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SPANNING TREES WITH A BOUNDED NUMBER OF LEAVES

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Abstract. In 1998, H. Broersma and H. Tuinstra proved that: Given a connected graph G with $n \geq 3$ vertices, if $d(u) + d(v) \geq n - k + 1$ for all non-adjacent vertices u and v of G ($k \geq 1$), then G has a spanning tree with at most k leaves. In this paper, we generalize this result by using implicit degree sum condition of t ($2 \leq t \leq k$) independent vertices and we prove what follows: Let G be a connected graph on $n \geq 3$ vertices and $k \geq 2$ be an integer. If the implicit degree sum of any t independent vertices is at least $\frac{t(n-k)}{2} + 1$ for ($k \geq t \geq 2$), then G has a spanning tree with at most k leaves.

Keywords: spanning tree, implicit degree, leaves.

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1. INTRODUCTION

In this paper, we consider only undirected, finite and simple graphs. Notation and terminology not defined here can be found in [2]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let H be a subgraph of G. For a vertex $u \in V(G)$, we define the *neighborhood* of u in H, denoted by $N_H(u)$, the set of vertices in H which are adjacent to u in G. The *degree* of u in H, denoted by $d_H(u)$, is $|N_H(u)|$. Let $P_H[u, v]$ denote a path between u and v in H and we call u and v end-vertices of $P_H[u, v]$. When $H = G$, we use $N(u)$, $d(u)$ and $P[u, v]$ in place of $N_G(u)$, $d_G(u)$ and $P_G[u, v]$, respectively.

Given a path P in G with two end-vertices a and b, let one of the end-vertices, say a, be the source vertex, the other end-vertex b, be the sink vertex. For a vertex x on P, we denote x^- the neighbor of x on P which is closer to the source vertex a and denote the other neighbor of x on P by x^+. We set $x^{+(h+1)} = (x^{+h})^+$ and $x^{-(h+1)} = (x^{-h})^-$ for $h \geq 1$. And for any $I \subseteq V(P)$, let $I^- = \{x \in V(P) : x^+ \in I\}$ and $I^+ = \{x \in V(P) : x^- \in I\}$. For two vertices $x, y \in V(P)$, $P[x, y]$ or xPy denotes the sub-path of P from x to y.

An *independent set* of a graph G is a subset of vertices such that no two vertices in the subset induce an edge of G. The cardinality of a maximum independent set in a graph G is called the *independence number* of G, denoted by $\alpha(G)$. For an integer $t \geq 1$, if $t \geq \alpha(G)$, then we denote

$$\sigma_t(G) = \min\left\{ \sum_{i=1}^{t} d(u_i) : \{u_1, u_2, \ldots, u_t\} \text{ is an independent set of } G \right\},$$

otherwise $\sigma_t(G) = +\infty$.

A *leaf* of a tree T is a vertex of T with degree one. A natural generalization of hamiltonian paths are spanning trees with small numbers of leaves. In 1998, Broersma and Tuinstra [3] proved the following theorem.

Theorem 1.1 ([3]). *Let G be a connected graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n - k + 1$ $(k \geq 2)$ for every pair of non-adjacent vertices u and v in G, then G has a spanning tree with at most k leaves.*

In 1989, Zhu, Li and Deng [12] found that though some vertices may have small degrees, we can use some large degree vertices to replace small degree vertices in the right position considered in the proofs, so that we may construct a longer cycle. This idea leads to the definition of implicit-degree. Since then, many results on cycles and paths using degree conditions have been extended by using implicit-degree conditions. There are two kinds of implicit-degrees of vertices introduced in [12]. Here we introduced these two definitions.

