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ON FRACTIONAL RANDOM DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. In this paper, we consider the existence and uniqueness of solutions of the fractional random differential equations with delay. Moreover, some kind of boundedness of the solution is proven. Finally, the applicability of the theoretical results is illustrated with some real world examples.

Keywords: sample path fractional integral, sample path fractional derivative, fractional differential equations, sample fractional random differential equations, Caputo fractional derivative, delay.

Mathematics Subject Classification: 26A33, 47H40, 60H25.

1. INTRODUCTION

The theory of fractional differential equations is an important branch of differential equation theory, which has been applied in physics, chemistry, biology and engineering and has emerged an important area of investigation in the last few decades. For some fundamental results in the theory of fractional calculus and fractional differential equations (see [4, 16, 18–20]). Random differential equations and random integral equations have been studied systematically in Ladde and Lakshmikantham [17] and Bharucha-Reid [3], respectively. They are good models in various branches of science and engineering since random factors and uncertainties have been taken into consideration. Hence, the study of the fractional differential equations with random parameters seem to be a natural one. We refer the reader to the monographs [3, 17, 26, 27], the papers [6–9, 12, 14] and the references therein.

Initial value problems for fractional differential equations with random parameters have been studied by V. Lupulescu and S.K. Ntouyas [24]. The basic tool in the study of the problems for random fractional differential equations is to treat it as a fractional differential equation in some appropriate Banach space. In [25], authors proved the existence results for a random fractional equation under a Carathéodory condition. Existence results for extremal random solution are also proved. For several other research results see [22, 23]. Continuing the work of the authors, in this paper, we consider the fractional random differential equations with delay as follows:

$$\begin{cases} \mathcal{D}^{\alpha} x(t,\omega) \overset{[0,a],\ \mathbb{P}-a.e.}{=} f_{\omega}(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega). \end{cases} \tag{1.1}$$

The random fractional differential equations with delay (1.1) is not new to the theory of random differential equations. When the random parameter ω is absent, the random equation (1.1) reduces to the fractional differential equations with delay,

$$\begin{cases} \mathcal{D}^{\alpha} x(t) = f(t,x_t) & \text{for } t \in [0,a], \\ x(t) = \varphi(t) & \text{for } t \in [-\sigma,0]. \end{cases} \tag{1.2}$$

The classical fractional differential equations with delay (1.2) has been studied in the literature by several authors. See for example, M. Benchohra, *et al.* [2], J. Deng, *et al.* [5] and the references therein.

In this paper, inspired and motivated by Lupulescu *et al.* [22–25] and Dhage [6–9], under the condition of the Lipschitzean right-hand side we obtain the existence and uniqueness of the solutions of the fractional random differential equations with delay. To prove this assertion we use an idea of successive approximations which has been applied in [21] for the fractional differential equations for this problem. The paper is organized as follows. In Section 2, we set up the appropriate framework on random processes and on fractional calculus. In Section 3, we prove the existence and uniqueness of solutions of the fractional random differential equations with delay. Moreover, some kind of boundedness of the solution is proven. Finally, in Section 4, an example is given to illustrate our results

2. PRELIMINARIES

In this section, we give some notations and properties related to the sample path fractional integral, the sample path fractional derivative and the metric space of random elements. The reader can see the detailed results in the monographs [17, 26, 27], the papers [24] and the references therein.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Let $([0,a], \mathcal{L}, \lambda)$ be a Lebesgue-measure space where $a > 0$ and let $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ be a product measurable function. A mapping $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is called sample path Lebesgue integrable on $[0,a]$ if $x(\cdot,\omega) : [0,a] \to \mathbb{R}^d$ is Legesgue integrable on $[0,a]$ for a.e. $\omega \in \Omega$.

Let $\alpha > 0$. If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is sample path Lebesgue integrable on $[0,a]$, then the fractional integral

$$I^{\alpha} x(t,\omega) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s,\omega)}{(t-s)^{1-\alpha}} ds$$

which will be called the sample path fractional integral of x, where Γ is Euler's Gamma function.

