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NON-FACTORIZABLE $\mathbb{C}$-VALUED FUNCTIONS INDUCED BY FINITE CONNECTED GRAPHS

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Abstract. In this paper, we study factorizability of $\mathbb{C}$-valued formal series at fixed vertices, called the graph zeta functions, induced by the reduced length on the graph groupoids of given finite connected directed graphs. The construction of such functions is motivated by that of Redei zeta functions. In particular, we are interested in (i) "non-factorizability" of such functions, and (ii) certain factorizable functions induced by non-factorizable functions. By constructing factorizable functions from our non-factorizable functions, we study relations between graph zeta functions and well-known number-theoretic objects, the Riemann zeta function and the Euler totient function.

Keywords: directed graphs, graph groupoids, Redei zeta functions, graph zeta functions, non-factorizable graphs, gluing on graphs.

Mathematics Subject Classification: 05E15, 11G15, 11R47, 11R56, 46L10, 46L40, 46L54.

1. INTRODUCTION

In [6], we established certain $\mathbb{C}$-valued functions induced by *directed graphs*. Combinatorial properties of graphs was considered in terms of analytic tools from *function theory*, and vice versa. Motivated by the construction of *Redei zeta functions* (e.g., [11]), we constructed so-called *graph zeta functions* as $\mathbb{C}$-valued functions with variables determined up to vertices of graphs and an arbitrary $\mathbb{C}$-variable. In [7], we further studied about the *factorizability* of graph zeta functions at fixed vertices. We recognized in [7] that there are certain graphs making their graph zeta functions be "not" factorizable in the sense of [7] (see below). In this paper, we further study *non-factorizability of graph zeta functions*.

We say that *a graph zeta function is not factorizable* if it does not have suitable (muti-)factors determined by certain subgraphs of the given graph, for each fixed vertex. And we say a given *graph is not factorizable* if the corresponding graph zeta functions are not factorizable for "all" vertices. It is interesting that a connected finite graph G, with its maximal *out-degree* $N \in \mathbb{N}$, having d-many vertices ($d \in \mathbb{N} \cup \{\infty\}$), is not factorizable if and only if it is graph-isomorphic to a subgraph K, having d-many vertices, of the circulant N-regular graph K_d^N with d-vertices, i.e. the non-factorizability of all finite-connected-graph zeta functions is completely characterized by a graph-theoretical invariance.

1.1. BACKGROUND

Recently, we found there are close connection between number-theoretic objects and operator-theoretic, and operator-algebraic objects. Moreover, number-theoretical techniques or tools can be applicable to the studies of operator-theoretic, free-probabilistic, and operator-algebraic structures. Conversely, free-probabilistic, operator-theoretic and operator-algebraic techniques and tools provide new models for studying number-theoretic problems (e.g., [2, 4, 5] and [12]). In this paper, by using the analytic-and-combinatorial number-theoretic techniques, we will characterize combinatorial objects, graphs, and corresponding algebraic objects, graph groupoids.

Independently, graphs and corresponding *graph groupoids* have close connections with operator-theoretic, representation-theoretic, and operator-algebraic objects. In particular, elements of graph groupoids are understood as operators induced by certain *projections* and *partial isometries*, assigned from vertices, respectively, edges of given graphs (e.g., [3] cited references therein) We cannot help emphasizing the importance of graphs and graph groupoids not only in mathematical fields but also in related science areas (e.g., [9] and [8]). Whenever a *graph G* is fixed, it has its corresponding algebraic structure $\mathbb{G}$, which is a *groupoid*, generated by edges of the *shadowed graph $\widehat{G}$ of G*, with its *multi-units* identified with the vertices of G. Recall that *groupoids* are algebraic structures consisting of sets with a single partially-defined binary operation equipped with "multi-units." (In our case, by attaching the *empty word* $\varnothing$ (if needed), we make the binary operation be well-defined.) For instance, every *group* is a groupoid with a "single" unit, the group identity. We call the groupoids induced by graphs, *graph groupoids*. They are understood as groupoidal version of *free groups*.

Graph groupoids are playing important roles not only in combinatorics and algebra, but also in noncommutatve function theory, dynamical systems, operator theory, (amalgamated) free probability, and operator algebra. Thus we cannot help emphasizing the importance of applications of graph groupoids in various scientific areas (e.g., [3]).

L-function theory (including the study of *Dirichlet series induced by arithmetic functions*) is one of the branches of *number theory* and mathematical *analysis*. Classically, *Dirichlet series* are constructed as $\mathbb{C}$-valued functions induced by *arithmetic functions*, i.e. if $f : \mathbb{N} \to \mathbb{C}$ is an *arithmetic function*, then one has the corresponding Dirichlet series $L_f : \mathbb{C} \to \mathbb{C}$ defined by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad s \in \mathbb{C}.$$

For example, the *Riemann zeta function* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is understood as the Dirichlet series $L_1(s)$, where 1 is the constant arithmetic function,

$$1(n) = 1 \text{ in } \mathbb{C} \text{ for all } n \in \mathbb{N}.$$

Let χ be a *Dirichlet character*, which is an arithmetic function satisfying that:

(i) there exists $N \in \mathbb{N}$ such that $\chi(n + N) = \chi(n)$,
(ii) $\chi(n) = 0$, whenever $\gcd(n, N) \neq 1$, and
(iii) $\chi(nm) = \chi(n)\chi(m)$ for all $n, m \in \mathbb{N}$.

The corresponding Dirichlet series $L_\chi(s)$ is said to be a (*Dirichlet*) *L-function of* χ (e.g., [10] and cited papers therein).

1.2. MOTIVATION

In [6], we attempted to connect the above two different topics, graph-groupoid theory and L-function theory, which seem completely un-related. For the graph groupoid $\mathbb{G}$ of a given graph G, we constructed a suitable order function ϖ, and then establish the corresponding *Redei-zeta-function-like* function

$$L_G(v, s) = \sum_{w \in \mathbb{G}_v} \frac{1}{\varpi(w)^s},$$

as in [11], where v is a fixed vertex and $\mathbb{G}_v$ is a certain subset of $\mathbb{G}$ determined by an arbitrarily fixed vertex v, and s is a $\mathbb{C}$-variable. Analytic, combinatorial, and algebraic properties of $\{L_G(v, s)\}_{v \in V(\widehat{G})}$ have been studied in [6].

In [7], we showed that there exists certain inner structure $\mathfrak{G}_v = \{G' \leqslant \widehat{G}\}$ of G such that

$$L_G(v, s) = \prod_{G' \in \mathcal{G}} L_{G'}(v, s),$$

where $\widehat{G}$ is the shadowed graph of G in the sense of Section 2.3 below. And we realized that there do exist graphs G such that $\mathfrak{G}_x = \{\widehat{G}\}$ for all $x \in V(G)$, and hence all graph zeta functions $L_G(x, s)$ are not factorizable.

1.3. OVERVIEW AND SKETCH OF MAIN RESULTS

We here further study factorizability of "finite", "connected" graphs, in terms of the factorizability of the corresponding graph zeta functions. To do that, we collect non-factorizable finite connected graphs, and construct other graphs induced by them. In particular, we construct factorizable graphs by "gluing" non-factorizable graphs, and study detailed properties of corresponding graph zeta functions.

In Section 2, we briefly introduce motivations and backgrounds for our proceeding works. In Sections 3 and 4, we construct graph zeta functions, and review fundamental properties of such functions (also, see [6]). In Sections 5 and 6, we consider factorizability of graph zeta functions, and those of graphs. We characterize the factorizability of graph zeta functions in terms of subgraphs in Theorem 5.6. And then, based on this factorizability, we characterize the non-factorizable graphs, see Theorem 6.3 (also, see [7]). In Section 7, the graph zeta functions of non-factorizable finite connected graphs are considered in detail. In particular, the specific forms of non-factorizable-graph zeta functions are obtained in Theorems 7.2, and certain subgraphs zeta functions of non-factorizable-graph zeta functions are characterized in Theorem 7.4. And hence, we refine the results in Theorem 7.5. In Section 8, we construct finite connected graphs by gluing non-factorizable graphs. And the corresponding glued-graph zeta functions are obtained in Theorem 8.3. And we obtain certain relations between our glued-graph zeta function and well-known number-theoretic objects, especially, the Riemann zeta function $\zeta(s)$, and the Euler totient function ϕ. See Theorems 8.6, 8.7, 8.8 and 8.9.

2. PRELIMINARIES

In this section, we introduce basic concepts and backgrounds of our proceeding study.

2.1. REDEI ZETA FUNCTIONS

In this section, we briefly introduce *Redei zeta functions*. For more about Redei zeta function, see [11].

Let $\mathbb{L}$ be an arbitrary lattice with its minimal element $0_{\mathbb{L}}$, equipped with its ordering $\leqslant$. And assume

$$f : \mathbb{L} \times \mathbb{L} \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

is a well-defined function satisfying

$$f(x, y) = \begin{cases} f(x, y) \text{ in } \mathbb{N} & \text{if } x \leqslant y, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for all $(x, y) \in \mathbb{L} \times \mathbb{L}$, and

$$f(x, y) = f(x, z)f(z, y), \tag{2.2}$$

whenever $x \leqslant z \leqslant y$ in $\mathbb{L}$. Such a function f satisfying (2.1) and (2.2) is called an *order function on the lattice* $\mathbb{L}$.

For a fixed order function f on $\mathbb{L}$, one may define a function

$$f_{x_0} : \mathbb{L} \to \mathbb{N}$$

by

$$f_{x_0}(x) = f(x_0, x), \quad x \in \mathbb{L}, \tag{2.3}$$

for an arbitrarily fixed $x_0 \in \mathbb{L}$. We call such a function f_{x_0} of (2.3) induced by f an *order function on* $\mathbb{L}$ *with its base* x_0.

Now, let A be a subset of $\mathbb{L}$, and let $M(A)$ be the sub-lattice generated by A in $\mathbb{L}$. A subset A is said to be a *Redei set* in $\mathbb{L}$ if for each element x in $M(A)$ there are only "finitely" many y in $M(A)$ such that $f(x, y) < \infty$.

Fix an arbitrary element x_0 in $\mathbb{L}$. The *Redei zeta function of A based on x_0* (see [11]) is defined by

$$L_{x_0}(s) \overset{def}{=} \sum_{a \in A} \frac{(-1)^{|A|}}{f_{x_0}(a)^s}. \tag{2.4}$$

Based on the construction (2.4) of Redei zeta function, we will establish our graph-L-functions in Sections 3 and 4 below. Our graph zeta functions are not exactly Redei zeta functions of (2.8), but the construction of them is highly motivated by them.

2.2. GRAPHS AND GRAPH GROUPOIDS

For more about directed graphs, graph groupoids and corresponding operator algebras, see [3]. We will use the same notations and definitions used there. In this section, we briefly introduce concepts from those references used for our study.

Let G be a directed graph $(V(G), E(G), s, r)$, where $V(G)$ is the vertex set, $E(G)$ is the edge set, and s and r are the functions from $E(G)$ onto $V(G)$, called the *source map*, respectively, the *range map*, indicating the initial vertex and the terminal vertex of each edge, respectively. All graphs in this paper will be automatically assumed to be directed. For convenience, we write $e = v_1 e v_2$, where $s(e) = v_1$ and $r(e) = v_2$ in $V(G)$, for all $e \in E(G)$.

The corresponding new graph G^{-1} is the "opposite" directed graph of G sharing same vertices, i.e. by replacing the orientation of edges of G backwardly, we obtain the graph G^{-1}. This new graph G^{-1} is said to be the *shadow of G*. It is trivial that

$$(G^{-1})^{-1} = G.$$

Let G_1 and G_2 be graphs. The *union* $G_1 \cup G_2$ is defined by a new graph G with

$$V(G) = V(G_1) \cup V(G_2), \tag{2.5}$$

and

$$E(G) = E(G_1) \cup E(G_2),$$

which preserves the directions of G_1 and G_2.

The *shadowed graph* $\widehat{G}$ of a given graph G is defined by the union $G \cup G^{-1}$, i.e.

$$V(\widehat{G}) = V(G) \cup V(G^{-1}) = V(G) = V(G^{-1}),$$

and

$$E(\widehat{G}) = E(G) \cup E(G^{-1}) = E(G) \sqcup E(G^{-1}).$$

Let $\widehat{G}$ be the shadowed graph of a given graph G. The set $FP(\widehat{G})$, consisting of all finite paths on $\widehat{G}$, is called the *finite path set of $\widehat{G}$*. All finite paths on $\widehat{G}$ is denoted by forms of words in edges of $E(\widehat{G})$.

Now, let $w = e_1 \ldots e_k \in FP(\widehat{G})$ for $k \in \mathbb{N}$. Then one can extend the maps s and r on $FP(\widehat{G})$ as follows:

$$s(w) = s(e_1) \quad \text{and} \quad r(w) = r(e_k).$$

If $s(w) = v_1$ and $r(w) = v_2$ in $V(\widehat{G})$, we also write

$$w = v_1 w, \text{ or } w = w v_2, \text{ or } w = v_1 w v_2,$$

for all $w \in FP(\widehat{G})$.

Define a set $\mathbb{F}^+(\widehat{G})$ by

$$\mathbb{F}^+(\widehat{G}) \overset{def}{=} \{\varnothing\} \cup V(\widehat{G}) \cup FP(\widehat{G}),$$

and define a binary operation $(\cdot)$ by

$$w_1 \cdot w_2 \overset{def}{=} \begin{cases} w_1 w_2 & \text{if } r(w_1) = s(w_2) \text{ in } V(\widehat{G}), \\ \varnothing & \text{otherwise} \end{cases} \tag{2.6}$$

for all $w_1, w_2 \in \mathbb{F}^+(\widehat{G})$, where $\varnothing$ is the *empty word*, representing the "undefinedness of $w_1 \cdot w_2$, as finite paths or vertices of $\widehat{G}$".

The operation $(\cdot)$ of (2.6) on $\mathbb{F}^+(\widehat{G})$ is called the *admissibility*. If $w_1 \cdot w_2 \neq \varnothing$ in $\mathbb{F}^+(\widehat{G})$, then w_1 and w_2 are said to be *admissible*; if $w_1 \cdot w_2 = \varnothing$, then they are said to be *not admissible*.

The algebraic pair $\mathbb{F}^+(\widehat{G}) = (\mathbb{F}^+(\widehat{G}), \cdot)$, equipped with the admissibility, is called the *free semigroupoid of* $\widehat{G}$.

