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ON STRONGLY SPANNING k-EDGE-COLORABLE SUBGRAPHS

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Abstract. A subgraph H of a multigraph G is called strongly spanning, if any vertex of G is not isolated in H. H is called maximum k-edge-colorable, if H is proper k-edge-colorable and has the largest size. We introduce a graph-parameter $sp(G)$, that coincides with the smallest k for which a multigraph G has a maximum k-edge-colorable subgraph that is strongly spanning. Our first result offers some alternative definitions of $sp(G)$. Next, we show that $\Delta(G)$ is an upper bound for $sp(G)$, and then we characterize the class of multigraphs G that satisfy $sp(G) = \Delta(G)$. Finally, we prove some bounds for $sp(G)$ that involve well-known graph-theoretic parameters.

Keywords: k-edge-colorable subgraph, maximum k-edge-colorable subgraph, strongly spanning k-edge-colorable subgraph, $[1, k]$-factor.

Mathematics Subject Classification: 05C70, 05C15.

1. INTRODUCTION

Let N denote the set of positive integers. In this paper we consider multigraphs. They are assumed to be finite, undirected and without loops, though they may contain multiedges. A multigraph without multiedges will be called a graph. If G is a multigraph, then for a vertex $x \in V(G)$ $d_G(x)$ denotes the degree of x in G. Moreover, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in G, respectively. A vertex is defined to be isolated in G, if its degree is zero. If G' is a subgraph of G, then we say that G' covers (misses) a vertex x of G, if $d_{G'}(x) \geq 1$ ($d_{G'}(x) = 0$). A subgraph is strongly spanning, if it covers all the vertices of the graph. A point that should be made clear here, is that if a vertex x of G is not a vertex of a subgraph G', then we assume that $d_{G'}(x) = 0$.

The length of a path P of a multigraph G is the number of edges lying on P. If a, b are non-negative integers, then a subgraph H of a multigraph G with $V(H) = V(G)$ is called an $[a, b]$-factor of G if for any vertex v of G $a \leq d_H(v) \leq b$. A subset E' of

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edges of a multigraph G is called matching, if $(V(G), E')$ is a $[0,1]$-factor of G. Clearly, matchings can be defined as a set of edges that contain no adjacent edges. Usually, a vertex that is (not) incident to an edge from a matching, is said to be covered (missed) by the matching. A matching is maximum, if it has the largest cardinality. A matching is perfect, if any vertex is incident to an edge from the matching.

A proper k-edge-coloring of a multigraph G is an assignment of colors from a set of k colors such that adjacent edges receive different colors. Observe that a proper k-edge-coloring of a multigraph G can be viewed as a partition of $E(G)$ into k matchings. Usually, these matchings into which $E(G)$ is partitioned, are called color-classes of the edge-coloring. The least integer k for which G has a proper k-edge-coloring is called the chromatic index of G and is denoted by $\chi'(G)$. Clearly, $\chi'(G) \geq \Delta(G)$ for any multigraph G, and the following classical theorems of Shannon and Vizing give non-trivial upper bounds for $\chi'(G)$:

Theorem 1.1 ([16])**.** *For every multigraph G*

$$\Delta(G) \leq \chi'(G) \leq \left[\frac{3\Delta(G)}{2}\right].$$

Theorem 1.2 ([19])**.** *For every multigraph G*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G),$$

where $\mu(G)$ denotes the maximum multiplicity of an edge in G.

Note that Shannon's theorem implies that if we consider a cubic multigraph G, then $3 \leq \chi'(G) \leq 4$, thus $\chi'(G)$ can take only two values. In 1981 Holyer proved that the problem of deciding whether $\chi'(G) = 3$ or not for cubic multigraphs G is NP-complete [8], thus the calculation of $\chi'(G)$ is already hard for cubic multigraphs.

For a multigraph G and $k \in N$, let

$$\nu_k(G) = \max\{|E(H_k)| : H_k \text{ is a proper } k\text{-edge-colorable subgraph of } G\}.$$

A proper k-edge-colorable subgraph of G containing $\nu_k(G)$ edges will be called a maximum k-edge-colorable subgraph. We define $\nu(G) = \nu_1(G)$.