For any vertex u of G, denote $N_1(u) = N(u) = \{v \in V(G) : uv \in E(G)\}$ and $N_2(u) = \{v \in V(G) : d(u,v) = 2\}$. Set $M_2 = \max\{d(v) : v \in N_2(u)\}$ and $m_2 = \min\{d(v) : v \in N_2(u)\}$. Assume that $d(u) = l + 1$ and $d_1 \leq d_2 \leq \ldots \leq d_l \leq d_{l+1} \leq \ldots$ is the degree sequence of the vertices in $N_1(u) \cup N_2(u)$. If $N_2(u) \neq \emptyset$ and $l \geq 1$, the two kinds of *implicit-degrees* of u, denoted by $id_1(u)$ and $id_2(u)$, respectively, are defined by

$$id_1(u) = \begin{cases} \max\{d_{l+1}, l+1\}, & \text{if } \ d_{l+1} > M_2, \\ \max\{d_l, l+1\}, & \text{otherwise} \end{cases}$$

and

$$id_2(u) = \begin{cases} \max\{m_2, l+1\}, & \text{if } \ m_2 \geq d_l, \\ id_1(u), & \text{otherwise.} \end{cases}$$

If $N_2(u) = \emptyset$ or $l < 1$, then define $id_1(u) = id_2(u) = d(u)$. By the above definitions, it is easy to check that $id_2(u) \geq id_1(u) \geq d(u)$.

Similar to the degree sum of independent vertices of a graph G, we can consider the implicit-degree sum of independent vertices of G. In [12], the authors gave a sufficient condition for a 2-connected graph to be hamiltonian by considering the implicit-degree sum of two non-adjacent vertices.

Theorem 1.2 ([12])**.** *Let G be a 2-connected graph on $n \geq 3$ vertices. If for each pair of non-adjacent vertices u and v, $id_2(u) + id_2(v) \geq n$, then G is hamiltonian.*

The introduce of implicit-degree is very important, since many classic results by considering degree conditions in graph theory can be generalized. We just give one example to show this. Fan's theorem [6] can be easily obtained from Theorem 1.2 and the authors in [12] gave a simple proof of this.

Coroallary 1.3 ([6])**.** *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices u and v with $d(u, v) = 2$, then G is hamiltonian.*

For more results using implicit-degree conditions, we refer to [4, 5] and [9]. Note that the degree condition in Theorem 1.1 can be seen as $\sigma_2(G) \geq \frac{2(n-k)}{2} + 1$. In this paper, using the first kind of implicit-degree of vertices, we will extend Theorem 1.1 by using the condition of implicit-degree sum of t independent vertices. Throughout this work, we will always use $id(u)$ to denote $id_1(u)$ and

$$i\sigma_t(G) = \min\left\{\sum_{i=1}^{t} id(u_i) : \{u_1, u_2, \dots, u_t\} \text{ is an independent set of } G\right\}$$

if $t \geq \alpha(G)$, otherwise $i\sigma_t(G) = +\infty$. We will show that the following result holds.

Theorem 1.4. *Let G be a connected graph on $n \geq 3$ vertices and $k \geq 2$ be an integer. If $i\sigma_t(G) \geq \frac{t(n-k)}{2} + 1$ $(k \geq t \geq 2)$, then G has a spanning tree with at most k leaves.*

Remark 1.5. This result is sharp, we can easily check it from the graph $K_{m,m+k}$. We can see that for $2 \leq t \leq k$,

$$i\sigma_t(K_{m,m+k}) = tm = \frac{t(|V(K_{m,m+k})| - k)}{2},$$

but any spanning tree in $K_{m,m+k}$ has at least $k + 1$ leaves.

The paper is organized as follows: in Section 2, we will give some preliminaries for the proof of Theorem 1.4; in Section 3, we will give the proof of Theorem 1.4.

2. PRELIMINARIES

In this section we will give some preliminaries for the proof of Theorem 1.4. First, we give a lemma on the property of σ_t.

Lemma 2.1 ([7])**.** *For any graph G and $t \geq 1$,*

$$\frac{\sigma_{t+1}(G)}{t+1} \geq \frac{\sigma_t(G)}{t}.$$

Then, we give a lemma on the upper bound of degree sum of two vertices on a given path.