For a fixed free semigroupoid $\mathbb{F}^+(\widehat{G})$, define a natural *reduction* by

$$w^{-1} w = v_2 \quad \text{and} \quad w w^{-1} = v_1, \tag{RR}$$

whenever $w = v_1 w v_2 \in \mathbb{F}^+(\widehat{G}) \setminus \{\varnothing\}$, with $v_1, v_2 \in V(\widehat{G})$. Then this reduction (RR) acts as a relation on the free semigroupoid $\mathbb{F}^+(\widehat{G})$.

Definition 2.1. The quotient set $\mathbb{G} = \mathbb{F}^+(\widehat{G}) /$ (RR), equipped with the inherited admissibility $(\cdot)$ from $\mathbb{F}^+(\widehat{G})$, is called the *graph groupoid of* G.

The graph groupoid $\mathbb{G}$ of G is indeed a categorial groupoid with its (multi-)units $V(\widehat{G}) = V(G)$ (e.g., [3, 9] and [8]). The subset of $\mathbb{G}$, consisting of all "reduced" finite paths, is denoted by $FP_r(\widehat{G})$. Notice that every graph groupoid $\mathbb{G}$ of a graph G is in fact a collection of all "reduced" words in the edge set $E(\widehat{G})$ of the shadowed graph $\widehat{G}$ under (RR).

Let K and G be graphs. The graph K is said to be a *subgraph of* G if K is a graph with

$$V(K) \subseteq V(G) \tag{2.7}$$

and

$$E(K) = \{e \in E(G) : \text{ there exists } e \in E(K) \text{ such that } e = v e v' \text{ for } v, v' \in V(K)\}.$$

We write this subgraph-inclusion by

$$K \leqslant G. \tag{2.8}$$

3. ORDER FUNCTIONS ON GRAPHS

Throughout this section, let G be a fixed graph with its graph groupoid $\mathbb{G}$. The (*reduced*) *length* $|\cdot|$ on $\mathbb{G}$ is naturally defined by

$$|w| = \begin{cases} 0 & \text{if } w \in \{\varnothing\} \cup V(\widehat{G}), \\ k & \text{if } w = e_1 \dots e_k \in FP_r(\widehat{G}), \end{cases} \tag{3.1}$$

with $e_1, \dots, e_k \in E(\widehat{G})$, for $k \in \mathbb{N}$ and $w \in \mathbb{G}$. Clearly,

$$\left|w^{-1}w\right| = 0 = \left|ww^{-1}\right|, \quad w \in \mathbb{G}.$$

Remark that, in general,

$$|w_1 w_2| \leqslant |w_1| + |w_2|, \quad w_1, w_2 \in \mathbb{G}.$$

Define the following quantities for each vertex v by

$$\deg_{out}(v) = |\{e \in E(G) : e = ve\}|, \quad \deg_{in}(v) = |\{e \in E(G) : e = ev\}|,$$

and

$$\deg(v) = \deg_{out}(v) + \deg_{in}(v)$$

in G, where $|X|$ means the *cardinality* of arbitrary sets X. They are called the *out-degree*, the *in-degree*, and the *degree* of v in G, respectively. A graph G is said to be *locally finite* if the degrees of all vertices of G are finite.

In the rest of this paper, all graphs are assumed to be "finite" and "connected".

Let G be a given graph with its graph groupoid $\mathbb{G}$, and let

$$N = \max\{\deg_{out}(v) : v \in V(G) \text{ in } G\}. \tag{3.2}$$

Define a new quantity N_2 by

$$N_2 \overset{def}{=} 2N + 1 \text{ in } \mathbb{N}, \tag{3.3}$$

where N is in the sense of (3.2). Note that

$$2N = \max\{\deg_{out}(v) : v \in V(\widehat{G}) \text{ in } \widehat{G}\},$$

where $\widehat{G}$ is the shadowed graph of G. Based on the finiteness of G, (3.2) and (3.3), one can define a function

$$\varpi : \mathbb{G} \to \mathbb{N}$$

by

$$\varpi(w) \overset{def}{=} N_2^{|w|}, \quad w \in \mathbb{G}, \tag{3.4}$$

where N_2 is in the sense of (3.3), and $|w|$ means the reduced length (3.1) on $\mathbb{G}$.

Now, let $w \in \mathbb{G}$. In particular, assume $w = w_1 w_2 \ldots w_k$ in $\mathbb{G}$ for some $w_1, \ldots, w_k \in \mathbb{G}$, for $k \in \mathbb{N}$. Then we write

$$w_j \hookrightarrow w \text{ in } \mathbb{G} \text{ for } j = 1, \ldots, k,$$

with axiomatization

$$\varnothing \hookrightarrow w \text{ for all } w \in \mathbb{G}.$$

Let w_1 and w_2 be nonempty elements in $\mathbb{G}$, and assume the "reduced" word $w_1 w_2$ is again a nonempty element of $FP_r(\widehat{G})$ in $\mathbb{G}$. Then there exist $w_1', w_2' \in FP_r(\widehat{G})$ such that:

(i) $w_j' \hookrightarrow w_j$ for $j = 1, 2$,
(ii) $w_1 w_2 = w_1' w_2'$ in $\mathbb{G}$,
(iii) w_j' are the maximal reduced words in $\mathbb{G}$, satisfying (i) and (ii), for $j = 1, 2$.

Definition 3.1. Let w_1' and w_2' be elements of $FP_r(\widehat{G})$ in $\mathbb{G}$, satisfying the above conditions (i), (ii), and (iii). Then they are said to be reduced embeddings of $w_1 w_2$.

The following proposition shows the relation between reduced embeddings and the function ϖ of (3.4) on $\mathbb{G}$.

Proposition 3.2 ([6]). *Let $\varpi : \mathbb{G} \to \mathbb{R}$ be a function* (3.4)*. Then ϖ is "conditionally" multiplicative, in the sense that*

$$\varpi(w_1 w_2) = \begin{cases} \varpi(w_1')\varpi(w_2') & \text{if } w_1 w_2 \in FP_r(\widehat{G}), \\ 1 & \text{if } w_1 w_2 \in \{\varnothing\} \cup V(\widehat{G}) \end{cases} \tag{3.5}$$

for all $w_1, w_2 \in \mathbb{G}$, where w_1' and w_2' are the reduced embeddings of $w_1 w_2$ in $\mathbb{G}$.

By (3.5), the function ϖ is "conditionally" multiplicative.

Definition 3.3. The map ϖ of (3.4) is called the ordering map of the graph groupoid $\mathbb{G}$ (or, of a graph G).

Remark that the finiteness of all our graphs guarantees the Redei-set-condition of Section 2.1 for ϖ.

4. GRAPH ZETA FUNCTIONS

Throughout this section, let G be a fixed finite connected graph with its graph groupoid $\mathbb{G}$, with corresponding quantity N_2 of (3.3) induced by N of (3.2).

4.1. GRAPH ZETA FUNCTIONS $L_G(\bullet, s)$

Define subsets $\mathbb{G}_{v_1}^{v_2}$ of the graph groupoid $\mathbb{G}$ by

$$\mathbb{G}_{v_1}^{v_2} \overset{def}{=} \{w \in \mathbb{G} : w = v_1 w v_2\} \tag{4.1}$$

for all $v_1, v_2 \in V(\widehat{G})$. Similarly, define subsets $\mathbb{G}_v$ and $\mathbb{G}^v$ of $\mathbb{G}$ by

$$\mathbb{G}_v = \bigcup_{x \in V(\widehat{G})} \mathbb{G}_v^x \quad \text{and} \quad \mathbb{G}^v = \bigcup_{x \in V(\widehat{G})} \mathbb{G}_x^v \tag{4.2}$$

for all $v \in V(\widehat{G})$, where $\mathbb{G}_v^x$ and $\mathbb{G}_x^v$ are subsets of $\mathbb{G}$ in the sense of (4.1).

Definition 4.1. Define a function $L_G : V(\widehat{G}) \times \mathbb{C} \to \mathbb{C}$ by

$$L_G(v, s) \overset{def}{=} \sum_{w \in \mathbb{G}_v} \frac{1}{\varpi(w)^s} \tag{4.3}$$

for $s \in \mathbb{C}$, where $\mathbb{G}_v$ are in the sense of (4.2). Such a function L_G is called the graph zeta function of G at v.

Define subsets $\mathcal{W}_n$ of $\mathbb{G}^\times \overset{def}{=} \mathbb{G} \setminus \{\varnothing\}$ by

$$\mathcal{W}_n \overset{def}{=} \{w \in \mathbb{G}^\times : |w| = n\} \tag{4.4}$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is not difficult to check that

$$\mathcal{W}_0 = V(\widehat{G})$$

and

$$\mathbb{G}^\times = \bigsqcup_{j=0}^{\infty} \mathcal{W}_j \quad \text{set-theoretically.}$$

Also, let

$$\mathbb{G}_{v_1}^{v_2}(n) \overset{def}{=} \mathbb{G}_{v_1}^{v_2} \cap \mathcal{W}_n \quad \text{and} \quad \mathbb{G}_v(n) \overset{def}{=} \mathbb{G}_v \cap \mathcal{W}_n$$

for all $v, v_1, v_2 \in V(\widehat{G})$ and $n \in \mathbb{N}_0$.

The following theorem allow us consider our graph zeta functions as certain (formal) series.

Proposition 4.2 ([6])**.** *Let L_G be the graph zeta function* (4.3) *of a graph G. Then*

$$L_G(v, s) = \sum_{k=0}^{\infty} \frac{|\mathbb{G}_v(k)|}{N_2^{k\ s}} = \sum_{n=1}^{\infty} \frac{\left|\mathbb{G}_v\left(\log_{N_2} n\right)\right|}{n^s}. \tag{4.5}$$

In particular, if $\log_{N_2} n \notin \mathbb{N}$, then $\mathcal{W}_{\log_{N_2} n}$ is empty by (4.4)*, and hence*

$$\left|\mathbb{G}_n\left(\log_{N_2} n\right)\right| = \left|\mathbb{G}_v \cap \mathcal{W}_{\log_{N_2} n}\right| = 0. \tag{4.6}$$

Define now functions $\eta_{v_1}^{v_2} : \mathbb{N} \to \mathbb{N}_0$ by

$$\eta_{v_1}^{v_2}(k) \overset{def}{=} \left|\mathbb{G}_{v_1}^{v_2}(k)\right| = \left|\mathbb{G}_{v_1}^{v_2} \cap \mathcal{W}_k\right| \tag{4.7}$$

for all $k \in \mathbb{N}$, where $v_1, v_2 \in V(\widehat{G})$. Similarly, define $\eta_v : \mathbb{N} \to \mathbb{N}_0$ by

$$\eta_v(k) \overset{def}{=} |\mathbb{G}_v(k)| = |\mathbb{G}_v \cap \mathcal{W}_k| \tag{4.8}$$

for all $k \in \mathbb{N}$ and $v \in V(\widehat{G})$. By the very constructions (4.7) and (4.8), one has that

$$\eta_v = \sum_{x \in V(\widehat{G})} \eta_v^x \text{ on } \mathbb{N}. \tag{4.9}$$

Thus, one has that

$$\begin{aligned} L_G(v,s) &= \sum_{n=1}^{\infty} n^{-s} \eta_v \left(\log_{N_2} n\right) = \sum_{n=1}^{\infty} n^{-s} \left(\sum_{x \in V(\widehat{G})} \eta_v^x \left(\log_{N_2} n\right) \right) \\ &= \sum_{x \in V(\widehat{G})} \left(\sum_{n=1}^{\infty} n^{-s} \eta_v^x \left(\log_{N_2} n\right) \right) \end{aligned} \tag{4.10}$$

by (4.6), (4.8) and (4.9). The above relation (4.10) motivates the following proposition, showing certain decomposition property of $L_G(v, s)$.

Proposition 4.3. *There exist $\mathbb{C}$-valued functions*

$$f_x(v,s) : V(\widehat{G}) \times \mathbb{C} \to \mathbb{C}$$

defined by

$$f_x(v,s) = \sum_{n=1}^{\infty} \frac{\eta_v^x \left(\log_{N_2} n\right)}{n^s} = \sum_{k=0}^{\infty} \frac{\eta_v^x\ (k)}{N_2^{k\ s}}$$

for all $x \in V(\widehat{G})$ such that

$$L_G(v,s) = \sum_{x \in V(\widehat{G})} f_x(v,s). \tag{4.11}$$

Proof. The proof of (4.11) is done by (4.10). □

4.2. GRAPH-ARITHMETIC ALGEBRAS

Let $\mathcal{A}$ be a set consisting of all arithmetic functions, i.e.

$$\mathcal{A} \overset{def}{=} \{f : \mathbb{N} \to \mathbb{C} : f \text{ is a function}\}.$$

Then, as we discussed in Section 2.1, the set $\mathcal{A}$ forms an algebra over $\mathbb{C}$, with the usual functional addition, and convolution. In fact, by defining the unary operation $f \mapsto f^*$, where

$$f^*(n) = \overline{f(n)} \text{ in } \mathbb{C} \text{ for all } n \in \mathbb{N},$$

one can understand $\mathcal{A}$ as a $*$-algebra, where $\overline{z}$ mean the conjugates of z for all $z \in \mathbb{C}$.

Definition 4.4. We call the $*$-algebra $\mathcal{A}$ the arithmetic(-functional-$*$-)algebra (over $\mathbb{C}$).

Let $\eta_{v_1}^{v_2}$ and η_v be in the sense of (4.7) and (4.8) for all $v_1, v_2, v \in V(\widehat{G})$. Since they are $\mathbb{N}_0$-valued in $\mathbb{C}$ with their domains $\mathbb{N}$, one can naturally regard them as arithmetic functions contained in $\mathcal{A}$.

For a fixed given graph G, define a subset $\mathcal{V}_G$ of the arithmetic algebra $\mathcal{A}$ by

$$\mathcal{V}_G = span_{\mathbb{C}} \left(\{\eta_{v_1}^{v_2} : v_1, v_2 \in V(\widehat{G})\} \right), \tag{4.12}$$

where $span_{\mathbb{C}}(X)$ mean the *vector spaces* generated (or spanned) by arbitrary sets X over $\mathbb{C}$. Then the set $\mathcal{V}_G$ of (4.12) is a subspace of $\mathcal{A}$, but we cannot guarantee it is a subalgebra of $\mathcal{A}$, because even though $\eta_1, \eta_2 \in \mathcal{V}_G$ and $\eta_1 * \eta_2 \in \mathcal{A}$, in general,

$$\eta_1 * \eta_2 \in \mathcal{A} \backslash \mathcal{V}_G.$$

Define now a "*conditional*" *convolution*

$$\circledast : \mathcal{V}_G \times \mathcal{V}_G \to \mathcal{A}$$

by a (non-closed) binary operation satisfying that

$$\eta_{v_1}^{v_2} \circledast \eta_{x_1}^{x_2} \overset{def}{=} \delta_{v_2, x_1} \eta_{v_1}^{v_2} * \eta_{x_1}^{x_2}, \tag{4.13}$$

under linearity on $\mathcal{V}_G$ for all $v_1, v_2, x_1, x_2 \in V(\widehat{G})$, where δ means the *Kronecker delta*. Clearly, $\eta_{v_1}^{v_2} \circledast \eta_{x_1}^{x_2}$ is contained in the arithmetic algebra $\mathcal{A}$, but still, in general, it is not contained in $\mathcal{V}_G$.

