. \end{aligned} \tag{4.14}$$

Taking into account (4.12), (4.14),

$$L'(t) \geq C_{12}\Big[L(t)\Big]^{\frac{1}{1-\alpha}}, \quad C_{12} > 0. \tag{4.15}$$

Integrating the both sides of (4.15) by variable τ from 0 to t, obtain the following

$$L(t) \geq \frac{1}{\left[L(0)^{\frac{\alpha}{\alpha-1}} - \frac{C_{12}\alpha}{1-\alpha}t\right]^{\frac{1-\alpha}{\alpha}}}. \tag{4.16}$$

Since

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int\limits_{\Omega_0} u_0(x)u_1(x)\,dx,$$

then, in virtue of $H(0) > 0$, by choosing sufficiently small $\varepsilon > 0$ it is possible to obtain $L(0) > 0$. Then from (4.16) we deduce the existence of such $T^* > 0$ that

$$\lim_{t \to T^* - 0} L(t) = +\infty.$$

We arrive at a contradiction with the statement that the function $L(t)$ is continuous on $[0, +\infty)$. So u cannot be aglobal solution of the problem (2.1)–(2.4) in the domain Q_T. The theorem is proved.

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