The quantitative aspect of the investigation of maximum k-edge-colorable subgraphs of multigraphs and particularly, r-regular multigraphs has attracted a lot of attention, previously. The basic problem that researchers were interested was the following: what is the proportion of edges of a multigraph (or an r-regular multigraph, and particularly, cubic multigraph), that we can cover by its k matchings?

For the case $k = 1$ in [7] an investigation is carried out in the class of cubic graphs, and in [4, 6, 13, 14, 20] for the general case. Let us also note that the relation between $\nu_1(G)$ and $|V|$ has also been investigated in the regular multigraphs of high girth [5].

The same is true for the case $k = 2, 3$. Albertson and Haas investigate these ratios in the class of cubic and 4-regular graphs in [1, 2], and Steffen investigates the problem in the class of bridgeless cubic multigraphs in [17]. Similar investigations are done in [15] for subcubic multigraphs. In [11] the problem is addressed in the class of cubic multigraphs. Finally, a best-possible bound is proved in [12] for the case $k = \Delta(G)$ in the class of all multigraphs.

It deserves to be mentioned that the quantitative line of the research was not the only one. Previously, a special attention was also paid to structural properties of maximum k-edge-colorable subgraphs, and sometimes this kind of results have helped researchers to get quantitative results. A typical example of a structural result is the one proved in [2], which states that in any cubic multigraph G there is a maximum 2-edge-colorable subgraph H, such that the multigraph $G\backslash E(H)$ is 2-edge-colorable. Recently, in [12] new such results are presented for maximum $\Delta(G)$-edge-colorable subgraphs of multigraphs G. In particular, it is shown there that any set of vertex-disjoint cycles of a multigraph G (particularly, any 2-factor) can be extended to a maximum $\Delta(G)$-edge-colorable subgraph of G if $\Delta(G) \geq 3$. Also, it is shown there that for any maximum $\Delta(G)$-edge-colorable subgraph H of G $|\partial_H(X)| \geq \lceil \frac{|\partial_G(X)|}{2} \rceil$ for each $X \subseteq V(G)$, where $\partial_K(X)$ is the set of edges of a multigraph K with exactly one end-vertex in X. Finally, in [3] it is shown that the edges of a cubic multigraph lying outside a maximum 3-edge-colorable subgraph form a matching. Though this result does not have a direct generalization, using the ideas of the proof of Vizing theorem for graphs from [21], in [12] it is shown that a graph G has a maximum $\Delta(G)$-edge-colorable subgraph H, such that the edges of G that do not belong to H form a matching.

In this paper, we concentrate on maximum k-edge-colorable subgraphs of multigraphs that are strongly spanning. In the beginning of the paper we introduce a graph-parameter $sp(G)$, that coincides with the smallest k for which a graph G has a maximum k-edge-colorable subgraph that is strongly spanning. We first give some alternative definitions of $sp(G)$. Then, we show that $\Delta(G)$ is an upper bound for $sp(G)$, and we proceed with the characterization of graphs G with $sp(G) = \Delta(G)$. Finally, we relate $sp(G)$ to some well-known graph-theoretic parameters.

Non-defined terms and concepts can be found in [10, 21].

2. THE MAIN RESULTS

We start with a lemma, that will allow us to look at our main parameter from various perspectives.

Lemma 2.1. *If a multigraph G has a k-edge-colorable subgraph that is strongly spanning, then it has a maximum k-edge-colorable subgraph that is strongly spanning, too.*

Proof. Let A_k be a k-edge-colorable subgraph that is strongly spanning. Consider all maximum k-edge-colorable subgraphs of G, and among them choose the ones that cover maximum possible number of vertices. From these subgraphs, choose a subgraph H_k such that $|E(A_k) \cap E(H_k)|$ is maximized. Let us show that H_k is a strongly spanning subgraph.

On the opposite assumption, consider a vertex u missed by H_k. Consider the vertices $u_1, \ldots, u_q$ $(q \geq 1)$ that are adjacent to u. Since H_k is a maximum k-edge-colorable subgraph of G, we have:

(a) $d_G(u_i) \geq k+1$ for $i = 1, \ldots, q$;
(b) $d_{H_k}(u_i) = k$ for $i = 1, \ldots, q$.