Construct now a subalgebra $\mathcal{A}_G$, generated by $\mathcal{V}_G$, under the conditional convolution $\circledast$ of (4.13), i.e.

$$\mathcal{A}_G \overset{def}{=} A\lg_{\mathbb{C}}^{\circledast}(\mathcal{V}_G) = \mathbb{C}_{\circledast}[\mathcal{V}_G], \tag{4.14}$$

where $A\lg_{\mathbb{C}}^{\circledast}(\mathcal{V}_G)$ means "the subalgebra $\mathbb{C}_{\circledast}[\mathcal{V}_G]$" of the algebra $\mathbb{C}[\mathcal{V}_G]$ generated by arbitrary subsets $\mathcal{V}_G$ in $\mathcal{A}$, "under $\circledast$".

Proposition 4.5 ([4]). *Let $\mathcal{A}_G$ be the algebra* (4.14) *of the arithmetic algebra $\mathcal{A}$ generated by the subspace $\mathcal{V}_G$ of* (4.12)*, under the conditional convolution $\circledast$ of* (4.13)*. Then it is a subalgebra of $\mathcal{A}$ over $\mathbb{C}$.*

Remark 4.6. Since two binary operations, the conditional convolution ($\circledast$) on $\mathcal{A}_G$ and the usual convolution ($*$) on $\mathcal{A}$, are defined differently, one may understand the "subset" $\mathcal{A}_G$ of $\mathcal{A}$ as an independent algebra induced from certain elements of $\mathcal{A}$ under the conditions dictated by the Kronecker delta of (4.13). However, we regard $\mathcal{A}_G$ as a subalgebra of $\mathcal{A}$, because

(i) $\mathcal{A}_G$ is an algebraic sub-structure of $\mathcal{A}$, and
(ii) the conditional convolution ($\circledast$) is nothing but the convolution ($*$) satisfying certain connection rules (in fact, the admissibility on $\mathbb{G}$) among elements η_v^x's of $\mathcal{A}$.

So, if there is no conflicts, we will use the term "subalgebra" for $\mathcal{A}_G$ in $\mathcal{A}$.

Remark here that the condition on $\circledast$ of (4.13) is preserving the combinatorial property of a given graph G.

Definition 4.7. Let $\mathcal{A}_G$ be the subalgebra (4.14) of the arithmetic algebra $\mathcal{A}$. We call $\mathcal{A}_G$, the G-arithmetic algebra. And the conditional convolution $\circledast$ of (4.13) on $\mathcal{A}_G$ is called the G-convolution on $\mathcal{A}_G$.

Let $\mathcal{A}_G$ be the G-arithmetic algebra with G-convolution $\circledast$, and let $\eta_v, \eta_x \in \mathcal{A}_G$, for $v, x \in V(\widehat{G})$, where η_y are in the sense of (4.8), for all $y \in V(\widehat{G})$. Then

$$\eta_v \circledast \eta_x(k) = \Big(\Big(\sum_{y \in V(\widehat{G})} \eta_v^y\Big) \circledast \Big(\sum_{z \in V(\widehat{G})} \eta_x^z\Big)\Big)(k)$$

by (4.9)

$$\begin{aligned}
&= \Big(\sum_{y,z \in V(\widehat{G})} \eta_v^y \circledast \eta_x^z\Big)(k) = \sum_{y,z \in V(\widehat{G})} \Big(\eta_v^y \circledast \eta_x^z(k)\Big) \\
&= \sum_{y,z \in V(\widehat{G})} \Big(\delta_{y,x} \eta_v^y * \eta_x^z\Big)(k) \\
&= \sum_{z \in V(\widehat{G})} \Big(\eta_v^x * \eta_x^z\Big)(k) = \Big(\eta_v^x * \Big(\sum_{z \in V(\widehat{G})} \eta_x^z\Big)\Big)(k) \\
&= \eta_v^x \circledast \eta_x(k) = \eta_v^x * \eta_x(k),
\end{aligned} \tag{4.15}$$

for all $v, x \in V(\widehat{G})$ and $k \in \mathbb{N}$.

Lemma 4.8. *Let $\mathcal{A}_G$ be the G-arithmetic algebra, and let $\eta_v, \eta_x \in \mathcal{A}_G$ for $v, x \in V(\widehat{G})$. Then*

$$\eta_v \circledast \eta_x = \eta_v^x * \eta_x. \tag{4.16}$$

Proof. The proof of (4.16) is done by (4.15). □

4.3. GRAPH ZETA FUNCTIONAL ALGEBRAS

Let G be a fixed locally finite and connected graph with its graph groupoid $\mathbb{G}$, and let $\mathcal{A}_G$ be the G-arithmetic algebra in $\mathcal{A}$. Define now a set $\mathcal{L}$ by the collection of all Dirichlet series induced by arithmetic functions, i.e.

$$\mathcal{L} \overset{def}{=} \left\{ \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \mid f \in \mathcal{A} \right\}. \tag{4.17}$$

We call $\mathcal{L}$ the (*Dirichlet-*)*L-functional algebra.* It is not difficult to check that there exists a morphism

$$\varphi : \mathcal{A} \to \mathcal{L}$$

defined by

$$\varphi(f) \overset{def}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \text{ in } \mathcal{L}, \quad f \in \mathcal{A}. \tag{4.18}$$

Then this morphism φ is a $*$-algebra-isomorphism up to the $\mathbb{C}$-variable s (see [4]), i.e. the arithmetic algebra $\mathcal{A}$ and the L-functional algebra $\mathcal{L}$ are $*$-isomorphic.

By understanding our G-arithmetic algebra $\mathcal{A}_G$ as a subalgebra of $\mathcal{A}$, one can determine a corresponding $*$-isomorphic subalgebra $\mathcal{L}_G$ of the L-functional algebra $\mathcal{L}$. Define

$$\mathcal{L}_G \overset{def}{=} \varphi\left(\mathcal{A}_G\right), \tag{4.19}$$

where φ is in the sense of (4.18) from $\mathcal{A}$ onto $\mathcal{L}$.

Definition 4.9. The subalgebra $\mathcal{L}_G$ of the L-functional algebra $\mathcal{L}$ is called the graph zeta functional algebra induced by G.

Notice here that the subalgebra $\mathcal{A}_G$ of $\mathcal{A}$ is dictated by the "conditionality" from the G-convolution $\circledast$. So, the (usual inherited) multiplication on $\mathcal{L}_G = \varphi\left(\mathcal{A}_G\right)$ (inherited from $\mathcal{L}$) is determined by the conditionality.

Theorem 4.10 ([4])**.** *Let $\eta_{v_1}^{v_2}, \eta_{x_1}^{x_2}$ be generating elements of the G-arithmetic algebra $\mathcal{A}_G$ for $v_j, x_j \in V(\widehat{G})$, $j = 1, 2$, and let $\varphi(\eta_{v_1}^{v_2}), \varphi(\eta_{x_1}^{x_2})$ be corresponding L-functions in $\mathcal{L}_G$. Then*

$$\left(\varphi(\eta_{v_1}^{v_2})\right)\left(\varphi(\eta_{x_1}^{x_2})\right) = \sum_{n=1}^{\infty} \frac{\delta_{v_2,x_1}\left(\eta_{v_1}^{v_2} \ * \ \eta_{x_1}^{x_2}\right)\left(\log_{N_2} n\right)}{n^s} \tag{4.20}$$

"in $\mathcal{L}_G$".

The above theorem characterizes the conditionality on the multiplication $(\cdot)$ on $\mathcal{L}_G$, in terms of $(\circledast)$ on $\mathcal{A}_G$ in $\mathcal{L}$.

To emphasize the above conditional multiplicativity (4.20) on the graph zeta functional algebra $\mathcal{L}_G$, we denote the multiplication on $\mathcal{L}_G$ by

$$\boxtimes : \mathcal{L}_G \times \mathcal{L}_G \to \mathcal{L}_G, \tag{4.21}$$

i.e. we want to handle the subalgebra $\mathcal{L}_G$ of $\mathcal{L}$, as an independent algebraic structure (under $\boxtimes$).

By (4.20) and (4.21), if $\eta_1, \eta_2 \in \mathcal{A}_G$, and if

$$L_1 = \sum_{n=1}^{\infty} \frac{\eta_1(n)}{n^s} \quad \text{and} \quad L_2 = \sum_{n=1}^{\infty} \frac{\eta_2(n)}{n^s} \text{ in } \mathcal{L}_G,$$

then

$$L_1 L_2 = L_1 \boxtimes L_2 = \sum_{n=1}^{\infty} \frac{\eta_1 \circledast \eta_2(n)}{n^s} \text{ "in } \mathcal{L}_G\text{".}$$

4.4. CONVERGENCE OF GRAPH ZETA FUNCTIONS

In the rest of this paper, we are no longer interested in the convergence of graph zeta functions. One may understand all our graph zeta functions are assumed to be convergent. But it is true that, under such conditions, we may lose lots of interesting analytic properties of graph zeta functions gotten from the pure algebraic-and-combinatorial properties of $\mathbb{G}$ (or of G). The following convergence condition (4.22) is obtained in [6].

Theorem 4.11 ([6]). *Let G be a finite connected graph, and $v \in V(G)$, and let $L_G(v,s)$ be the graph zeta function based at v. If*

$$\mathrm{Re}(s) > \log_{N_2}\left(\lim_{n\to\infty}\frac{\eta_v(n+1)}{\eta_v(n)}\right), \tag{4.22}$$

then $L_G(v,s)$ is convergent absolutely, as an infinite series in $\mathcal{C}$.

5. FACTORIZATIONS OF GRAPH ZETA FUNCTIONS

As before, we fix a finite connected graph G with its graph groupoid $\mathbb{G}$, and let $L_G(v,s)$ be graph zeta functions at $v \in V(\widehat{G})$. In this section, we further study factorizability of graph zeta functions (at vertices) which have been considered in [7].

Let w be a loop reduced finite path in $FP_r(\widehat{G})$ in $\mathbb{G}$. Since w is a loop, w^m are all loop reduced finite paths, for all $m \in \mathbb{Z}$. Moreover, whenever such a loop w is chosen in $\mathbb{G}$, there exists a unique loop finite path l, such that: (i) $w = l^n$, for some $n \in \mathbb{N}$, and (ii) there is no other loop finite path y such that $y \hookrightarrow l$.

Let l be the loop satisfying the conditions (i) and (ii) of the above paragraph. Then we call l a *basic loop* (*reduced finite path*).

Let $w^o = vw^o \in FP_r(\widehat{G})$ be a reduced finite paths of $\mathbb{G}$ initial from the vertex v, and let

$$w^o = ve_1e_2e_3\ldots e_n, \text{ with } e_1, e_2, \ldots, e_n \in E(\widehat{G})$$

for $n \in \mathbb{N}$. Moreover, assume that w^o is "*maximal from v*", in the sense that w^o is a reduced word in $E(\widehat{G})$ satisfying the following two conditions (5.1) and (5.2):

if there is a loop reduced finite path w' embedded in w^o (i.e. $w' \hookrightarrow w^o$ or $w' = w^o$), then w' is basic in $\mathbb{G}$, (5.1)

there does not exist w satisfying (5.1) such that $w^o \hookrightarrow w$ "properly" in $\mathbb{G}$. Equivalently, if such w exists, then $w = w^o$. (5.2)

Definition 5.1. Assume that w^o is maximal from v in the sense of (5.1) and (5.2). Then w^o is said to be a ray from v (or a v-ray).

See [7] for precise examples for v-rays. Especially, in [7], we considered "locally-finite" graphs (instead of handling finite graphs), and hence, it is possible that a ray w^o cannot be contained in $\mathbb{G}$, if G is an infinite graph under (5.1) and (5.2). However, in this paper, we assumed all given graphs G are finite, so rays are regarded as reduced finite paths in $\mathbb{G}$ under (5.1) and (5.2).

Now, let $u^o = ve_1e_2e_3\cdots e_n$ be a ray from a fixed vertex v in $\mathbb{G}$, where $e_1, \ldots, e_n \in E(\widehat{G})$ for some $n \in \mathbb{N}$, and let

$$x_j = r(e_j) = s(e_{j+1}), \quad j = 1, \ldots, n-1,$$

with $x_n = r(e_n)$. Then one can construct the corresponding sequence U^o of vertices from the v-ray u^o,

$$U^o = (v = x_0, x_1, x_2, x_3, \ldots, x_n). \tag{5.3}$$

We call the sequence U^o of (5.3), the *ray-vertex sequence of* u^o. For each embedded pair $W_j^o = (x_j, x_{j+1})$ of U^o, there may be multi-edges connecting x_j to x_{j+1} in the edge set $E(\widehat{G})$ of the shadowed graph $\widehat{G}$, $j = 1, \ldots, n-1$.

Let G be given and v, a fixed vertex in $V(\widehat{G})$, and let w^o be the v-ray. Let W^o be the corresponding ray-vertex sequence of w^o as in (5.3). Now, construct a subgraph G_{w^o} of the "shadowed" graph $\widehat{G}$ of G by a new independent graph with

$$V(G_{w^o}) = \{v = x_0, x_1, x_2, x_3, \ldots, x_n\}, \tag{5.4}$$

and

$$E(G_{w^o}) = \{e \in E(\widehat{G}) : e = x_j e x_{j+1}, j = 0, 1, \ldots, n-1\}.$$

Definition 5.2. Let G be a graph, and $v \in V(G)$, and let w^o be the v-ray with its ray-vertex sequence W^o. Then subgraph G_{w^o} of (5.4) is called a ray subgraph of G from v.

The following proposition is proven by the connectedness of our graph G.