If $x(\cdot,\omega) : [0,a] \to \mathbb{R}^d$ is Lebesgue integrable on $[0,a]$ for a.e. $\omega \in \Omega$, then $t \mapsto I^\alpha x(t,\omega)$ is also Lebesgue integrable on $[0,a]$ for a.e. $\omega \in \Omega$. A mapping $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is said to be a Carathéodory function if $t \mapsto x(t,\omega)$ is continuous for a.e. $\omega \in \Omega$ and $\omega \mapsto x(t,\omega)$ is measurable for each $t \in [0,a]$. We recall that a Carathéodory function is a product measurable function (see [13]), then function $(t,\omega) \mapsto I^\alpha x(t,\omega)$ is also a Carathéodory function (see [24]).

A mapping $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is said to have a sample path derivative at $t \in (0,a)$ if the function $t \mapsto x(t,\omega)$ has a derivative at t for a.e. $\omega \in \Omega$. We will denote by $\frac{d}{dt}x(t,\omega)$ or by $x'(t,\omega)$ the sample path derivative of $x(\cdot,\omega)$ at t. We say that $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is sample path differentiable on $[0,a]$ if $x(\cdot,\cdot)$ has a sample path derivative for each $t \in (0,a)$ and possesses a one-sided sample path derivative at the end points 0 and a.

If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is sample path absolutely continuous on $[0,a]$, that is, $t \mapsto x(t,\omega)$ is absolutely continuous on $[0,a]$ for a.e. $\omega \in \Omega$, then the sample path derivative $x'(t,\omega)$ exists for λ-a.e. $t \in [0,a]$. Let $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ be sample path absolutely continuous on $[0,a]$ and let $\alpha \in (0,1]$. Then, for λ-a.e. and for a.e. $\omega \in \Omega$, we define the Caputo sample path fractional derivative of x by

$$\mathcal{D}^\alpha x(t,\omega) = I^{1-\alpha} x'(t,\omega) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s,\omega)}{(t-s)^\alpha} ds.$$

Obviously, if $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}$ is sample path differentiable on $[0,a]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,a]$, then $\mathcal{D}^\alpha x(t,\omega)$ exists for every $t \in [0,a]$ and $t \mapsto \mathcal{D}^\alpha x(t,\omega)$ is continuous on $[0,a]$.

If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is a Carathéodory function, then

$$\mathcal{D}^\alpha I^\alpha x(t,\omega) = x(t,\omega)$$

for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$.

If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}$ is sample path absolutely continuous on $[0,a]$, then

$$I^\alpha \mathcal{D}^\alpha x(t,\omega) = x(t,\omega) - x(0,\omega) \tag{2.1}$$

for λ-a.e. and for a.e. $\omega \in \Omega$.

If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is sample path differentiable on $[0,a]$ and $t \mapsto x'(t,\omega)$ is continuous on $[0,a]$, then (2.1) holds for all $t \in [0,a]$ and for a.e. $\omega \in \Omega$. If $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is sample path absolutely continuous on $[0,a]$, then

$$t \mapsto y(t,\omega) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x(s,\omega)}{(t-s)^\alpha} ds$$

is also sample path absolutely continuous on $[0,a]$. Moreover, for λ-a.e. and for a.e. $\omega \in \Omega$ and a.e. $\omega \in \Omega$, we have that

$$y(t,\omega) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(s,\omega)}{(t-s)^\alpha} ds = \mathcal{D}^\alpha x(t,\omega) + \frac{x(0,\omega)}{t^\alpha \Gamma(1-\alpha)}.$$

Let the space $C = C([0,a],\mathbb{R}^d)$ of all continuous functions from $[0,a]$ from $\mathbb{R}^d$ endowed with the uniform metric

$$d(x,y) = \sup_{t\in[0,a]} \|x(t) - y(t)\|,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$.

If $x : \Omega \to C$ is a random element, then the function $x(\cdot,\cdot) : [0,a] \times \Omega \to \mathbb{R}^d$ is a Carathéodory function. If $x : \Omega \to C$ is a random element, then the function $x(\cdot,\omega) : [0,a] \to \mathbb{R}^d$ is said to be a realization or a trajectory of the random element x, corresponding to the outcome $\omega \in \Omega$.