Let v_i be any neighbour of the vertex u_i $(1 \leq i \leq q)$ with $d_{H_k}(v_i) \geq 1$. Note that (a) implies that such a vertex v_i exists, moreover, it is different from u. Let us show that

(c) $d_{H_k}(v_i) = 1$.

Now if $d_{H_k}(v_i) \geq 2$, then define a subgraph H'_k of G as follows:

$$H'_k = (H_k \backslash \{(u_i, v_i)\}) \cup \{(u, u_i)\}.$$

Clearly H'_k is a maximum k-edge-colorable subgraph of G. Moreover, H'_k covers more vertices of G than H_k does, which contradicts the choice of H_k. Thus (c) must hold.

We are ready to complete the proof of the lemma. Since A_k is a strongly spanning subgraph, there is an edge $e = (u, w) \in E(A_k)$. By (b), we have $d_{H_k}(w) = k$, thus there is an edge $f = (w, z) \in E(H_k)$ such that $f \notin E(A_k)$. Consider a subgraph H''_k of G defined as follows:

$$H''_k = (H_k \backslash \{f\}) \cup \{e\}.$$

Clearly H''_k is a maximum k-edge-colorable subgraph of G. Due to (c), H''_k covers maximum possible number of vertices, like H_k does. However,

$$|E(A_k) \cap E(H''_k)| > |E(A_k) \cap E(H_k)|,$$

which contradicts the choice of H_k. The proof of the Lemma 2.1 is complete.

□

Next, we prove the following theorem.

Theorem 2.2. *For $k \in N$ and a multigraph G without isolated vertices, the following conditions are equivalent:*

(a) *G contains a $[1, k]$-factor,*
(b) *G contains a k-edge-colorable subgraph that is strongly spanning,*
(c) *G contains a maximum k-edge-colorable subgraph that is strongly spanning.*

Proof. Since a maximum k-edge-colorable subgraph is a k-edge-colorable subgraph, (c) implies (b). Moreover, since a strongly spanning k-edge-colorable subgraph is a $[1, k]$-factor, (b) implies (a). By Lemma 2.1, we already have that (b) implies (c). Thus, it suffices to show that (a) implies (b).

Let H be a $[1, k]$-factor of G. Let T be a sub-forest of H with $V(T) = V(H) = V(G)$. Clearly, T is a strongly spanning subgraph of G. Since T is $\Delta(T)$-edge-colorable and $\Delta(T) \leq \Delta(H) \leq k$, we have that T is k-edge-colorable. Hence (a) implies (b). The proof of Theorem 2.2 is complete. □

Corollary 2.3. *If a multigraph has a perfect matching, then for all $k \geq 1$ it has a maximum k-edge-colorable subgraph that is strongly spanning.*

We are ready to introduce our main parameter. If G is a multigraph without isolated vertices, then define

$$sp(G) = \min\{k : G \text{ has a maximum } k\text{-edge-colorable subgraph} \\ \text{that is strongly spanning}\}.$$

Observe that due to Theorem 2.2, $sp(G)$ coincides with the least k for which G has a k-edge-colorable subgraph that is strongly spanning. Similarly, $sp(G)$ represents the smallest k for which G has a $[1, k]$-factor.

A multigraph G without isolated vertices can be viewed as a $[1, \Delta(G)]$-factor of G, thus we have

$$1 \leq sp(G) \leq \Delta(G). \tag{2.1}$$

The following theorem of Tutte characterizes multigraphs G with $sp(G) = 1$.

Theorem 2.4 (see [10, Theorem 3.1.1])**.** *A multigraph G has a perfect matching, if and only if for any $S \subseteq V(G)$ one has $o(G - S) \leq |S|$, where for a multigraph H $o(H)$ denotes the number of components of H that contain odd number of vertices.*

We will also need the Tutte-Berge formula, which can be shown to be equivalent to the mentioned theorem of Tutte (see Theorem 3.1.14 from [10]).