Proposition 5.3 ([7]). *Let G be a graph with its shadowed graph $\widehat{G}$, and let v be a fixed graph. Let*

$$\mathcal{G}_v = \{K \leqslant \widehat{G} : K \textit{ is a ray subgraph of } \widehat{G} \textit{ from } v\}, \tag{5.5}$$

i.e. the family $\mathcal{G}_v$ is the set of all ray subgraphs of G from v. Then

$$\widehat{G} \textit{ is graph-isomorphic to } \textstyle\bigcup_{K \in \mathcal{G}_v} \widehat{K}, \tag{5.6}$$

and

$$\mathbb{G} \textit{ is groupoid-isomorphic to } \underset{K \in \mathcal{G}_v}{\oplus} \mathbb{K}, \tag{5.7}$$

where $\widehat{K}$ mean shadowed graphs of K, and $\mathbb{K}$ are graph groupoids of K, for all $K \in \mathcal{G}_v$.

Remark that the above relations (5.6) and (5.7) do not hold in general without connectedness of G.

Now, let $\mathcal{G}_v$ be as in (5.5), and assume $K_1 \neq K_2 \in \mathcal{G}_v$. Suppose

$$(V(K_1) \setminus \{v\}) \cap (V(K_2) \setminus \{v\}) \neq \varnothing.$$

Then construct the graph union $K_1 \cup K_2$. If there exists $K_3 \in \mathcal{G}_v$ such that

$$K_3 \neq K_1 \quad \text{and} \quad K_3 \neq K_2$$

and

$$(V(K_3) \setminus \{v\}) \cap (V(K_1 \cup K_2) \setminus \{v\}) \neq \varnothing,$$

then construct the graph union

$$(K_1 \cup K_2) \cup K_3 = K_1 \cup K_2 \cup K_3,$$

as a subgraph of G. Do this processes until it ends for $\mathcal{G}_v$.

Definition 5.4. The process in the very above paragraph is called the *v-ray-subgraph unionization* for a fixed vertex v, and the resulted subgraph induced by $\mathcal{G}_v$ of (5.5) are said to by *v-ray-subgraph unions* in the shadowed graph $\widehat{G}$ of G. Then one obtains a new family $\mathfrak{G}_v$ of subgraphs of G,

$$\mathfrak{G}_v = \{K \leqslant \widehat{G} : K \text{ is a } v\text{-ray-subgraph union}\}. \tag{5.8}$$

This family $\mathfrak{G}_v$ of (5.8) is called the v-ray factorization of $\widehat{G}$.

For instance, let

$$G = \quad \bullet \rightarrow \bullet \leftarrow \underset{v}{\bullet} \leftarrow \underset{\circlearrowleft}{\bullet} \rightrightarrows \bullet \rightleftarrows \bullet.$$

Then the v-ray subgroup unions $\mathcal{G}_v$ of (5.5) is

$$\mathcal{G}_v = \{K_1, K_2, K_3, K_4, K_5\},$$

where

$$K_j = \quad \underset{v}{\bullet} \rightarrow \underset{\circlearrowleft}{\bullet} \rightarrow \bullet \rightarrow \bullet, \quad j = 1, 2, 3, 4,$$

and

$$K_5 = \quad \bullet \leftarrow \bullet \leftarrow \underset{v}{\bullet}$$

in $\widehat{G}$. Then we obtain the v-ray factorization $\mathfrak{G}_v$ of (5.8),

$$\mathfrak{G}_v = \{G_1, G_2\},$$

where

$$G_1 = K_5 = \quad \bullet \leftarrow \bullet \leftarrow \underset{v}{\bullet},$$

and

$$G_2 = \bigcup_{j=1}^{4} K_j = \quad \underset{v}{\bullet} \rightarrow \underset{\circlearrowleft}{\bullet} \rightrightarrows \bullet \rightrightarrows \bullet,$$

in the shadowed graph $\widehat{G}$ of G.

Definition 5.5. Let $\mathfrak{G}_v$ be the v-ray factorization (5.8) induced by the v-ray subgraph unions $\mathcal{G}_v$ of (5.5). If $\mathfrak{G}_v = \{\widehat{G}\}$, then the graph G (or the shadowed graph $\widehat{G}$, or the graph groupoid $\mathbb{G}$) is not v-ray factorizable. Otherwise, one says G (resp., $\widehat{G}$, resp., $\mathbb{G}$) is v-ray factorizable.

Assume now that G is v-ray factorizable, i.e. the family $\mathfrak{G}_v$ of (5.8) has more than one element. Let $K_1 \neq K_2 \in \mathfrak{G}_v$. Then it is not difficult to verify that

$$V(K_1) \cap V(K_2) = \{v\} \quad \text{and} \quad E(K_1) \cap E(K_2) = \varnothing \tag{5.9}$$

in $\mathbb{G}$, i.e. the distinct elements of the v-ray factorization $\mathfrak{G}_v$ share their only common element v as (proper) subgraphs of $\widehat{G}$.

Proposition 5.6. *Let $\mathfrak{G}_v$ be the v-ray factorization* (5.8) *for a fixed vertex v of a given graph G. Assume that G is v-ray factorizable, and $K_1 \neq K_2 \in \mathfrak{G}_v$. Then*

$$\mathbb{K}_1^{\times} \cap \mathbb{K}_2^{\times} = \{v\} \text{ in } \mathbb{G}, \tag{5.10}$$

where $\mathbb{K}_j$ are the graph groupoids of K_j, as subgroupoids of $\mathbb{G}$, and $\mathbb{K}_j^{\times} = \mathbb{K}_j \backslash \{\varnothing\}$, $j = 1, 2$.

Proof. The proof of (5.10) is done by (5.9). □

Consider now the following proposition.

Proposition 5.7. *Suppose K is a subgraph of G, and let*

$$l_K(x, s) = \sum_{w \in \mathbb{K}_x} \frac{1}{\varpi(w)^s},$$

where $\mathbb{K}_x = \mathbb{K} \cap \mathbb{G}_x$, where $\mathbb{K}$ is the graph groupoid of K (as a subgroupoid of $\mathbb{G}$), for all vertices x in $\mathbb{K}$. If $v \in V(K)$ in $V(G)$, then

$$l_K(v, s) = \sum_{n=1}^{\infty} \frac{\left|(\mathbb{K}_v)\left(\log_{N_2} n\right)\right|}{n^s} \text{ in } \mathcal{L}_G. \tag{5.11}$$

Proof. The proof of (5.11) is trivial by the very construction of $l_K(x, s)$ as an embedded part of $L_G(x, s)$, by (5.10). □

Notice here that, by the construction of the graph zeta-functional algebra $\mathcal{L}_G$ (by that of the G-arithmetic algebra $\mathcal{A}_G$), indeed, the function $l_K(v, s)$ is contained in $\mathcal{L}_G$, whenever $K \leqslant G$. Note also that the functions $l_K(v, s)$ generated by subgraphs K of G are regarded as independent graph zeta functions $L_{K_0}(v, s)$, where K_0 is a graph with its shadowed graph K.

In $\mathcal{L}_G$, if $v \in V(G)$, but $v \notin V(K)$, then $l_K(v, s)$ is automatically assumed to be the zero element $0_{\mathcal{L}_G}$, and if $v \in V(K)$, then $l_K(v, s)$ is determined as above in (5.11) in $\mathcal{L}_G$.

Proposition 5.8 ([7])**.** *Let K_1 and K_2 be connected subgraphs of the shadowed graph $\widehat{G}$ of G with*

$$V_{1,2} = V(K_1) \cap V(K_2) \neq \varnothing,$$

and let $v \in V_{1,2}$. Let $l_{K_1}(v, s)$ and $l_{K_2}(v, s)$ be elements of the graph zeta functional algebra $\mathcal{L}_G$, as in (5.11)*. Then*

$$l_{K_1}(v, s) \boxtimes l_{K_2}(v, s) = l_{K_1 \cup K_2}(v, s), \tag{5.12}$$

where $K_1 \cup K_2$ is the graph union of K_1 and K_2, as a subgraph of G, and where $\boxtimes$ is in the sense of (4.21)*.*

With help of (5.12), we obtain the following theorem.

Theorem 5.9. *Let $\mathfrak{G}_v$ be the v-ray factorization induced by $\mathcal{G}_v$, for a fixed vertex v of a given graph G. Assume that $\mathfrak{G}_v$ is v-ray factorizable and suppose K_1 and K_2 are distinct elements in $\mathfrak{G}_v$, as subgraphs of $\widehat{G}$. Then*

$$l_{K_1 \cup K_2}(v,s) = (l_{K_1}(v,s))\,(l_{K_2}(v,s)) \ \text{ in } \mathcal{L}_G, \tag{5.13}$$

where l_{K_1}, l_{K_2} and $l_{K_1 \cup K_2}$ are in the sense of (5.11) *in $\mathcal{L}_G$. (Remark here that $K_1 \cup K_2$ is again a subgraph of $\widehat{G}$, but it is "not" contained in $\mathfrak{G}_v$.)*

Proof. By (5.12), we have that

$$l_{K_1 \cup K_2}(v,s) = l_{K_1}(v,s) \boxtimes l_{K_2}(v,s),$$

where $\boxtimes$ is in the sense of (4.21) on the graph zeta functional algebra $\mathcal{L}_G$. However, one has

$$\mathbb{K}_1^{\times} \cap \mathbb{K}_2^{\times} = \{v\},$$

by (5.10), where $\mathbb{K}_j$ are the graph groupoids of K_j and $\mathbb{K}_j^{\times} = \mathbb{K}_j \setminus \{v\}$, $j = 1, 2$. So, if

$$w_1 = w_1 v \in V(K_1) \quad \text{and} \quad w_2 = v w_2 \in V(K_2),$$

then the reduced finite path $w_1 w_2$ in $\mathbb{K}_{1,2}$ which is the graph groupoid of the graph union $K_1 \cup K_2$ (as a subgroupoid of $\mathbb{G}$) satisfies that

$$w_1' = w_1 \quad \text{and} \quad w_2' = w_2,$$

where w_j' means the reduced embeddings in $\mathbb{K}_{1,2}$ in the sense of (3.5), and hence

$$w_1 w_2 = w_1^o w_2^o \text{ in } \mathbb{K}_{1,2},$$

satisfying

$$|w_1 w_2| = |w_1^o w_2^o| = |w_1^o| + |w_2^o| = |w_1| + |w_2|\,.$$

It shows that

$$\varpi(w_1 w_2) = \varpi(w_1)\varpi(w_2),$$

by (3.5). It guarantees that, for the fixed vertex v,

$$l_{K_1}(v,s) \boxtimes l_{K_2}(v,s) = (l_{K_1}(v,s))\,(l_{K_2}(v,s))$$

in $\mathcal{L}_G$. □

The characterization (5.13) shows that if K_1 and K_2 are distinct elements of the v-ray factorization $\mathfrak{G}_v$, then the "conditional" product of $l_{K_1}(v,s)$ and $l_{K_2}(v,s)$ of $\mathcal{L}_G$ becomes the "usual" functional multiplication of them.

Therefore, we obtain the following factorizability, by (5.13), with help of (5.10) and (5.11).

Theorem 5.10. *Let G be a finite connected graph and let v be an arbitrarily chosen vertex in $V(\widehat{G})$. Let $\mathfrak{G}_v$ be the v-ray factorization* (5.8)*. Then*

$$L_G(v,s) = \prod_{K \in \mathfrak{G}_v} l_K(v,s), \tag{5.14}$$

in the graph zeta functional algebra $\mathcal{L}_G$, where $l_K(v,s)$ are in the sense of (5.11)*, and $\prod$ means the usual functional multiplication.*

Proof. Suppose first that the v-ray factorization $\mathfrak{G}_v$ is not factorizable in the sense that $\mathfrak{G}_v = \{\widehat{G}\}$. Then, clearly, we have

$$L_G(v,s) = \prod_{K \in \mathfrak{G}_v = \{G\}} l_K(v,s) = l_{\widehat{G}}(v,s) = L_G(v,s).$$

So, the relation (5.14) holds.

Assume now that $\mathfrak{G}_v$ is factorizable. And let

$$\mathfrak{G}_v = \{K_1, K_2, \ldots, K_m\} \quad \text{for some} \quad m \in \mathbb{N}, m \neq 1.$$

Then, by the construction of K_j, their graph groupoids $\mathbb{K}_j$ only share their common elements v as subgroupoids of $\mathbb{G}$ for $j = 1, \ldots, m$.

Notice that, by (5.6) and (5.7), we have

$$\widehat{G} = \bigcup_{K \in \mathfrak{G}_v} \widehat{K} \quad \text{and} \quad \mathbb{G} = \bigoplus_{K \in \mathfrak{G}_v} \mathbb{K},$$

So,

$$L_G(v,s) = l_{\widehat{G}}(v,s) = l_{\bigcup_{K \in \mathfrak{G}_v} K}(v,s),$$

where $L_G(v,s)$ means our graph zeta function, and $l_{\widehat{G}}(v,s)$ means the element of $\mathcal{L}_G$ in the sense of (5.11)

$$= \underset{K \in \mathfrak{G}_v}{\boxtimes} (l_K(v,s))$$

by (5.12)

$$= \prod_{K \in \mathfrak{G}_v} (l_K(v,s))$$

by (5.13). Therefore, the graph zeta function $L_G(v,s)$ at a fixed vertex v is factorizable with its factors, the subgraph zeta functions $l_K(v,s)$, for the v-ray-subgraph unions K, at v. □

Definition 5.11. Let $L_G(v,s)$ be a graph zeta function at $v \in V(G)$. We say that $L_G(v,s)$ is factorizable at v if the v-ray factorization $\mathfrak{G}_v$ is factorizable; otherwise, $L_G(v,s)$ is not factorizable at v.

6. NON-FACTORIZABLE GRAPHS

In this section, we consider non-factorizable graph zeta functions more in detail. First, we introduce the following example, providing a motivation.

Example 6.1. *Let K_d^n be the n-regular circulant graph with d-vertices, for $n \in \mathbb{N}$, $d \in \mathbb{N}\setminus\{1\}$, i.e.*

$$K_d^n = \begin{array}{ccccc} & & \overset{v_1}{\bullet} & & \\ & \swarrow & & \nwarrow & \\ \overset{v_2}{\bullet} & & & & \overset{v_d}{\bullet} \\ \downarrow & & & & \uparrow \\ \underset{v_3}{\bullet} & \rightarrow & \cdots & \rightarrow & \underset{v_{d-1}}{\bullet} \end{array}. \tag{6.1}$$

where each arrow "$\rightarrow$" in (6.1) actually means n-multi-edges. Then, for any arbitrarily fixed vertex v_j, the v_j-ray factorization $\mathfrak{G}_{v_j}$ satisfies that

$$\mathfrak{G}_{v_j} = \{\widehat{K_d^n}\}, \quad j = 1, \ldots, d.$$

Therefore, the graph zeta function $L_{K_d^n}(v_j, s)$ are not factorizable, for all $j = 1, \ldots, d$. Furthermore,

$$L_{K_d^n}(v_1, s) = L_{K_d^n}(v_2, s) = \cdots = L_{K_d^n}(v_d, s) \text{ in } \mathcal{L}_{K_d^n}.$$

So, one can classify the following special types of finite connected graphs.