Let $M(C)$ be the space of all probability measures on $\mathcal{B}(C)$. If $x : \Omega \to C$ is a random element in C, then the probability measure μ_x, defined by

$$\mu_x = \mathbb{P}(x^{-1}(B)) = \mathbb{P}(\{\omega \in \Omega, x(\omega) \in B\}), \quad B \in \mathcal{B}(C),$$

is called the distribution of x. Since C has been a complete and separable metric space, then it is well known that $M(C)$ is a complete and a separable metric space with respect to the Prohorov metric $D : M(C) \to [0,\infty)$ given by

$$D(\mu,\eta) = \inf\{\epsilon > 0, \mu(B) \le \eta(B^\epsilon) + \epsilon\}, \quad B \in \mathcal{B}(C),$$

where $B^\alpha = \{x \in C; \inf_{y\in B} d(x,y) < \epsilon\}$.

Let $R(\Omega,C)$ be the metric space of all random elements in C. A sequence of random variables $\{x_n\} \subset R(\Omega,C)$ is said to converge almost everywhere (a.e.) to $x \in R(\Omega,C)$ if there exists $N \subset R$ such that $\mathbb{P}(N) = 0$, and $\lim_{n\to\infty} d(x_n(\omega),x(\omega)) = 0$ for every $\omega \in \Omega \setminus N$. We use the notation $x_n \to x$ a.e. for almost everywhere convergence. If $\{x_n\} \subset R(\Omega,C)$ is a ρ convergent sequence, then it is known that $\{x_n\}$ is a ρ-Cauchy sequence, where $\rho(x,y) = D(\mu,\eta)$ is a metric on the set of random elements in C (see [11]).

Lemma 2.1 (see [24]). *Let $e_n : [0,a] \times \Omega \to \mathbb{R}^d$ be a sequence of Carathéodory positive functions and let u, v be positive constants such that $e_0(t,\omega) \le u$ for every $n \ge 1$,*

$$e_n(t,\omega) \le u + \frac{v}{\Gamma(\alpha)} \int_0^t \frac{e_{n-1}(s,\omega)}{(t-s)^{1-\alpha}} ds$$

for all $t \in [0,a]$ and $\omega \in \Omega$. Then for every $n \ge 1$,

$$e_n(t,\omega) \le u E_\alpha(v a^\alpha), \quad \omega \in \Omega,$$

where $E_\alpha = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$ is the Mittag-Leffler function.

For a positive number σ, we denote by C_σ the space $C([-\sigma,0],\mathbb{R}^d)$. Also, we denote by

$$\|x-y\|_{C_\sigma} = \sup_{s\in[-\sigma,0]} \|x-y\|$$

the metric on the space C_σ. Let $x(\cdot) \in C([-\sigma,\infty),\mathbb{R}^d)$. Then, for each $t \in [0,\infty)$we denote by x_t the element of C_σ defined by $x_t(s) = x(t+s)$ for $s \in [-\sigma,0]$.

3. MAIN RESULTS

In this section, we consider the fractional random differential equations with delay as follows :

$$\begin{cases} \mathcal{D}^\alpha x(t,\omega) \overset{[0,a],\ \mathbb{P}-a.e.}{=} f_\omega(t,x_t), \\ x(t,\omega) \overset{[-\sigma,0]}{=} \varphi(t,\omega). \end{cases} \tag{3.1}$$

where $x_0 : \Omega \to \mathbb{R}^d$ is a random vector, $\mathcal{D}^\alpha x$ is the Caputo fractional derivative of x with respect to the variable t, and $f : \Omega \times [0,a] \times C_\sigma \to \mathbb{R}^d$ is a given function.

Lemma 3.1. *Let a $(\lambda\times\mathbb{P})$ - measurable function $x : [-\sigma,a]\times\Omega \to \mathbb{R}^d$ be a sample path Lebesgue integrable on $[-\sigma,a]$. Then $x(t,\omega)$ is a sample solution of* (3.1) *if and only if $x(\cdot,\omega)$ is a continuous on $[-\sigma,a]$ for $\mathbb{P}-a.e.$ $\omega\in\Omega$ and it satisfies the following random integral equation:*

$$\begin{aligned} x(t,\omega) &\overset{[-\sigma,0]}{=} \varphi(t,\omega), \\ x(t,\omega) &\overset{[0,a],\ \mathbb{P}-a.e.}{=} \varphi(0,\omega) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds. \end{aligned} \tag{3.2}$$

We shall consider the fractional random differential equations with delay (3.1) assuming that the following assumptions are satisfied.

(f1) The mapping $f_\omega(\cdot,\cdot) : [0,a]\times C_\sigma \to \mathbb{R}^d$ are measurable and continuous for each $\omega\in\Omega$.