Theorem 2.5 (Tutte-Berge formula)**.** *For any multigraph G*

$$\max_{S \subseteq V(G)} (o(G - S) - |S|) = |V(G)| - 2\nu(G).$$

Now, let us characterize the class of multigraphs with $sp(G) = \Delta(G)$. Clearly, if $G_1, \ldots, G_t$ are components of G, then $sp(G) = \max\{sp(G_1), \ldots, sp(G_t)\}$. Thus, a multigraph G satisfies the equality $sp(G) = \Delta(G)$ if and only if some of its components satisfies the same equality, and the maximum degree among those components coincides with the maximum degree of G. This observation enables us to focus on the characterization of connected multigraphs G that satisfy $sp(G) = \Delta(G)$.

Lemma 2.6. *If G is a connected multigraph with $sp(G) = \Delta(G)$, then either G is an odd cycle or G is a tree.*

Proof. Let G be a counter-example to this statement minimizing $|E(G)|$. Let us show that G is unicyclic, that is, G contains exactly one cycle.

Since G is not a tree, it must contain a cycle. Let us assume that G contains at least two cycles, and let e be an edge of G lying on a cycle of G. Observe that

$$sp(G) \leq sp(G - e) \leq \Delta(G - e) \leq \Delta(G).$$

Taking into account that $sp(G) = \Delta(G)$, we have that $sp(G - e) = \Delta(G - e)$. Since $G - e$ is connected and $|E(G - e)| = |E(G)| - 1 < |E(G)|$, we have that $G - e$ is either a tree or an odd cycle. Now, if $G - e$ is a tree, then G must be unicyclic [21], which we assumed to be not the case. Hence $G - e$ is an odd cycle. However, this case is also impossible since if $G - e$ is an odd cycle, then $\Delta(G) = 3$ and $sp(G) \leq 2$, and therefore $sp(G) < \Delta(G)$, which contradicts the choice of G. We conclude that G is unicyclic.

Let C be the cycle of G. Observe that since G is not a cycle ($G \neq C$), it must contain a vertex of degree one.

Let us show that any degree one vertex of G is adjacent to a vertex of C. On the opposite assumption, we can consider a vertex u of G such that $d_G(u) = p + 1 \geq 2$ and u is adjacent to $p \geq 1$ vertices of degree one. Let $u_1, \ldots, u_p$ be the degree one neighbours of u, and let v be the other neighbour of u. Observe that since G is not a tree, v is not of degree one. Let G_1 be the component of $G - (u, v)$ containing the vertex v. Clearly, C is a cycle of G_1. We need to consider two cases.

Case 1. $G_1 = C$. In this case, we have that $\Delta(G) = \max\{d_G(v), d_G(u)\} = \max\{3, p+1\}$ and $sp(G) \leq \max\{2, p\}$, hence $sp(G) < \Delta(G)$, which contradicts the choice of G.

Case 2. $G_1 \neq C$. Since G_1 is connected, G_1 contains a cycle and $|E(G_1)| < |E(G)|$, we have that $sp(G_1) \leq \Delta(G_1) - 1 < \Delta(G)$. Hence $sp(G) \leq \max\{sp(G_1), p\} < \Delta(G)$, since $\Delta(G) \geq p + 1$, which contradicts the choice of G.

The considered two cases imply that any degree one vertex of G is adjacent to a vertex of C. Observe that this implies that all vertices of G that are of degree at least two, lie on C. We are ready to complete the proof of the lemma. For this purpose we consider the following two cases, and in each of them we exhibit a contradiction.

Case 1. G contains two degree two vertices that are adjacent. Let u and v be adjacent degree two vertices of G, and let u_1 and v_1 be the other ($\neq v$ and $\neq u$) neighbours of u and v, respectively. Consider the multigraph G' obtained from G by removing the vertices u and v, and adding an edge connecting u_1 and v_1. Since G' is connected and $|E(G')| < |E(G)|$, we have that $sp(G') \leq \Delta(G') - 1 = \Delta(G) - 1$. Let H' be a strongly spanning $(\Delta(G) - 1)$-edge-colorable subgraph of G'. Consider a subgraph H of G obtained from H' as follows:

$$H = \begin{cases} (H' \backslash \{(u_1, v_1)\}) \cup \{(u, u_1), (v, v_1)\}, & \text{if } (u_1, v_1) \in E(H'); \\ H' \cup \{(u, v)\}, & \text{if } (u_1, v_1) \notin E(H'). \end{cases}$$

It is easy to see that H is a strongly spanning $(\Delta(G) - 1)$-edge-colorable subgraph of G, hence $sp(G) \leq \Delta(G) - 1$ contradicting the choice of G.