Definition 6.2. If a graph G induces non-factorizable graph zeta functions $L_G(x, s)$, for "all $x \in V(G)$," then the graph G is said to be non-factorizable.

By the above example, the n-regular circulant graphs K_d^n with d-vertices are non-factorizable graphs, for all $n \in \mathbb{N}, d \in \mathbb{N}\setminus\{1\}$.

More generally, we obtained the following theorem.

Theorem 6.3 ([7])**.** *Let G be a graph with the quantity N of (3.2), and $d = |V(G)|$ in $\mathbb{N}$. Then G is non-factorizable if and only if the shadowed graph $\widehat{G}$ of G is graph-isomorphic to the shadowed graph $\widehat{K}$ of a "connected" subgraph K of the circulant N-regular K_d^N with d-vertices of (6.1) whose vertex set satisfies $V(K) = V(K_d^N)$, i.e.*

$$\text{a graph } G \text{ is non-factorizable with } N \text{ of } (3.2) \tag{6.2}$$

if and only if

$$\text{there exists connected } K \leqslant K_{|V(G)|}^N \text{ with same } N \text{ such that } \widehat{G} \overset{Graph}{=} \widehat{K}.$$

The above statement (6.2) characterizes "finite and connected" non-factorizable graphs having more than 1 vertices. Without loss of generality, every non-factorizable graph G with N of (3.2) and $d = |V(G)| \in \mathbb{N} \cup \{\infty\}$ is regarded as a subgraph K of the circulant N-regular graph K_d^N with d-vertices such that $|V(K)| = d$.

7. GRAPH ZETA FUNCTIONS FOR NON-FACTORIZABLE GRAPHS

Let G be a finite connected graph with the quantity $N \in \mathbb{N}$ of (3.2), and $|V(G)| = d \in \mathbb{N} \cup \{\infty\}$. In this section, based on the main results of Section 6, we study graph zeta functions $L_G(\bullet, s)$, where G is non-factorizable. In particular, we are interested in the cases where given non-factorizable graphs G are (graph-isomorphic to) the circulant N-regular graph K_d^N with d-vertices.

Recall that, by the characterization (6.2), a finite connected graph G is non-factorizable if and only if there exists a subgraph K of the circulant N-regular $K_{|V(G)|}^N$ such that $\widehat{K}$ and $\widehat{G}$ are graph-isomorphic.

Proposition 7.1. *Let K_d^1 be the circulant* (1*-regular*) *graph with d-vertices $v_1, \ldots, v_d$, for all $d \in \mathbb{N} \cup \{\infty\}$. Then*

$$L_{K_d^1}(v_j, s) = \frac{3^s + 1}{3^s - 1} \tag{7.1}$$

for all $j = 1, \ldots, d$.

Proof. One can check that, for any $d \in \mathbb{N} \cup \{\infty\}$,

$$\left|\left(\mathbb{K}_d^1\right)_{v_j}(0)\right| = 1 \quad \text{and} \quad \left|\left(\mathbb{K}_d^1\right)_{v_j}(k)\right| = 2$$

for all $k \in \mathbb{N}$ and $j = 1, \ldots, d$, where $\mathbb{K}_n^1$ are the graph groupoids of circulant 1-regular graphs K_n^1 with n-vertices for all $n \in \mathbb{N}$. Therefore, we obtain that

$$K_{K_d^1}(v_j, s) = \sum_{k=0}^{\infty} \frac{\left|\left(\mathbb{K}_d^1\right)_v(k)\right|}{3^{k\,s}} = 1 + \sum_{k=1}^{\infty} \frac{\left|\left(\mathbb{K}_d^1\right)_v(k)\right|}{3^{k\,s}} = 1 + 2\left(\frac{1}{3^s - 1}\right)$$

for all $j = 1, \ldots, d$, since N_2 of (3.3) satisfies that $N_2 = 2(1) + 1 = 3$, where N is in the sense of (3.2). ☐

Let us extend the formula (7.1) to the general cases.

Theorem 7.2. *Let K_d^N be the circulant N-regular graph with d-vertices, for $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{\infty\}$. Then*

$$L_{K_d^N}(v, s) = 1 + \frac{2N}{(2N+1)^s}\left(1 - \frac{N}{(2N+1)^s}\right)^{-1} \tag{7.2}$$

for all $v \in V(K_d^N)$.

Proof. Observe that

$$\left|\left(\mathbb{K}_d^N\right)_v(1)\right| = |\{v\}| = 1$$

and

$$\left|\left(\mathbb{K}_d^N\right)(k)\right| = 2N^k \text{ for all } k \in \mathbb{N},$$

where $\mathbb{K}_n^m$ are the graph groupoids of the circulant m-regular graphs K_n^m with n-vertices, for all $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$.

Thus, for each vertex v of K_d^N,

$$L_{K_d^N}(v,s) = \sum_{k=0}^{\infty} \frac{\left|\left(\mathbb{K}_d^N\right)(k)\right|}{(2N+1)^{ks}} = 1 + \sum_{k=1}^{\infty} \frac{2N^k}{(2N+1)^{ks}},$$

in $\mathcal{L}_{K_d^N}$. So, the formula (7.2) holds. □

Consider now (proper) connected subgraphs $K_d^{N:1}$ of the circulant N-regular graph K_d^N of d-vertices with

$$V(K_d^{N:1}) = V(K_d^N)$$

and

$$E(K_d^{N:1}) = \left\{ e \in E(K_d^N) \cap \left(\mathbb{K}_d^N\right)_1 \,\middle|\, \begin{array}{c} e \text{ is a unique edge connecting} \\ \text{pairs of vertices in } V(K_d^{N:1}) \end{array} \right\}.$$

For example, if

$$K_2^2 = \quad \bullet \underset{e_2}{\overset{e_1}{\rightrightarrows}} \bullet,$$

then

$$K_2^{2:1} = \quad \bullet \overset{e_1}{\to} \bullet \quad \text{or} \quad \bullet \underset{e_2}{\to} \bullet.$$

Recall that a graph G is said to be *simplicial* if

(i) G has no loop-edges, and
(ii) if there is an edge $e = vex$ connecting a vertex v to a vertex x, then it is the only edge connecting v to x, for $v, x \in V(G)$.

So, the subgraphs $K_d^{N:1}$ are simplicial (sub)graphs in K_d^N. We call the subgraphs $K_d^{N:1}$ of K_d^N, the *simplicial subgraphs of* K_d^N.

Remark 7.3. There are N^d-many simplicial subgraphs of K_d^N.

The following theorem demonstrates not only the similarity of

$$l_{K_d^{N:1}}(v,s) \in \mathcal{L}_{K_d^N} \quad \text{and} \quad L_{K_d^1}(v,s) \in \mathcal{L}_{K_d^1},$$

but also the difference of them.

Theorem 7.4. *Let K_d^N be the circulant N-regular graph with d-vertices, for $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{\infty\}$, and let $K_d^{N:1}$ be a simplicial subgraph of K_d^N. Let $l_{K_d^{N:1}}(x,s)$ be elements of $\mathcal{L}_{K_d^N}$ in the sense of* (5.11), *for all $x \in V(K_d^{N:1}) = V(K_d^N)$. Then*

$$l_{K_d^{N:1}}(v,s) = 1 + \frac{2}{(2N+1)^s}\left(1 - \frac{1}{(2N+1)^s}\right)^{-1}, \tag{7.3}$$

in $\mathcal{L}_{K_d^N}$, for all $v \in V(K_d^{N:1})$.

Proof. By (5.9), one can get that

$$l_{K_d^{N:1}}(v,s) = \sum_{w \in (\mathbb{K}_d^{N:1})_v} \frac{1}{\varpi(w)^s},$$

with

$$\varpi(w) = (2N+1)^{|w|} \text{ for all } w \in \mathbb{K}_d^{N:1} \subseteq \mathbb{K}_d^N,$$

where $l_{K_d^{N:1}}(\bullet, s)$ are subgraph zeta functions in $\mathcal{L}_{K_d^N}$ in the sense of (5.11), and the quantity $2N+1$ is in the sense of (3.3). Then it is not difficult to check that, for any $v \in V(K_d^{N:1}) = V(K_d^N)$,

$$l_{K_d^{N:1}}(v,s) = \sum_{k=0}^{\infty} \frac{\left|\left(\mathbb{K}_d^{N:1}\right)_v (k)\right|}{(2N+1)^{k\,s}}$$

by (5.11)

$$= 1 + \sum_{k=1}^{\infty} \frac{2}{(2N+1)^{ks}}$$

because

$$\left|(\mathbb{K}_d^{N:1})_x \ (0)\right| = |\{x\}| = 1,$$

and

$$\left|\left(\mathbb{K}_d^{N:1}\right)_x (k)\right| = 2,$$

for all $x \in V(K_d^{N:1})$, as in (7.1), and $k \in \mathbb{N}$.

Therefore, one has that

$$l_{K_d^{N:1}}(v,s) = 1 + \frac{2}{(2N+1)^s}\left(1 - \frac{1}{(2N+1)^s}\right)^{-1},$$

similar to (7.2). So, the computation (7.3) holds true in $\mathcal{L}_{K_d^N}$. □

Let $K_d^{N:1}$ be a simplicial subgraph of K_d^N as above, for $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{\infty\}$. Then, we have

$$l_{K_d^{N:1}}(v,s) = 1 + \frac{2}{(2N+1)^s}\left(1 - \frac{1}{(2N+1)^s}\right)^{-1},$$

by (7.3), for all $v \in V(K_d^{N:1})$, and

$$L_{K_d^1}(x,s) = 1 + \frac{2}{3^s}\left(1 - \frac{1}{3^s}\right)^{-1},$$

by (7.1), for all $x \in V(K_d^1)$.

Observe now that the functions $l_{K_d^{N:1}}(\bullet, s)$ are obtained by replacing 3 to $2N+1$ in the computations of the graph zeta functions $L_{K_d^1}(\bullet, s)$.

By (7.2), one has

$$L_{K_d^N}(x,s) = \frac{N_2^s + N}{N_2^s - N}$$

for all $x \in V(K_d^N)$, and, by (7.3), we have

$$l_{K_d^{N:1}}(v,s) = \frac{N_2^s + 1}{N_2^s - 1}$$

for all $v \in V(K_d^N) = V(K_d^{N:1})$, in $\mathcal{L}_{K_d^N}$, where N and N_2 are in the sense of (3.2) and (3.3).

Theorem 7.5. *Let $L_{K_d^N}(x,s)$ be the graph zeta function of K_d^N, and let $l_{K_d^{N:1}}(v,s)$ be the elements of $\mathcal{L}_{K_d^N}$ in the sense of (5.11) for the simplicial subgraphs $K_d^{N:1}$ in $\mathcal{L}_{K_d^N}$, for all $x, v \in V(K_d^N)$. Now, let us understand $L_{K_d^N}(x,s)$ and $l_{K_d^{N:1}}(v,s)$ as complex-valued functions in s on $\mathbb{C}$. Then there exists a $\mathbb{C}$-valued function*

$$f(s) = \left(\frac{(2N+1)^s + N}{(2N+1)^s + 1}\right)\left(\frac{(2N+1)^s - 1}{(2N+1)^s - N}\right),$$

such that

$$L_{K_d^N}(x,s) = (f(s))\left(l_{K_d^{N:1}}(v,s)\right) \text{ for all } x, v \in V(K_d^N). \tag{7.4}$$

Proof. By the straightforward computation, one has

$$L_{K_d^N}(x,s) = \left(\frac{(2N+1)^s + N}{(2N+1)^s + 1}\right)\left(\frac{(2N+1)^s - 1}{(2N+1)^s - N}\right)\left(l_{K_d^{N:1}}(v,s)\right),$$

as $\mathbb{C}$-valued functions. □

The above relation (7.4) shows the relations between our graph-zeta-functional study and complex function theory.

8. GLUING NON-FACTORIZABLE GRAPHS AND GRAPH ZETA FUNCTIONS

In this section, we study how to construct factorizable graphs from non-factorizable graphs, and consider corresponding graph zeta functions.

Let $\mathfrak{NF}$ be the collection of all non-factorizable finite connected graphs,

$$\mathfrak{NF} \overset{def}{=} \left\{ G \,\middle|\, \begin{array}{c} G \text{ is a non-factorizable, finite,} \\ \text{connected graphs} \end{array} \right\}. \tag{8.1}$$

Define now a sub-family,

$$\mathfrak{NF}_d^N = \left\{ K \in \mathfrak{NF} \,\middle|\, \begin{array}{c} K \text{ is a graph with } N \text{ of (3.2),} \\ \text{and } d\text{-many vertices} \end{array} \right\}, \tag{8.2}$$

of $\mathfrak{NF}$, for $d, N \in \mathbb{N}$.

Without loss of generality, we define the families (8.1) and (8.2) "up to graph-isomorphisms", i.e. if G_1 and G_2 are graph-isomorphic in $\mathfrak{NF}$ (or $\mathfrak{NF}_d^N$), then we regard them as the same element in $\mathfrak{NF}$ (resp., in $\mathfrak{NF}_d^N$, for all $d, N \in \mathbb{N}$).

Then, under the above assumption, one can re-define $\mathfrak{NF}_d^N$ by the identical family $\mathfrak{K}_d^N$,

$$\mathfrak{K}_d^N \overset{def}{=} \left\{ K \in \mathfrak{NF} \,\middle|\, \begin{array}{c} K \text{ is a subgraph of } \widehat{K_d^N} \\ \text{with } V(K) = V(K_d^N) \end{array} \right\}, \tag{8.3}$$

under graph-isomorphisms, where K_d^N are the N-regular circulant graph with d-vertices, by the non-factorizability (6.2).