(f2) There exists a Carathéodory function $L : [0,a]\times\Omega \to \mathbb{R}^d$ such that

$$\|f_\omega(t,\Phi) - f_\omega(t,\Psi)\| \le L(t,\omega)\,\|\Phi-\Psi\|_{C_\sigma}$$

for every $t\in[0,a]$, $\Phi,\Psi\in C_\sigma$ and $\mathbb{P}-a.e.$ $\omega\in\Omega$.

(f3) There exists a non-negative constant M such that

$$\|f_\omega(t,\Phi)\| \le M$$

for every $t\in[0,a]$, $\Phi\in C_\sigma$ and $\mathbb{P}-a.e.$ $\omega\in\Omega$.

Theorem 3.2. *Let $f : \Omega\times[0,a]\times C_\sigma \to \mathbb{R}^d$ satisfy the assumptions* (f1)–(f2). *Then the fractional random differential equations with delay* (3.1) *has a unique solution on* $[0,a]$.

Proof. To prove the theorem we apply the method of successive approximations. So, we define the functions $x^n : [-\sigma, a] \to \mathbb{R}^d$, $n = 0, 1, 2, 3, \ldots$ as follows: for every $t \in [0, a]$, every $\omega \in \Omega$ let us put

$$x^0(t,\omega) = \begin{cases} \varphi(t,\omega) & \text{for } t \in [-\sigma, 0], \\ \varphi(0,\omega) & \text{for } t \in [0, a]. \end{cases} \tag{3.3}$$

and

$$x^n(t,\omega) = \begin{cases} \varphi(t,\omega) & \text{for } t \in [-\sigma, 0], \\ \varphi(0,\omega) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \dfrac{f_\omega(s, x_s^{n-1})}{(t-s)^{1-\alpha}} ds & \text{for } t \in [0, a]. \end{cases} \tag{3.4}$$

For $n = 1$ and every $t \in [0, a]$, every $\omega \in \Omega$, we have

$$\begin{aligned} \left\|x^1(t,\omega) - x^0(t,\omega)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|f_\omega(s, x_0(\omega))\|}{(t-s)^{1-\alpha}} ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \leq \frac{Mt^\alpha}{\Gamma(1+\alpha)} \leq \frac{Ma^\alpha}{\Gamma(1+\alpha)} < \infty. \end{aligned} \tag{3.5}$$

In particular,

$$\begin{aligned} \left\|x^2(t,\omega) - x^1(t,\omega)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\left\|f_\omega(s, x_s^1) - f_\omega(s, x_0(\omega))\right\|}{(t-s)^{1-\alpha}} ds \\ &= \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{\left\|x_s^1(\cdot,\omega)) - x_0(\cdot,\omega)\right\|_{C_\sigma}}{(t-s)^{1-\alpha}} ds \\ &= \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{s\in[-\sigma,0]} \left\|x^1(r,\omega)) - x_0(r,\omega)\right\| ds \\ &\leq \frac{\hat{L}(\omega)M}{\Gamma(\alpha)\Gamma(1+\alpha)} \int_0^t \frac{s^\alpha}{(t-s)^{1-\alpha}} ds \leq \frac{\hat{L}(\omega)M}{\Gamma(1+2\alpha)} t^{2\alpha} \\ &\leq \frac{\hat{L}(\omega)M}{\Gamma(1+2\alpha)} a^{2\alpha}, \quad t \in [0, a], \end{aligned}$$

where $\hat{L}(\omega) = \sup_{t\in[0,a]} L(t,\omega)$.

Further, if we assume that

$$\left\|x^n(t,\omega) - x^{n-1}(t,\omega)\right\| \leq \frac{M[\hat{L}(\omega)t]^{n\alpha}}{\Gamma(1+n\alpha)} \leq \frac{\hat{M}[L(\omega)a]^{n\alpha}}{\Gamma(1+n\alpha)}, \quad t \in [0, a],$$

then we have

$$\begin{aligned}\left\|x^{n+1}(t,\omega)-x^{n}(t,\omega)\right\| &\leq \frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t \frac{M[\hat{L}(\omega)s]^{n\alpha}}{\Gamma(1+n\alpha)}ds\\ &\leq \frac{M[\hat{L}(\omega)t]^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \leq \frac{M[\hat{L}(\omega)a]^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \qquad (3.6)\end{aligned}$$

for every $t\in[0,a]$ and $\mathbb{P}-a.e.\ \omega\in\Omega$.