Case 2. G contains no two degree two vertices that are adjacent. Observe that this case includes the case when there are no degree two vertices in G. For each degree two vertex u of G choose the edge (u, u') incident to u such that u' is the next neighbour of u in the direction of clockwise circumvention of C, and let M be the matching of G that contains all such edges (u, u'). Consider a subgraph H of G obtained as follows: all edges of G that are incident to a degree one vertex add to H, and add M to H, too. Clearly, H is a strongly spanning $(\Delta(G) - 1)$-edge-colorable subgraph of G, hence $sp(G) \leq \Delta(G) - 1$ contradicting the choice of G.

The proof of the Lemma 2.6 is complete. □

Lemma 2.6 implies that in order to characterize the connected multigraphs G with $sp(G) = \Delta(G)$, we can focus on trees. For this purpose, for an arbitrary tree T, we introduce the following two sets:

$$A = \{v \in V(T) : d_T(v) = \Delta(T)\}, \quad B = V(T)\backslash A.$$

Lemma 2.7. *Let T be a tree with $|V(T)| \geq 3$. Then for any $v \in B$ there is a $(\Delta(T) - 1)$-edge-colorable subgraph H of G, such that either $V(H) = V(T)$ or $V(T)\backslash V(H) = \{v\}$.*

Proof. We will give a method for the construction of such a subgraph. We start with $H = \emptyset$. Consider the following partition of vertices of T:

$$V_0 = \{v\}, V_1 = \{u : (v,u) \in E(T)\}, \ldots, V_p = \{u : (z,u) \in E(T) \text{ and } z \in V_{p-1}\}.$$

Now, add all edges (z,u) to H, such that $u \in V_p$ and $z \in V_{p-1}$. Observe that for any $w \in V(H) \cap V_{p-1}$ one has $d_H(w) \leq \Delta(T) - 1$ since w has one neighbour in V_{p-2}. After this, remove all edges that we have added to H and the vertices incident to them from T. Repeat this process until $V(T)$ becomes empty or $V(T) = \{v\}$.

It can be easily seen that the components of the resulting subgraph H of T are stars, such that their centers are of degree at most $\Delta(T) - 1$. Hence H is $(\Delta(T) - 1)$-edge-colorable. Moreover, it meets the requirements of the lemma. □

In the following two corollaries, for a tree T let H denote the subgraph from Lemma 2.7.

Corollary 2.8. *If T is a tree with $|V(T)| \geq 3$ and $sp(T) = \Delta(T)$, then*

$$V(T)\backslash V(H) = \{v\}.$$

Corollary 2.9. *If T is a tree with $|V(T)| \geq 3$ and the subgraph H does not cover v, then there is a $\Delta(T)$-edge-colorable subgraph H' of T, such that H' is strongly spanning and $d_{H'}(v) = 1$.*

Now, we introduce an operation that will help us to characterize the trees T with $sp(T) = \Delta(T)$. Let T_1 be a tree with $|V(T_1)| \geq 3$, and let $K_{1,p}$ be a star with $p \geq 2$. Consider the tree $T = T_1 \circ K_{1,p}$ obtained from T_1 and $K_{1,p}$ by identifying a degree one vertex of $K_{1,p}$ with a vertex $v \in B = B(T_1)$. First, we establish some properties of the operation $\circ$.