For instance, if

$$K_3^2 = \begin{array}{ccc} \bullet & \rightrightarrows & \bullet \\ & \nwarrow\nwarrow & \downdownarrows \\ & & \bullet \end{array},$$

then the collection $\mathfrak{K}_3^2$ of (8.3) contains not only K_3^2, itself, but also, the graphs

$$\begin{array}{ccc} \bullet & \rightrightarrows & \bullet \\ & \nwarrow & \downdownarrows \\ & & \bullet \end{array}, \quad \begin{array}{ccc} \bullet & \rightarrow & \bullet \\ & \nwarrow\nwarrow & \downdownarrows \\ & & \bullet \end{array}, \quad \begin{array}{ccc} \bullet & \rightrightarrows & \bullet \\ & \nwarrow\nwarrow & \downarrow \\ & & \bullet \end{array}, \tag{8.4}$$

$$\begin{array}{ccc} \bullet & \rightarrow & \bullet \\ & \nwarrow\nwarrow & \downarrow \\ & & \bullet \end{array}, \quad \begin{array}{ccc} \bullet & \rightarrow & \bullet \\ & \nwarrow & \downdownarrows \\ & & \bullet \end{array}, \quad \begin{array}{ccc} \bullet & \rightrightarrows & \bullet \\ & \nwarrow & \downarrow \\ & & \bullet \end{array}, \quad \begin{array}{ccc} \bullet & \rightarrow & \bullet \\ & \nwarrow & \downarrow \\ & & \bullet \end{array},$$

and all other graphs obtained by changing the direction of each edge of the graphs of (8.4) one-by-one. Then, by (6.2), this family $\mathfrak{K}_3^2$ of (8.3) is identical to the family $\mathfrak{NF}_3^2$ of (8.2), up to graph-isomorphisms.

We naturally obtain the following classification theorem by (6.2).

Proposition 8.1. *Let* $\mathfrak{NF}$ *and* $\mathfrak{NF}_d^N$ *be given as in* (8.1) *and* (8.2)*, respectively, for* $(d, N) \in \mathbb{N} \times \mathbb{N}$*. Then*

$$\mathfrak{NF} = \bigsqcup_{(d,N) \in \mathbb{N}_\infty \times \mathbb{N}} \mathfrak{K}_d^N, \tag{8.5}$$

where $\mathfrak{K}_d^N$ *are in the sense of* (8.3)*.*

Proof. By the very definitions (8.1) and (8.2),

$$\mathfrak{NF} = \bigsqcup_{(d,N) \in \mathbb{N} \times \mathbb{N}} \mathfrak{NF}_d^N.$$

By the characterization (6.2), the set $\mathfrak{NF}_d^N$ is equipotent to $\mathfrak{K}_d^N$ of (8.3), for all $d, N \in \mathbb{N}$. Thus, the classification (8.5) holds. □

8.1. GLUING ON $\mathfrak{NF}$

Let G_1 and G_2 be arbitrary two finite connected graphs such that

$$V(G_1) \cap V(G_2) = \varnothing = E(G_1) \cap E(G_2).$$

Fix vertices $v_j \in V(G_j)$ for $j = 1, 2$. Now, let us identify these two vertices as a single ideal vertex, denoted by $v_{1,2}$. Such an identified vertex $v_{1,2}$ is called the *glued vertex of* v_1 *and* v_2.

Understand now the graphs G_1 and G_2 as the corresponding graphs G_1' and G_2' with

$$V(G_j') = (V(G_1) \setminus \{v_j\}) \cup \{v_{1,2}\} \tag{8.6}$$

and

$$E(G_j') = E(G_j),$$

with identification: if $e \in E(G_j)$ satisfies either $e = v_j e$ or $e = ev_j$, then identify it as $e = v_{1,2}e$, respectively, $e = ev_{1,2}$ in $E(G_j')$, for all $j = 1, 2$.

Definition 8.2. Let G_j' be the graphs in the sense of (8.6) induced by fixed graphs G_j, for $j = 1, 2$, where $v_{1,2}$ is the glued vertex of v_1 and v_2. Then the graph union $G_{1,2} = G_1' \cup G_2'$ is called the glued graph of G_1 and G_2 with its glued vertex $v_{1,2}$.

For example, let

$$G_1 = \quad \bullet \to \bullet_{v_1}$$

and

$$G_2 = \quad \begin{array}{ccc} {}_{v_2}\bullet & \to & \bullet \\ & \searrow & \\ & & \bullet, \end{array}$$

then

$$G_{1,2} = \quad \begin{array}{cccc} \bullet \to & \underset{v_{1,2}}{\bullet} & \to & \bullet \\ & & \searrow & \\ & & & \bullet. \end{array}$$

The above construction of glued graphs is called the *gluing (process) on graphs.*

We do gluings on the family $\mathfrak{NF}$ of (8.1), satisfying (8.5). Take now $K_1 \neq K_2$ in $\mathfrak{NF}$. More precisely, let

$$K_j \in \mathfrak{NF}_{d_j}^{N_j} \text{ for } j = 1, 2.$$

(Note here that (d_j, N_j) are not necessarily distinct, if $N_j \neq 1$ in $\mathbb{N}$. Remark that

$$\mathfrak{NF}_d^1 = \mathfrak{K}_d^1 = \{K_d^1\}$$

for all $d \in \mathbb{N}$.)

Fix vertices $v_j \in K_j$, for $j = 1, 2$, and glue these vertices to the glued vertex $v_{1,2}$, and with respect to this glued vertex $v_{1,2}$, construct the glued graph $K_{1,2}$ of K_1 and K_2.

Theorem 8.3. *Let $K_{d_j}^{N_j}$ be the N_j-regular circulant graph with d_j-vertices in $\mathfrak{NF}_{d_j}^{N_j}$ in $\mathfrak{NF}$, for $j = 1, 2$, and a fixed $N \in \mathbb{N}$. Let $K_{1,2}$ be the glued graph with its glued vertex $v_{1,2}$. Then*

$$L_{K_{1,2}}(v_{1,2}, s) = \left(\frac{(2N_1+1)^s + N_1}{(2N_1+1)^s - N_1}\right)\left(\frac{(2N_2+1)^s + N_2}{(2N_2+1)^s - N_2}\right) \text{ in } \mathcal{L}_{K_{1,2}}, \tag{8.7}$$

where $L_G(\bullet, s)$ means the graph zeta functions at vertices of G in $\mathcal{L}_G$, for any locally finite connected graphs G.

Proof. If K_j' are in the sense of (8.6) induced from $K_{d_j}^{N_j} \in \mathfrak{NF}_{d_j}^{N}$, for $j = 1, 2$, the corresponding graph zeta functions $L_{K_j'}(\bullet, s)$ satisfy

$$L_{K_j'}(x, s) = L_{K_{d_j}^{N_j}}(y, s), \text{ as } \mathbb{C}\text{-valued functions}, \tag{8.8}$$

for all $x \in V(K_j')$ and $y \in V(K_j)$, by the non-factorizability (6.2) of $K_{d_j}^{N_j}$, $j = 1, 2$.

Now, by (5.13), one has that

$$L_{K_{1,2}}(v_{1,2}, s) = \left(L_{K_1'}(v_{1,2}, s)\right)\left(L_{K_2'}(v_{1,2}, s)\right), \tag{8.9}$$

since the $v_{1,2}$-ray factorization $\mathfrak{G}_{v_{1,2}} = \{K_1', K_2'\}$, by the non-factorizability of $K_{d_1}^{N_1}$ and $K_{d_2}^{N_2}$.

Therefore, by (8.8) and (8.9), we obtain that

$$L_{K_{1,2}}(v_{1,2}, s) = (L_{K_1}(v_1, s))(L_{K_2}(v_2, s)) \text{ in } \mathcal{L}_{K_{1,2}}.$$

By applying the computation (7.3), one can get the formula (8.7). □

By (8.7), we obtain the following corollary.

Corollary 8.4. *Let $K_{d_j}^{N_j} \in \mathfrak{NF}_{d_j}^{N_j}$ be the N_j-regular circulant graphs with d_j-vertices, for $j = 1, \ldots, n$, for $n \in \mathbb{N}\backslash\{1\}$. Fix vertices $v_j \in V(K_{d_j}^{N_j})$, for $j = 1, \ldots, n$, and construct the iterated glued vertex $v_{1,\ldots,n}$, i.e. identify all vertices $v_1, \ldots, v_n$. Let $K_{1,\ldots,n}$ be the iterated glued graph with its glued vertex $v_{1,\ldots,n}$. Then*

$$L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) = \prod_{j=1}^{n}\left(\frac{(2N_j+1)^s + N_j}{(2N_j+1)^s - N_j}\right) \text{ in } \mathcal{L}_{K_{1,\ldots,n}}. \tag{8.10}$$

Proof. The formula (8.10) is obtained by induction on (8.7). □

Also, by (8.7) and (8.10), the following rough estimation among $\mathbb{C}$-valued functions is obtained.

Corollary 8.5. *Let $K_j \in \mathfrak{NF}_{d_j}^{N_j}$ in $\mathfrak{NF}$, for $j = 1, 2$, and let $K_{1,2}$ be the glued graph of K_1 and K_2 with its glued vertex $v_{1,2}$. Let $L_{K_{1,2}}(v_{1,2}, s)$ be the graph zeta function at $v_{1,2}$. Then*

$$\left|L_{K_{1,2}}(v_{1,2}, s)\right| \leqslant \left|\left(\frac{(2N_1+1)^s + N_1}{(2N_1+1)^s - N_1}\right)\right|\left|\left(\frac{(2N_2+1)^s + N_2}{(2N_2+1)^s - N_2}\right)\right|. \tag{8.11}$$

Proof. Similar to the proof of (8.7), one has that

$$L_{K_{1,2}}(v_{1,2},s)=\left(L_{K_1'}(v_1,s)\right)\left(L_{K_2'}(v_2,s)\right)=\left(L_{K_1}(v_1,s)\right)\left(L_{K_2}(v_2,\ s)\right). \tag{8.12}$$

So,

$$\left|L_{K_{1,2}}(v_{1,2},s)\right|=\left|\left(L_{K_1}(v_1,s)\right)\left(L_{K_2}(v_2,s)\right)\right|$$

by (8.12)

$$\leqslant \left|\left(L_{K_1}(v_1,\ s)\right)\right|\left|\left(L_{K_2}(v_2,s)\right)\right| \tag{8.13}$$
$$\leqslant \left|\left(L_{K_{d_1}^{N_1}}(x,s)\right)\right|\left|\left(L_{K_{d_2}^{N_2}}(v,s)\right)\right|$$

for any $x\in V(K_{d_1}^{N_1})$ and $v\in V(K_{d_2}^{N_2})$, by the non-factorizability of $K_{d_j}^{N_j}$, for all $j=1,2$.

By (7.3), one has

$$L_{K_d^N}(x,v)=\frac{(2N+1)^s+N}{(2N+1)^s-N}\text{ for all }x\in V(K_d^N),$$

for all $d,N\in\mathbb{N}$. So, the estimation (8.13) goes to

$$\left|L_{K_{1,2}}(v_{1,2},s)\right|\leqslant\left|\left(\frac{(2N_1+1)^s+N_1}{(2N_1+1)^s-N_1}\right)\right|\left|\left(\frac{(2N_2+1)^s+N_2}{(2N_2+1)^s-N_2}\right)\right|.$$

Therefore, we obtain the estimation (8.11). □

8.2. GLUED GRAPHS AND $l_{K_d^{N:1}}(\bullet,s)$

In this section, we apply our gluing process of Section 8.1, to other functions $l_{K_d^{N:1}}(\bullet,s)$ in the sense of (7.3) in $\mathcal{L}_{K_d^N}$. Recall now that the simplicial subgraphs $K_d^{N:1}$ of K_d^N introduced in Section 7 is also contained in $\mathfrak{NF}$, because $K_d^{N:1}$, themselves, are independent non-factorizable graphs. Indeed, for any $K_d^N\in\mathfrak{NF}_d^N$, one obtains $K_d^{N:1}\in\mathfrak{NF}_d^1$ in $\mathfrak{NF}$.

Also, recall that

$$l_{K_d^{N:1}}(x,s)=\sum_{w\in\left(\mathbb{K}_d^{N:1}\right)_x}\frac{1}{\varpi(w)^s}=\frac{(2N+1)^s+1}{(2N+1)^s-1}, \tag{8.14}$$

in $\mathcal{L}_{K_d^N}$, by (7.3), for all $x\in V(K_d^{N:1})=V(K_d^N)$ and $(d,N)\in\mathbb{N}\times\mathbb{N}$.

Now, let us take two graphs

$$K_j=K_{d_j}^{N_j:1}\in\mathfrak{NF}_{d_j}^{N_j}, \tag{8.15}$$

and let $l_{K_{d_j}^{N_j:1}}(v_j,s)$ be the corresponding elements in $\mathcal{L}_{K_{d_j}^{N_j}}$ in the sense of (7.11), for $j=1,\ 2$.

Suppose $K_{1,2}$ is the glued graph of $K_{d_1}^{N_1:1}$ and $K_{d_2}^{N_2:1}$. Then one has

$$L_{K_{1,2}}(v_{1,2}, s) = \left(l_{K_{d_1}^{N_1:1}}(v,\ s)\right)\left(l_{K_{d_2}^{N_2:1}}(x, s)\right)$$

by (8.12)

$$\begin{aligned}
&= \left(1 + \frac{2}{(2N_1+1)^s}\left(1 - \frac{1}{(2N_1+1)^s}\right)^{-1}\right)\left(1 + \frac{2}{(2N_2+1)^s}\left(1 - \frac{1}{(2N_2+1)^s}\right)^{-1}\right) \\
&= 1 + \frac{2}{(2N_1+1)^s}\left(1 - \frac{1}{(2N_1+1)^s}\right)^{-1} + \frac{2}{(2N_2+1)^s}\left(1 - \frac{1}{(2N_2+1)^s}\right)^{-1} \\
&\quad + \left(\frac{2}{(2N_1+1)^s}\right)\left(\frac{2}{(2N_2+1)^s}\right)\left(1 - \frac{1}{(2N_1+1)^s}\right)^{-1}\left(1 - \frac{1}{(2N_2+1)^s}\right)^{-1} \\
&= l_{K_{d_1}^{N_1:1}}(v, s) + \left(l_{K_{d_2}^{N_2:1}}(x, s) - 1\right) \\
&\quad + \left(\frac{2}{(2N_1+1)^s}\right)\left(\frac{2}{(2N_2+1)^s}\right)\left(1 - \frac{1}{(2N_1+1)^s}\right)^{-1}\left(1 - \frac{1}{(2N_2+1)^s}\right)^{-1}.
\end{aligned} \tag{8.16}$$

Define a new $\mathbb{C}$-valued function $\varphi_{K_{1,2}}(s)$ by

$$\varphi_{K_{1,2}}(s) \overset{def}{=} L_{K_{1,2}}(v_{1,2}, s) - l_{K_{d_1}^{N_1:1}}(v_1, s) - \left(l_{K_{d_2}^{N_2:1}}(v_2, s) - 1\right). \tag{8.17}$$

Now, let $K_{d_j}^{N_j:1} \in \mathfrak{N}\mathfrak{F}_{d_j}^{N_j}$, and choose vertices $v_j \in V(K_{d_j}^{N_j:1})$, for $j = 1, 2, 3$. By doing iterated gluing processes, obtain the glued graph $K_{1,2,3}$ of $K_{d_1}^{N_1:1}, K_{d_2}^{N_2:1}$ and $K_{d_3}^{N_3:1}$ with its glued vertex $v_{1,2,3}$ of v_1, v_2 and v_3. Then, by (5.13) and (8.12), one obtains

$$L_{K_{1,2,3}}(v_{1,2,3}, s) = \prod_{j=1}^{3}\left(l_{K_{d_j}^{N_j:1}}(v_j, s)\right), \tag{8.18}$$

since the $v_{1,2,3}$-ray factorization $\mathfrak{G}_{v_{1,2,3}} = \{K_1', K_2', K_3'\}$, where each element K_j' is graph-isomorphic to $K_{d_j}^{N_j:1}$, for all $j = 1, 2, 3$, and the corresponding graph groupoids $\mathbb{K}_1', \mathbb{K}_2'$ and $\mathbb{K}_3'$ share their only common element $v_{1,2,3}$, the glued vertex.