Lemma 2.2. *Let P be a path with two end-vertices a and b and let b be the sink vertex of P. For any two vertices x, y not on P, if $N_P(x)^- \cap N_P(y) = \emptyset$, then $d_P(x) + d_P(y) \leq |V(P)| + 1$.*

Proof. Since $N_P(x)^- \cap N_P(y) = \emptyset$, we have $N_P(x)^- \cup N_P(y) \subseteq V(P)$ and $|N_P(x)^-| + |N_P(y)| \leq |V(P)|$. Note that there is no vertex a^- and x can be adjacent to a, $|N_P(x)| \leq |N_P(x)^-| + 1$. Thus

$$d_P(x) + d_P(y) = |N_P(x)| + |N_P(y)| \leq |V(P)| + 1. \qquad \square$$

3. PROOF OF THEOREM 1.4

In this section we will prove Theorem 1.4 by contradiction. Let G be a graph satisfying the condition of Theorem 1.4 and suppose to the contrary that any spanning tree of G has at least $k + 1$ leaves.

Let $\mathcal{T}$ denote the set of spanning trees of G. We choose a spanning tree $T \in \mathcal{T}$ satisfying the followings conditions:

(1) the number of leaves in T is as small as possible; let X be the set of leaves of T, and
(2) $\sum_{v \in X} d(v)$ is as large as possible, subject to (1).

Claim 3.1. *X is an independent set of G.*

Proof. Denote $X = \{v_1, v_2, \ldots, v_l\}$, $l \geq k + 1 \geq 3$. Suppose to the contrary that there exist two vertices v_i, v_j such that $v_i v_j \in E(G)$. Since $l \geq 3$, there must exist a vertex of T with degree at least 3 in T. Let u be the unique vertex which has at least 3 neighbors on T and is closest to v_j on T. Denote the path $v_j T u = v_j u_1 \ldots u_s u$. Let T' be the spanning tree obtained from T by deleting the edge $u_s u$ and adding the edge $v_i v_j$. It is easy to check that v_i, v_j are not leaves of T' and u_s is a new leaf of T'. Thus T' has at least one leaf less than that of T, a contradiction to the choice of T. $\square$

Claim 3.2. *For each vertex v_j in X, we have $id(v_j) = d(v_j)$.*

Proof. Suppose to the contrary that there exists a vertex $v_j \in X$ such that $id(v_j) > d(v_j)$. Denote $d(v_j) = s$ and $N(v_j) = \{x_1, x_2, \ldots, x_s\}$. Let $d_1 \leq d_2 \leq \ldots \leq d_{s-1} \leq d_s \leq \ldots$ be the degree sequence of vertices in $N_1(v_j) \cup N_2(v_j)$. Note that $s \geq 2$, otherwise $id(v_j) = d(v_j)$, a contradiction. Without loss of generality, we assume that x_1 is the neighbor of v_j on T. For $2 \leq i \leq s$, we denote x_i' be the neighbor of x_i on T which is closet to v_j. Since X is an independent set, there exists a vertex $x_m'' \in T \setminus \{x_2', x_3', \ldots, x_s'\}$ such that x_m'' is adjacent to x_m on T and $x_m'' v_j \notin E(G)$, that means $x_m'' \in N_2(v_j)$.

By the definition of implicit-degree of v_j, $id(v_j) = d_{s-1}$ or $id(v_j) = d_s$. Thus we will continue the proof by discussing these two cases.