Lemma 2.10. *Let T_1 be a tree with $|V(T_1)| \geq 3$, and let $K_{1,p}$ be a star with $p \geq 2$. If $T = T_1 \circ K_{1,p}$ then:*

(a) *if $p < sp(T_1) = \Delta(T_1)$, then $sp(T) \neq \Delta(T)$;*
(b) *if $p \leq sp(T_1) < \Delta(T_1)$, then $sp(T) \neq \Delta(T)$;*
(c) *if $sp(T_1) < p$, then $sp(T) \neq \Delta(T)$;*
(d) *if $p = sp(T_1) = \Delta(T_1)$, then $sp(T) = \Delta(T)$.*

Proof. Let $L = \max\{\Delta(T_1), p\}$. Clearly, $\Delta(T) = L$. Suppose that the tree T has been obtained from T_1 and $K_{1,p}$, by identifying the vertices $w \in B = B(T_1)$, and the degree one vertex $u \in V(K_{1,p})$. Moreover, let z be the center of $K_{1,p}$.

(a) Since $\Delta(T_1) > p$, then $\Delta(T) = \Delta(T_1)$. Let us show that $sp(T) \leq \Delta(T) - 1$. As $w \in B = B(T_1)$, Corollary 2.8 implies that there is a $(\Delta(T_1) - 1)$-edge-colorable subgraph H_1 of T_1, such that $V(T_1)\backslash V(H_1) = \{w\}$. Consider the subgraph H of T obtained from H_1 by adding $E(K_{1,p})$ to it. Clearly, H is $(\Delta(T) - 1)$-edge-colorable subgraph of T, hence $sp(T) \leq \Delta(T) - 1 < \Delta(T)$.

(b) Clearly, $\Delta(T) = \Delta(T_1)$. Let us show that $sp(T) \leq sp(T_1) < \Delta(T)$. Take a strongly spanning $sp(T_1)$-edge-colorable subgraph H_1 of T_1. Consider the subgraph H of T obtained from H_1 by adding $E(K_{1,p})\backslash\{(u, z)\}$ to it. Clearly, H is a strongly spanning $sp(T_1)$-edge-colorable subgraph of T. Hence $sp(T) \leq sp(T_1)$.

(c) Let us show that $sp(T) \leq p - 1 < \Delta(T)$. Take a strongly spanning $sp(T_1)$-edge-colorable subgraph H_1 of T_1. Consider the subgraph H of T obtained from H_1 by adding $E(K_{1,p})\backslash\{(u, z)\}$ to it. Clearly, H is a strongly spanning $(p - 1)$-edge-colorable subgraph of T. Hence $sp(T) \leq p - 1$.

(d) Clearly, $\Delta(T) = \Delta(T_1) = p$. Suppose that $k = sp(T) < \Delta(T) = p$, and let H be a strongly spanning k-edge-colorable subgraph of T. Set: $H_1 = H \cap E(T_1)$.

Observe that $(w, z) \notin E(H)$, as otherwise $E(K_{1,p}) \subseteq E(H)$ and hence all edges of $K_{1,p}$ would have to be colored, which would mean that $k = p$. This implies that H_1 is a strongly spanning k-edge-colorable subgraph of T_1, hence $sp(T_1) \leq k < p = \Delta(T_1)$, which contradicts our assumption.

□

We are ready to characterize the trees T with $sp(T) = \Delta(T)$. For that purpose, for any two trees T' and T'', we write $T' \to T''$, if T'' can be obtained from T' by the application of Lemma 2.10(d).

Theorem 2.11. *A tree T satisfies $sp(T) = \Delta(T)$, if and only if, there is a sequence of trees $T_0, T_1, \ldots, T_m$ $(m \geq 0)$, such that T_0 is a star, $T_m = T$, $sp(T_j) = \Delta(T_j)$ for $j = 0, 1, \ldots, m$ and $T_0 \to T_1 \to \ldots \to T_m$.*

Proof. If T is a star, then clearly $sp(T) = \Delta(T)$. On the other hand, if T is obtained from a star T_0 by applying Lemma 2.10(d), then by Lemma 2.10(d), all intermediate trees T_j satisfy $sp(T_j) = \Delta(T_j)$. Hence $sp(T) = \Delta(T)$.

Now, assume that T satisfies $sp(T) = \Delta(T)$. Let us show the existence of the corresponding sequence of trees. If T is a star, we are done. Otherwise, assume that T is not a star. Then, there is a vertex z of T, that is of degree $p \geq 2$, such that z is adjacent to exactly $p - 1$ vertices of degree one. Let T' be the tree obtained from T by removing the vertex z and all its neighbours that are of degree one. Moreover, let w be the vertex of T' such that $(z, w) \in E(T)$. Let us show that $T = T' \circ K_{1,p}$.