Then, as in (8.16), the product which is the right-hand side of (8.18) contains the term

$$\left(\prod_{j=1}^{3}\frac{2}{(2N_j+1)^s}\right)\left(\prod_{j=1}^{3}\left(1 - \frac{1}{(2N_j+1)^s}\right)^{-1}\right),$$

i.e.

$$\prod_{j=1}^{3}\left(l_{K_{d_j}^{N_j:1}}(v_j, s)\right) = \left(\prod_{j=1}^{3}\frac{2}{(2N_j+1)^{\ s}}\right)\left(\prod_{j=1}^{3}\left(1 - \frac{1}{(2N_j+1)^s}\right)^{-1}\right) + [\text{Rest terms}].$$

Define a $\mathbb{C}$-valued function $\varphi_{K_{1,2,3}}(s)$ by

$$\varphi_{K_{1,2,3}}(s) = L_{K_{1,2,3}}(v_{1,2,3}, s) - [\text{Rest terms}], \tag{8.19}$$

as in (8.17), where [Rest terms] of (8.19) is from the right-hand side of (8.18), as in the above paragraph.

Inductively, one can have the glued graph $K_{1,\ldots,n}$ of $K_{d_j}^{N_j:1}$ with its glued vertex $v_{1,\ldots,n}$, for $j = 1, \ldots, n$, for some $n \in \mathbb{N}\backslash\{1\}$, and the corresponding $\mathbb{C}$-valued function $\varphi_{K_{1,\ldots,n}}(s)$,

$$\varphi_{K_{1,\ldots,n}}(s) = \left(\prod_{j=1}^{n} \frac{2}{(2N_j+1)^s}\right)\left(\prod_{j=1}^{n}\left(1 - \frac{1}{(2N_j+1)^s}\right)^{-1}\right). \tag{8.20}$$

Define now new functions $\zeta_{K_{1,\ldots,n}}(s)$ by

$$\zeta_{K_{1,\ldots,n}}(s) = \left(\prod_{j=1}^{n} \frac{2}{(2N_j+1)^s}\right)^{-1} \varphi_{K_{1,\ldots,n}}(s), \tag{8.21}$$

where $\varphi_{K_{1,\ldots,n}}(s)$ are in the sense of (8.20). So, by the very definition (8.21), the functions $E_{K_{1,\ldots,n}}(s)$ are simply identical to

$$\zeta_{K_{1,\ldots,n}}(s) = \prod_{j=1}^{n}\left(1 - \frac{1}{(2N_j+1)^s}\right)^{-1}. \tag{8.22}$$

Definition 8.6. The functions $\zeta_{K_{1,\ldots,n}}(s)$ of (8.22) are called the Riemann-zeta parts of the glued graphs $K_{1,\ldots,n}$ of $K_{d_j}^{N_j:1} \in \mathfrak{N}\mathfrak{F}_{d_j}^{N_j}$ for $j = 1, \ldots, n$ and $n \in \mathbb{N}\backslash\{1\}$.

Let N_j be "distinct" numbers in $\mathbb{N}$ such that $2N_j + 1$ are "odd primes" for $j = 1, \ldots, n$ and for some $n \in \mathbb{N}$, and let

$$K_j = K_{d_j}^{N_j:1} \in \mathfrak{N}\mathfrak{F}_{d_j}^{N_j}, \text{ for } j = 1, \ldots, n, \tag{8.23}$$

as above.

From below, let us take N_j as natural numbers, making $2N_j + 1$ be prime in $\mathbb{N}$. In such a case, we denote these primes $2N_j+1$ by p_j. If N_j has odd prime $p_j = 2N_j+1$ in $\mathbb{N}$, we will say N_j *has odd prime property.*

For instance, if $N_j = 2$, then $2N_j + 1$ becomes a prime 5 in $\mathbb{N}$, but if $N_j = 4$, then $2N_j + 1$ becomes a composite number 9. So, the quantity 2 has odd prime property, but the quantity 5 does not have odd prime property.

The Riemann-zeta part $\zeta_{K_{1,\ldots,n}}$ of the graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ at the glued vertex $v_{1,\ldots,n}$ for the glued graph $K_{1,\ldots,n}$ of the graphs $K_1, \ldots, K_n$ of (8.23) satisfies that

$$\zeta_{K_{1,\ldots,n}} = \prod_{j=1}^{n}\left(1 - \frac{1}{p_j^s}\right)^{-1}$$

for odd primes $p_1, \ldots, p_n$, with $p_j = 2N_j + 1$.

Lemma 8.7. *Let $K_j = K_{d_j}^{N_j:1}$ be the elements (8.23) of $\mathfrak{NF}_{d_j}^{N_j}$ for $j = 1, \ldots, n$, and $n \in \mathbb{N}$, and let $K_{1,\ldots,n}$ be the glued graph of $K_1, \ldots, K_n$ with its glued vertex $v_{1,\ldots,n}$. Let $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ be the corresponding graph zeta function at $v_{1,\ldots,n}$ in $\mathcal{L}_{K_{1,\ldots,n}}$, and $\zeta_{K_{1,\ldots,n}}(s)$, the Riemann-zeta part of it as a $\mathbb{C}$-valued function in the sense of* (8.22)*. If N_j have odd prime property in $\mathbb{N}$, for all $j = 1, \ldots, n$, then*

$$\zeta_{K_{1,\ldots,n}}(s) = (\zeta(s)) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right) \right). \tag{8.24}$$

where $\zeta(s)$ is the Riemann zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

Proof. The proof of (8.24) is clear by the very construction (8.22) of Riemann-zeta parts where N_j have odd prime property, for all $j = 1, \ldots, n$, i.e. under hypothesis, one has that

$$\begin{aligned}
\zeta_{K_{1,\ldots,n}}(s) &= \prod_{j=1}^{n} \left(1 - \frac{1}{p_j^s}\right)^{-1} \\
&= \left(\prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \right) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right)^{-1} \right)^{-1} \\
&= (\zeta(s)) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right)^{-1} \right)^{-1} \\
&= (\zeta(s)) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right) \right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, and $\mathcal{P}$ is the set of all primes in $\mathbb{N}$. □

So, if we choose suitably big number n and n-many quantities N_j, having odd prime property, for $j = 1, \ldots, n$, then the Riemann-zeta part $\zeta_{K_{1,\ldots,n}}(s)$ of the glued-graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ at the glued vertex $v_{1,\ldots,n}$ approximates to the Riemann zeta function $\zeta(s)$. It demonstrates the connections between our (non-factorizable-)graph zeta functions and the Riemann zeta function $\zeta(s)$.

By (8.24), one can get the following corollary.

Theorem 8.8. *Under the same hypothesis with the above lemma, we obtain that*

$$\begin{aligned}
&\left(\prod_{j=1}^{n} \frac{2}{p_j^s} \right)^{-1} \left(L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) - [\textit{Rest terms}] \right) \\
&= (\zeta(s)) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right) \right),
\end{aligned} \tag{8.25}$$

where $[$Rest terms$]$ is in the sense of (8.19).

Proof. The proof of (8.25) is identical to that of (8.24), because the left-hand side of (8.25) is nothing but the Riemann-zeta part $\zeta_{K_{1,\ldots,n}}(s)$. $\square$

By expressing (8.24) as (8.25), we can directly see the relation between the glued-graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ at the glued vertex $v_{1,\ldots,n}$ and the Riemann zeta function $\zeta(s)$, whenever $K_j = K_{d_j}^{N_j:1}$, with $2N_j + 1 = p_j$ are odd primes, for $j = 1, \ldots, n$.

Independently, recall now an arithmetic function $\phi \in \mathcal{A}$, defined by

$$\phi(n) = |\{k \in \mathbb{N} : k \leqslant n, \ \ \gcd(k, \ n) = 1\}| \tag{8.26}$$

for all $n \in \mathbb{N}$, where "gcd" means the *greatest common divisor*, i.e. the image $\phi(n)$ is counting the number of all relative primes with a fixed number n, which are less than or equal to n, for all $n \in \mathbb{N}$. This arithmetic function ϕ of (8.26) is the well-known *Euler totient function.*

For any $n \in \mathbb{N}$, one has

$$\phi(n) = n \left(\prod_{p \in \mathcal{P}, \ p|n} \left(1 - \frac{1}{p}\right) \right). \tag{8.27}$$

So, we obtain the following number-theoretic characterization of a specific functional value $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, 1)$ of the graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ of the glued graph $K_{1,\ldots,n}$ of $K_1, \ldots, K_n$ in the sense of (8.23).

Theorem 8.9. *Let $K_j = K_{d_j}^{N_j:1}$ be non-factorizable subgraphs of $K_{d_j}^{N_j}$ in $\mathfrak{N}\mathfrak{F}_{d_j}^{N_j}$ in the sense of* (8.23) *for all $j = 1, \ldots, n$, and $n \in \mathbb{N}\backslash\{1\}$. Assume that N_j have odd prime property, for $j = 1, \ldots, n$. Let $K_{1,\ldots,n}$ be the iterated glued graph of $K_1, \ldots, K_n$ with its glued vertex $v_{1,\ldots,n}$, and $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$, the corresponding graph zeta function at $v_{1,\ldots,n}$ in $\mathcal{L}_{K_{1,\ldots,n}}$. If $\zeta_{K_{1,\ldots,n}}(s)$ is the Riemann-zeta part of $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$, then there exists $t_0 \in \mathbb{N}$ such that*

$$t_0 = \prod_{j=1}^{n} p_j = \prod_{j=1}^{n} (2N_j + 1), \tag{8.28}$$

and

$$\zeta_{K_{1,\ldots,n}}(1) = t_0 \left(\phi(t_0)\right)^{-1}.$$

Proof. By (8.22), one has

$$\zeta_{K_{1,\ldots,n}}(1) = \prod_{j=1}^{n} \left(1 - \frac{1}{p_j^1}\right)^{-1} = \prod_{j=1}^{n} \left(1 - \frac{1}{p_j}\right)^{-1}$$

where $p_j = 2N_j + 1$ for all $j = 1, \ldots, n$

$$= \frac{1}{\prod_{j=1}^{n} \left(1 - \frac{1}{p_j}\right)} = \frac{t_0}{t_0 \left(\prod_{j=1}^{n} \left(1 - \frac{1}{p_j}\right)\right)}$$

where $t_0 = \prod_{j=1}^{n} p_j$ in $\mathbb{N}$

$$= \frac{t_0}{\phi(t_0)} = t_0 \left(\phi(t_0)\right)^{-1},$$

i.e. there exists $t_0 = \prod_{j=1}^{n} (2N_j + 1)$ such that

$$\zeta_{K_{1,\ldots,n}}(1) = t_0 \left(\phi(t_0)^{-1}\right).$$

□

The above characterization (8.28) shows that the ratio between a natural number t_0, and the corresponding Euler totient functional value $\phi(t_0)$, can be measured by the functional value $\zeta_{K_{1,\ldots,n}}(1)$ of a Riemann-zeta part $\zeta_{K_{1,\ldots,n}}(s)$ of the glued-graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ at the glued vertex $v_{1,\ldots,n}$ of certain non-factorizable graphs $K_1, \ldots, K_n$.

8.3. MORE ABOUT (8.25)

In this section, we concentrate on refining the relation (8.25), which is equivalent to (8.24). In (8.25) and (8.28), we showed a relation between glued-graph zeta functions induced by non-factorizable graphs, and the Riemann zeta function. Even though the formulas (8.24) and (8.25) are interesting, it seems not clear enough to understand in details, because of [Rest terms] of (8.19), in (8.25). So, here, we analyze [Rest terms] of (8.19) as a summand in (8.25).

First, recall that how we obtain the [Rest terms] in (8.19). Let

$$K_j = K_{d_j}^{N_j:1} \in \mathfrak{NF}_{d_j}^{N_j}$$

be non-factorizable graphs, for $j = 1, \ldots, n$, for some fixed $n \in \mathbb{N}\setminus\{1\}$, and let $K_{1,\ldots,n}$ be the corresponding iterated glued graph of K_j's with its glued vertex $v_{1,\ldots,n}$, having its graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ at the vertex $v_{1,\ldots,n}$. Assume further that N_j have odd prime property with

$$p_j = 2N_j + 1 \in \mathcal{P}, \text{ in } \mathbb{N}$$

for all $j = 1, \ldots, n$, where $\mathcal{P}$ is the set of all primes in $\mathbb{N}$.

By the factorizability (5.13), one has

$$L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) = \prod_{j=1}^{n} \left(l_{K_j}(v_j, s)\right) \tag{8.29}$$

with

$$l_{K_j}(v_j, s) = 1 + \frac{2}{p_j^s}\left(1 - \frac{1}{p_j^s}\right)^{-1} = \frac{p_j^s + 1}{p_j^s - 1}$$

for all $j = 1, \ldots, n$, by (7.3), where $l_{K_j}(v_j, s)$ are in the sense of (5.11). Let

$$\alpha_j \overset{denote}{=} \alpha_j(s) \overset{def}{=} \frac{2}{p_j^s}\left(1 - \frac{1}{p_j^s}\right)^{-1}, \quad j = 1, \ldots, n, \tag{8.30}$$

i.e.

$$l_{K_j}(v_j, s) = 1 + \alpha_j, \quad j = 1, \ldots, n,$$

by (8.29).