Case 1. $id(v_j) = d_{s-1}$.
Since $\{x_2', x_3', \ldots, x_s'\} \subseteq N_1(v_j) \cup N_2(v_j)$, $d(x_2'), d(x_3'), \ldots, d(x_s')$ are in the degree sequence of vertices in $N_1(v_j) \cup N_2(v_j)$. Thus there must exist a vertex

$x'_p \in \{x'_2, x'_3, \ldots, x'_s\}$ such that $d(x'_p) \geq d_{s-1} = id(v_j) > d(v_j)$. We can claim that x'_p is a vertex of degree 2 on the spanning tree T. Otherwise, we can obtain a new spanning tree T' from T by deleting the edge $x_p x'_p$ and adding the edge $v_j x_p$. Note that v_j is not a leaf of T' and x'_p is of degree at least 2 in T'. Thus T' has less leaves than T, a contradiction. Now, denote by T^* the spanning tree obtained from T by deleting the edge $x_p x'_p$ and adding the edge $v_j x_p$. Then v_j is not a leaf on T^* while x'_p is a new leaf. Let X^* denote the set of leaves of T^*, then $|X^*| = |X|$. However,

$$\sum_{u \in X^*} d(u) - \sum_{w \in X} d(w) = d(x'_p) - d(v_j) \geq id(v_j) - d(v_j) > 0,$$

a contradiction to the choice of T.

Case 2. $id(v_j) = d_s$.
Note that by the definition of $id(v_j)$, $d_s > M_2$, here $M_2 = \max\{d(u) : u \in N_2(v_j)\}$. Since $x''_m \in N_2(v_j)$, we can get that $d(x''_m) \leq M_2 < d_s = id(v_j)$. Since $x''_m, x'_2, x'_3, \ldots, x'_s$ are s vertices in $N_1(v_j) \cup N_2(v_j)$, there must exist a vertex, say $x'_p \in \{x'_2, x'_3, \ldots, x'_s\}$ such that $d(x'_p) \geq d_s = id(v_j) > d(v_j)$. Similarly as in Case 1, we can obtain a new spanning tree T^* from T by deleting the edge $x'_p x_p$ and adding the edge $v_j x_p$. Then v_j is not a leaf in T^* while x'_p is a new leaf of T^*. Let X^* denote the set of leaves of T^*, then $|X^*| = |X|$. However,

$$\sum_{u \in X^*} d(u) - \sum_{w \in X} d(w) = d(x'_p) - d(v_j) \geq id(v_j) - d(v_j) > 0,$$

a contradiction to the choice of T.

This completes the proof of Claim 3.2. □

Now we calculate degree sum in G of any two leaves v_i, v_j of the spanning tree T. Denote by P_{ij} the path connecting v_i and v_j on the spanning tree T.

Claim 3.3. $N_{P_{ij}}(v_i)^- \cap N_{P_{ij}}(v_j) = \emptyset$.

Proof. Suppose to the contrary that $N_{P_{ij}}(v_i)^- \cap N_{P_{ij}}(v_j) \neq \emptyset$ and assume $v \in N_{P_{ij}}(v_i)^- \cap N_{P_{ij}}(v_j)$. Since $l \geq 3$ and T is connected, there must exist a vertex u on the sub-path $v_i^+ P_{ij} v_j^-$ such that $d_T(u) \geq 3$. Consider the new spanning tree T' obtained from T by adding the edges $v_i v^+$ and $v_j v$ and deleting the edges vv^- and uu^+ if $u \in V(v_i P_{ij} v)$ or uu^- if $u \in V(v P_{ij} v_j)$. That is $T' = (T \setminus \{vv^+, uu^+\}) \cup \{v_i v^+, v_j v\}$ if $u \in V(v_i P_{ij} v)$ or $T' = (T \setminus \{vv^+, uu^-\}) \cup \{v_i v^+, v_j v\}$ if $u \in V(v P_{ij} v_j)$. It is easy to check that v_i, v_j are not leaves of T' and u^+ is the new leaf if $u \in V(v_i P_{ij} v)$ or u^- is the new leaf if $u \in V(v P_{ij} v_j)$. Thus T' has at least one leaf less than that of T. This is contrary to the choice of T. □