Clearly, it suffices to show that $w \in B = B(T')$. Suppose that $w \in A = A(T')$, that is $d_{T'}(w) = \Delta(T')$. Then, clearly, $\Delta(T) = \max\{d_T(w), d_T(z)\} = \max\{\Delta(T') + 1, p\}$. Consider a strongly spanning subgraph H of T obtained from any strongly spanning $\Delta(T')$-edge-colorable subgraph of T' by adding all edges incident to z except (z, w).

It is not hard to see that H is $\max\{\Delta(T'), p-1\}$-edge-colorable, hence $sp(T) \leq \max\{\Delta(T'), p-1\} < \max\{\Delta(T')+1, p\} = \Delta(T)$ contradicting the choice of T.

Lemma 2.10 implies that T' and p satisfy the conditions of Lemma 2.10(d). Hence, $T' \to T$. By induction, there is a sequence of trees $T_0, T_1, \ldots, T_m$ ($m \geq 0$), such that T_0 is a star, $T_m = T'$, $sp(T_j) = \Delta(T_j)$ for $j = 0, 1, \ldots, m$ and $T_0 \to T_1 \to \ldots \to T_m$. Consider the sequence of trees $T_0, T_1, \ldots, T_m, T_{m+1}$, where $T_{m+1} = T$. Observe that it meets the requirements of the theorem. The proof of the Theorem 2.11 is complete. □

Now we turn to the problem of finding some bounds for $sp(G)$ in terms of well-known graph theoretic parameters.

Thomassen has shown that any almost regular multigraph G (that is, a multigraph G with $\Delta(G) - \delta(G) \leq 1$) has a $[1,2]$-factor [18], hence we have:

Proposition 2.12. *For any almost regular multigraph G, $sp(G) \leq 2$.*

Corollary 2.13. *Any regular multigraph has a maximum 2-edge-colorable subgraph that is strongly spanning.*

Corollary 2.14. *Any cubic multigraph has a maximum 2-edge-colorable subgraph that is strongly spanning.*

Let us note that the statement of the last corollary for bridgeless cubic multigraphs first appeared in the proof of Theorem 4.1 from [17]. However, an attentive reader probably has already realized that the proof given in [17] is wrong.

Retaining the notations of [17], let us, first explain, what is wrong there. The gap is that when the author removes the edges e_1 and e_2 from a maximum 2-edge-colorable subgraph H and adds the edges (v, u_1) and (v, u_2) to it to get a new maximum 2-edge-colorable subgraph H', he may leave the other ($\neq u_1$ and $\neq u_2$, respectively) end-vertices isolated, so after this operation one can not conclude that $V(H') = V(H) \cup \{v\}$ as it is done there.

Below we offer a generalization of Proposition 2.12. Our proof requires the following result of Lovász:

Theorem 2.15 ([9])**.** *If G is a multigraph with $\Delta(G) \leq s+t-1$, then G can be partitioned into two subgraphs H and L, such that $\Delta(H) \leq s$ and $\Delta(L) \leq t$.*

Theorem 2.16. *For any multigraph G without isolated vertices $sp(G) \leq \Delta(G) - \delta(G) + 2$.*

Proof. For a multigraph G take $s = \Delta(G) - \delta(G) + 2$ and $t = \delta(G) - 1$. Observe that $\Delta(G) = s + t - 1$. Apply Lovász's theorem. As a result we have two subgraphs H and L, such that $\Delta(H) \leq s$ and $\Delta(L) \leq t$.

Since $\Delta(L) \leq t = \delta(G) - 1$, we have $\delta(H) \geq 1$. On the other hand, $\Delta(H) \leq s = \Delta(G) - \delta(G) + 2$. Thus H is a $(1, \Delta(G) - \delta(G) + 2)$-factor, which proves the theorem. □

Let us note that this bound is tight, since any regular multigraph without a perfect matching achieves it. It can be shown that this bound can be improved by one if G is non-regular (that is, $\Delta(G) \neq \delta(G)$). However, we will not prove this, because below we will prove a significantly better bound for $sp(G)$.