Recall now that, for arbitrary $x_1, \ldots, x_n \in \mathbb{C}$ and $n \in \mathbb{N}$, the polynomial

$$F(z) = \prod_{j=1}^{n} (z + x_j) \in \mathbb{C}[z]$$

satisfies that

$$F(z) = 1 + \sum_{k=1}^{n} \sigma_k z^{n-k}, \tag{8.31}$$

where the coefficients

$$\sigma_k = \sigma_k(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\prod_{l=1}^{k} x_{i_l} \right).$$

By defining

$$\sigma_0 = \sigma_0(x_1, \ldots, x_n) = 1,$$

the expression (8.31) can be re-written by

$$F(z) = \sum_{k=0}^{n} \sigma_k z^{n-k}. \tag{8.32}$$

Generally, if we understand $x_1, \ldots, x_n$ as arbitrary algebraically independent indeterminants, then the coefficients σ_k of (8.31) and (8.32) are said to be the *elementary symmetric polynomials*. Recall that a n-variable function $f(x_1, \ldots, x_n)$ is *symmetric* if

$$f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) = f(x_1, \ldots, x_n)$$

for all $\sigma \in S_n$, where S_n is the *symmetric group over* $\{1, \ldots, n\}$.

It is trivial that, for $x_1, \ldots, x_n \in \mathbb{C}$, the coefficients $\sigma_k = \sigma_k(x_1, \ldots, x_n)$ of (8.31) are symmetric for all $k = 1, \ldots, n$.

Now, apply such computations and notations to our cases.

By (8.29),

$$l_{K_j}(v_j, s) = 1 + \alpha_j,$$

where $\alpha_j = \alpha_j(s)$ are in the sense of (8.30), for all $j = 1, \ldots, n$. So, one has that

$$\prod_{j=1}^{n} l_{K_j}(v_j, s) = \prod_{j=1}^{n} (1 + \alpha_j) = \sum_{k=0}^{n} \sigma_k(\alpha_1, \ldots, \alpha_n) 1^{n-k}$$

by (8.31) or (8.32)

$$= \sum_{k=0}^{n} \sigma_k(\alpha_1, \ldots, \alpha_n),$$

where $\sigma_k(\alpha_1, \ldots, \alpha_n)$ are in the sense of (8.31) for all $k = 1, \ldots, n$.

Theorem 8.10. *Let K_j be the simplicial subgraphs $K_{d_j}^{N_j:1}$ of $K_{d_j}^{N_j}$ in $\mathfrak{N}\mathfrak{F}_{d_j}^{N_j}$ for $j = 1, \ldots, n$, and let $K_{1,\ldots,n}$ be the iterated glued graph of them with its glued vertex $v_{1,\ldots,n}$. Then*

$$L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) = \sum_{k=0}^{n} \sigma_k(\alpha_1, \ldots, \alpha_n), \tag{8.33}$$

where σ_k are in the sense of (8.32) *and α_k are in the sense of* (8.30) *for all $k = 1, \ldots, n$. Moreover, one obtains that*

$$\varphi_{K_{1,\ldots,n}}(s) = \sigma_n(\alpha_1, \ldots, \alpha_n) = \frac{2^n}{\left(\prod_{j=1}^{n} p_j\right)^s} \left(\prod_{j=1}^{n} \left(1 - \frac{1}{p_j^s}\right)^{-1} \right) \tag{8.34}$$

and

$$\zeta_{K_{1,\ldots,n}}(s) = 2^{-n} \left(\prod_{j=1}^{n} p_j \right)^{-s} \sigma_n(\alpha_1, \ \ldots, \ \alpha_n),$$

where $\varphi_{K_{1,\ldots,n}}(s)$ is in the sense of (8.20) *and $\zeta_{K_{1,\ldots,n}}(s)$ is the Riemann-zeta part of $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, \ s)$ in the sense of* (8.21).

Proof. The proof of (8.33) is done by (8.31). Indeed, by the construction (8.20) of

$$\varphi_{K_{1,\ldots,n}}(s) \quad \text{from} \quad L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s),$$

one has

$$\begin{aligned}
\varphi_{K_{1,\ldots,n}}(s) &= \prod_{k=1}^{n} \alpha_k = \sigma_n(\alpha_1, \ldots, \alpha_n) = \prod_{j=1}^{n} \frac{2}{p_j^s} \left(1 - \frac{1}{p_j^s}\right)^{-1} \\
&= \frac{2^n}{\left(\prod_{j=1}^{n} p_j\right)^s} \left(\prod_{j=1}^{n} \left(1 - \frac{1}{p_j^s}\right)^{-1} \right).
\end{aligned}$$

Thus, the first formula of (8.34) holds true. Since the formula (8.34) holds, the Riemann-zeta part $\zeta_{K_{1,\ldots,n}}(s)$ satisfies

$$\begin{aligned}
\zeta_{K_{1,\ldots,n}}(s) &= \left(\prod_{j=1}^{n} \left(1 - \frac{1}{p_j^s}\right)^{-1} \right) = \left(\frac{2^n}{\left(\prod_{j=1}^{n} p_j\right)^s} \right)^{-1} \left(\varphi_{K_{1,\ldots,n}}(s)\right) \\
&= 2^{-n} \left(\prod_{j=1}^{n} p_j \right)^{-s} \sigma_n(\alpha_1, \ldots, \alpha_n),
\end{aligned}$$

i.e. one obtains that

$$\zeta_{K_{1,\ldots,n}}(s) = \frac{\left(\prod_{j=1}^{n} p_j\right)^s}{2^n} \sigma_n \left(1 - \frac{1}{(2N_1+1)^s}, 1 - \frac{1}{(2N_2+1)^s}, \ldots, 1 - \frac{1}{(2N_n+1)^s}\right). \tag{8.35}$$

□

By the above theorem, in particular, by the formulas (8.34) and (8.35), we obtain the following conclusion.

Corollary 8.11. *The* [*Rest terms*] *of* (8.19) *is identified with*

$$[\textit{Rest terms}] = \sum_{k=0}^{n-1} \sigma_k(\alpha_1, \ldots, \alpha_n), \tag{8.36}$$

where $\sigma_k(\alpha_1, \ldots, \alpha_n)$ *are in the sense of* (8.34), $k = 1, \ldots, n-1$.

This means that by (8.34), we can not only clarify [Rest terms] of (8.19) by (8.36), but also re-characterize the Riemann-zeta parts $\zeta_{K_{1,\ldots,n}}$ of $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ by (8.35).

Now, under the same hypothesis, assume $k \neq n$, and let

$$\sigma_k = \sigma_k(\alpha_1, \ldots, \alpha_n) \text{ for } k \neq n,$$

i.e.

$$\sigma_k = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\prod_{l=1}^{k} \alpha_{i_l} \right) \text{ for such } k. \tag{8.37}$$

Proposition 8.12. *Let us denote the summands* $\prod_{l=1}^{k} \alpha_{i_l}$ *of* (8.37) *simply by* $\sigma^{i_1,\ldots,i_k}$, *for* $k \lneqq n$. *Then*

$$\sigma^{i_1,\ldots,i_k} = \sigma_k\left(\alpha_{i_1}, \alpha_{i_2}, \ \ldots, \ \alpha_{i_k}\right) = \left(\frac{2^n}{\left(\prod_{l=1}^{k} \ p_{i_l}\right)^s} \right) \left(\zeta_{K_{i_1,i_2,\ldots,i_k}}(s) \right), \tag{8.38}$$

and hence, σ_k *of* (8.37) *satisfies that*

$$\sigma_k = \sigma_k(\alpha_1, \ldots, \alpha_n) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\frac{2^n}{\left(\prod_{l=1}^{k} \ p_{i_l}\right)^s} \right) \left(\zeta_{K_{i_1,i_2,\ldots,i_k}}(s) \right), \tag{8.39}$$

where $\zeta_{K_{i_1,\ldots,i_k}}(s)$ *are the Riemann-zeta parts of the glued-graph zeta function* $L_{K_{i_1,\ldots,i_k}}(v_{i_1,\ldots,i_k}, s)$ *with its glued vertex* $v_{i_1,\ldots,i_k}$, *where* $K_{i_1,\ldots,i_k}$ *is the glued graph of* $K_{i_l} = K_{d_{i_l}}^{N_{i_l}:1} \in \mathfrak{N}\mathfrak{F}_{d_{i_l}}^{N_{i_l}}$, *for* $l = 1, \ldots, k$.

Proof. For $k \in \{1, \ldots, n-1\}$,

$$\begin{aligned}
\sigma^{i_1,\ldots,i_k} &= \sigma_k\left(\alpha_{i_1}, \ \alpha_{i_2}, \ \ldots, \ \alpha_{i_k}\right) \\
&= \frac{2^n}{\left(\prod_{l=1}^{k} \ p_{i_l}\right)^s} \left(\prod_{l=1}^{k} \left(1 - \frac{1}{p_{i_l}^s}\right)^{-1} \right) = \left(\frac{2^n}{\left(\prod\limits_{l=1}^{k} \ p_{i_l}\right)^s} \right) \left(\zeta_{K_{i_1,i_2,\ldots,i_k}}(s) \right).
\end{aligned}$$

So, the formula (8.38) holds. Thus, by (8.37) and (8.38), one obtains that

$$\begin{aligned}\sigma_k = \sigma_k(\alpha_1, \ldots, \alpha_n) &= \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\prod_{l=1}^{k} \alpha_{i_l} \right) \\ &= \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \sigma_k \left(\alpha_{i_1}, \ \alpha_{i_2}, \ \ldots, \ \alpha_{i_k} \right) \\ &= \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\left(\frac{2^n}{\left(\prod_{l=1}^{k} \ p_{i_l} \right)^s} \right) \left(\zeta_{K_{i_1, i_2, \ldots, i_k}}(s) \right) \right)\end{aligned}$$

for all $k = 1, \ldots, n-1$. Therefore, the formula (8.39) holds. □

So, by (8.38) and (8.39), one obtains the following result of this section.

Theorem 8.13. *Let $K_j = K_{d_j}^{N_j:1} \in \mathfrak{N}\mathfrak{F}_{d_j}^{N_j:1}$ be non-factorizable graphs for $j = 1, \ldots, n$ and $n \in \mathbb{N} \setminus \{1\}$, and let $K_{1,\ldots,n}$ be the iterated glued graph of $K_1, \ldots, K_n$ with its glued vertex $v_{1,\ldots,n}$. Assume that N_j have odd prime property, $j = 1, \ldots, N$. Then the graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ satisfies that*

$$L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) = \sum_{k=0}^{n} \left(\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \left(\left(\frac{2^n}{\left(\prod_{l=1}^{k} \ p_{i_l} \right)^s} \right) \left(\zeta_{K_{i_1, i_2, \ldots, i_k}}(s) \right) \right) \right), \tag{8.40}$$

where $\zeta_{K_{i_1,\ldots,i_k}}(s)$ are the Riemann-zeta parts of the glued graph zeta functions $L_{K_{i_1,\ldots,i_k}}(v_{i_1,\ldots,i_k}, s)$, for all k-tuples $(i_1, \ldots, i_k)$, such that $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, and where $p_j = 2N_j + 1$, for all $j = 1, \ldots, n$.

Proof. The proof of (8.40) is done by (8.33), (8.38) and (8.39). □

8.4. THE RIEMANN-ZETA PARTS $\zeta_{K_{1,\ldots,n}}(s)$

In this section, we fix $N_1, \ldots, N_n \in \mathbb{N}$, having odd prime property with

$$p_j = 2N_j + 1 \in \mathcal{P}, \quad j = 1, \ldots, n,$$

for $n \in \mathbb{N} \setminus \{1\}$. And let

$$K_j = K_{d_j}^{N_j:1} \in \mathfrak{N}\mathfrak{F}_{d_j}^{N_j} = \mathfrak{K}_{d_j}^{N_j},$$

be simplicial subgraphs of $K_{d_j}^{N_j}$ for all $j = 1, \ldots, n$, generating the iterated glued graph $K_{1,\ldots,n}$ with its glued vertex $v_{1,\ldots,n}$. Then one can have the corresponding graph zeta function $L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s)$ equipped with its Riemann-zeta part $\zeta_{K_{1,\ldots,n}}(s)$ in the sense of (8.22). By (8.34) and (8.35),

$$\zeta_{K_{1,\ldots,n}}(s) = \frac{\left(\prod_{j=1}^{n} p_j \right)^s}{2^n} \sigma_n \left(1 - \frac{1}{p_1^s}, \ 1 - \frac{1}{p_2^s}, \ \ldots, \ 1 - \frac{1}{p_n^s} \right)$$

satisfying

$$\zeta_{K_{1,\ldots,n}}(s) = (\zeta(s)) \left(\prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right)\right), \tag{8.41}$$

by (8.24), where $\zeta(s)$ is the Riemann zeta function. By (8.41), clearly, one can verify that if $n \to \infty$, then

$$\zeta_{K_{1,\ldots,n}}(s) \to 2\zeta(s).$$

Equivalently to (8.41), one has that

$$\frac{\zeta_{K_{1,\ldots,n}}(s)}{\zeta(s)} = \prod_{q \in \mathcal{P} \setminus \{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right). \tag{8.42}$$

Therefore, one can obtain the following ratio determined by primes.

Theorem 8.14. *Under the above hypothesis, we have*

$$\frac{\left(\prod_{j=1}^{n} \frac{2}{p_j^s}\right)^{-1} \left(L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) - [\textit{Rest terms}]\right)}{\zeta(s)} = \prod_{q \in \mathcal{P}\setminus\{p_1,\ldots,p_n\}} \left(1 - \frac{1}{q^s}\right), \tag{8.43}$$

where $[\textit{Rest terms}]$ *is in the sense of* (8.19), *characterized by* (8.36).

Proof. The proof of (8.43) is done by the identity

$$\zeta_{K_{1,\ldots,n}}(s) = \left(\prod_{j=1}^{n} \frac{2}{p_j^s}\right)^{-1} \left(L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) - [\text{Rest terms}]\right),$$

by (8.42). □

So, by (8.40) and (8.43), we have the following corollary.

Corollary 8.15. *Under the same hypothesis with the above theorem, we obtain the following relation between our glued-graph zeta function of non-factorizable graphs at the glued vertex, and the Riemann zeta function* $\zeta(s)$.

$$L_{K_{1,\ldots,n}}(v_{1,\ldots,n}, s) = \sum_{k=0}^{n} \left(\sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} \sigma^{i_1,\ldots,i_k}(s)\right), \tag{8.44}$$

with

$$\sigma^{i_1,\ldots,i_k}(s) = \left(\left(\frac{2^n}{\left(\prod_{l=1}^{k} p_{i_l}\right)^s}\right)\left((\zeta(s))\left(\prod_{q \in \mathcal{P} \setminus \{p_{i_1},\ldots,p_{i_k}\}} \left(1 - \frac{1}{q^s}\right)\right)\right)\right).$$

Proof. The proof of (8.44) is done by (8.40) and (8.43). □

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