Set $P'_{ij} = P_{ij} \setminus \{v_i, v_j\}$. Then by Claim 3.3 and Lemma 2.2, we have

$$d_{P'_{ij}}(v_i) + d_{P'_{ij}}(v_j) \leq |P'_{ij}| + 1 = |P_{ij}| - 1.$$

By Claim 3.1, $v_i v_j \notin E(G)$. Therefore, we can get that

$$d_{P_{ij}}(v_i) + d_{P_{ij}}(v_j) = d_{P'_{ij}}(v_i) + d_{P'_{ij}}(v_j) \leq |P_{ij}| - 1. \tag{3.1}$$

Next we consider the vertices in $T - V(P_{ij})$. Since T is a spanning tree and P_{ij} is a path of T, we can decompose each component of $T - V(P_{ij})$ into paths in the following ways: each time when the component, say C, is not a path, we take a path P' as long as possible in C such that one end-vertex of P' is adjacent to a vertex of P_{ij} in T and continue this process in $C - V(P')$. Assume we decompose all the components of $T - V(P_{ij})$ into paths $\mathcal{P} = \{P_1, P_2, \ldots\}$.

Now, we calculate the degree sum of v_i and v_j on each path P_s. By the choice of each P_s, there must be one end-vertex of P_s, say v_s, is a leaf of T. Thus the end-vertex v_s is not adjacent to v_i or v_j.

Claim 3.4. *The other end-vertex of P_s, denoted by x_s, is not adjacent to v_i or v_j.*

Proof. Suppose to the contrary that x_s is adjacent to v_i or v_j, without of loss generality, say $v_i x_s \in E(G)$. Denote by x'_s the vertex of degree at least 3 which is adjacent to x_s on the spanning tree T. By deleting the edge $x'_s x_s$ and adding the edge $v_i x_s$, we can obtain a new spanning tree which has less leaves than T, a contradiction. □

By using the similar method as in Claim 3.3, we can get that $N_{P_s}(v_i)^- \cap N_{P_s}(v_j) = \emptyset$. Therefore, by Lemma 2.2, we can prove that

$$d_{P_s \setminus \{v_s, x_s\}}(v_i) + d_{P_s \setminus \{v_s, x_s\}}(v_j) \leq |P_s \setminus \{v_s, x_s\}| + 1 = |P_s| - 1.$$

Thus by Claim 3.4, we can get that

$$d_{P_s}(v_i) + d_{P_s}(v_j) = d_{P_s \setminus \{v_s, x_s\}}(v_i) + d_{P_s \setminus \{v_s, x_s\}}(v_j) \leq |P_s| - 1. \tag{3.2}$$

Since T has at least l leaves and P_{ij} has two leaves in $\mathcal{P}$, there are at least $l - 2 \geq k - 1$ paths. Thus by inequation (3.1) and inequation (3.2), we have

$$\begin{aligned} d(v_i) + d(v_j) &= d_{P_{ij}}(v_i) + d_{P_{ij}}(v_j) + \sum_{P_s \in \mathcal{P}} d_{P_s}(v_i) + d_{P_s}(v_j) \\ &\leq |P_{ij}| - 1 + \sum_{P_s \in \mathcal{P}} (|P_s| - 1) \leq n - k. \end{aligned}$$

Therefore, we can get that

$$d(v_1) + d(v_i) \leq n - k, \quad i = 2, 3, \ldots, l$$

and

$$d(v_i) + d(v_{i+1}) \leq n - k, \quad i = 1, 2, \ldots, l - 1.$$

Thus, for $2 \leq t \leq k$, we have

$$\sum_{i=1}^{t} d(v_i) \leq \frac{t(n-k)}{2}.$$

By Claim 3.2, $\sum_{i=1}^{t} id(v_i) = \sum_{i=1}^{t} d(v_i)$, we have

$$i\sigma_t(G) \leq \sum_{i=1}^{t} id(v_i) = \sum_{i=1}^{t} d(v_i) \leq \frac{t(n-k)}{2},$$

a contradiction. This completes the proof of Theorem 1.4. □

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