Our next bound is formulated in terms of $\nu(G)$. Its proof requires Theorem 2.1.9 from [22]:

Theorem 2.17 ([22])**.** *Let $b > a \geq 1$. Then a multigraph G has an $[a,b]$-factor, if and only if for all $S \subseteq V(G)$ $\sum_{i=0}^{a-1}(a-i)p_i(G-S) \leq b|S|$, where $p_i(G-S)$ is the number of vertices of degree i in the multigraph $G-S$.*

Theorem 2.18. *For any multigraph G without isolated vertices $sp(G) \leq |V(G)| -2 \cdot \nu(G) + 1$.*

Proof. By Theorem 2.17, it suffices to show that for each $S \subseteq V(G)$ $p_0(G-S) \leq (|V(G)| - 2 \cdot \nu(G) + 1)|S|$, where $p_0(G-S)$ is the number of isolated vertices of $G-S$. Clearly, we can assume that S is non-empty. Observe that by Tutte-Berge formula, we have:

$$\begin{aligned} p_0(G-S) &\leq o(G-S) \leq |S| + (|V(G)| - 2 \cdot \nu(G)) \\ &\leq |S| + |S|(|V(G)| - 2 \cdot \nu(G)) = (|V(G)| - 2 \cdot \nu(G) + 1)|S|. \end{aligned}$$ □

Note that any multigraph with a perfect or a near-perfect matching (a matching missing exactly one vertex) achieves this bound.

Now, we prove the following improvement of Theorem 2.16:

Theorem 2.19. *For any multigraph G without isolated vertices $sp(G) \leq 1 + \left\lfloor \frac{\Delta(G)}{\delta(G)} \right\rfloor$. Moreover, if G is non-regular, then $sp(G) \leq \left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil$.*

Proof. Note that since $1 + \left\lfloor \frac{\Delta(G)}{\delta(G)} \right\rfloor > 1$, by Theorem 2.17 it suffices to show that for each $S \subseteq V(G)$

$$p_0(G-S) \leq \left(1 + \left\lfloor \frac{\Delta(G)}{\delta(G)} \right\rfloor\right)|S|,$$

where $p_0(G-S)$ is the number of isolated vertices of $G-S$.

Observe that the $p_0(G-S)$ isolated vertices are connected to vertices of S, thus

$$\delta(G) \cdot p_0(G-S) \leq \Delta(G) \cdot |S|,$$

which proves the required bound.

For the proof of the second statement, observe that since G is non-regular, then $\left\lceil \frac{\Delta(G)}{\delta(G)} \right\rceil > 1$, thus Theorem 2.17 is applicable. The rest is the same as above. □

Let us note that there are examples of multigraphs such that the difference between the upper bound offered by Theorem 2.19 and $sp(G)$ is arbitrarily large. To see this, let H be an r-regular multigraph containing a perfect matching F. Consider a multigraph G obtained from H by replacing one edge of F by a path of length three. Observe that G contains a perfect matching, hence $sp(G) = 1$, however the bound offered by Theorem 2.19 is $\lceil \frac{r}{2} \rceil$.

In Theorem 2.18, we have shown that an upper bound for $sp(G)$ is provable in terms of the difference between $|V(G)|$ and $\nu(G)$. It is natural to wonder, whether such a bound is possible to prove in terms of the ratio of $|V(G)|$ and $\nu(G)$. The following proposition shows the impossibility of such a bound.

Proposition 2.20. *For any positive integers a, b there is a tree G with $sp(G) > a(\frac{|V(G)|}{\nu(G)})^b$.*

Proof. Let n be any positive integer with $n \geq 4$. Set: $k = an^b$ and $x = 2k$. Consider the tree G obtained from a path of length x and the star $K_{1,k}$ by identifying the center of the star to one of end-vertices of the path. Observe that: $|V(G)| = 3an^b + 1$, $\nu(G) = an^b + 1$ and $sp(G) = an^b$. Clearly, we have that $sp(G) > a(\frac{|V(G)|}{\nu(G)})^b$. □

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