For every $n=0,1,2,\ldots$, the functions $x^n(\cdot,\omega):[-\sigma,a]\to\mathbb{R}^d$ are continuous on $[-\sigma,a]$ for every $\omega\in\Omega$. Indeed, since $\varphi\in C_\sigma$, $x^0(t,\omega)$ is continuous on $[-\sigma,a]$ for every $\omega\in\Omega$. Next, we assume that $x^k(\cdot,\omega)$, $k=0,1,2,\ldots,n-1$, are continuous on $[-\sigma,a]$ for every $\omega\in\Omega$. Then for $t\in[-\sigma,a]$ and $h>0$ small enough such that $t+h\in(0,a]$ we have

$$\begin{aligned}\|x^n(t+h,\omega)-x^n(t,\omega)\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t+h}\frac{f_\omega(s,x_s^{n-1})}{(t+h-s)^{1-\alpha}}ds-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}}\right\|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t\left|\frac{1}{(t+h-s)^{1-\alpha}}-\frac{1}{(t-s)^{1-\alpha}}\right|\left\|f_\omega(s,x_s^{n-1})\right\|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_t^{t+h}\frac{1}{(t+h-s)^{1-\alpha}}\left\|f_\omega(s,x_s^{n-1})\right\|ds. \qquad (3.7)\end{aligned}$$

On the other hand, by assumptions (f2)-(f3) we have

$$\begin{aligned}\left\|f_\omega(t,x_t^{n-1})\right\| &\leq \left\|f_\omega(t,x_t^{n-1})-f_\omega(t,x_t^0)\right\|+\left\|f_\omega(t,x_t^0)\right\|\\ &\leq M+\hat{L}(\omega)\left\|x_t^{n-1}(\cdot,\omega)-x_t^0(\cdot,\omega)\right\|_{C_\sigma}\\ &= M+\hat{L}(\omega)\sup_{r\in[-\sigma,0]}\left\|x_t^{n-1}(r,\omega)-x_t^0(r,\omega)\right\|\\ &= M+\hat{L}(\omega)\sup_{r\in[-\sigma,0]}\left\|x^{n-1}(t+r,\omega)-x^0(t+r,\omega)\right\|\\ &= M+\hat{L}(\omega)\sup_{s\in[t-\sigma,t]}\left\|x^{n-1}(s,\omega)-x^0(s,\omega)\right\|\\ &\leq M+\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{1}{(t-s)^{1-\alpha}}\sup_{\gamma\in[0,s]}\left\|f_\omega(\gamma,x^{n-2}(\gamma,\omega))\right\|ds.\end{aligned}$$

For every $n\geq 1$, every $t\in[0,a]$ and every $\omega\in\Omega$, we get

$$\sup_{t\in[0,a]}\left\|f_\omega(t,x_t^{n-1})\right\|\leq M+\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{1}{(t-s)^{1-\alpha}}\sup_{\gamma\in[0,s]}\left\|f_\omega(\gamma,x^{n-2}(\gamma,\omega))\right\|ds. \qquad (3.8)$$

Using Lemma 4.1 of [24] and (3.8), we obtain

$$\left\|f_\omega(t, x_t^{n-1})\right\| \leq ME_\alpha(\hat{L}(\omega)a^\alpha). \tag{3.9}$$

From (3.7) and (3.9) we get

$$\begin{aligned}\|x^n(t+h,\omega) - x^n(t,\omega)\| &\leq ME_\alpha(\hat{L}(\omega)a^\alpha)\int_0^t \Big|\frac{1}{(t+h-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}}\Big|ds \\ &+ ME_\alpha(\hat{L}(\omega)a^\alpha)\int_t^{t+h}\frac{1}{(t-s)^{1-\alpha}}ds \\ &\leq \frac{ME_\alpha(\hat{L}(\omega)a^\alpha)\big(2h^\alpha + |(t+h)^\alpha - t^\alpha|\big)}{\Gamma(1+\alpha)} \overset{\mathbb{P}.a.e}{\longrightarrow} 0 \quad \text{as } h \to 0^+.\end{aligned}$$

Similarly, for $t \in [0,a]$ and $h > 0$ small enough such that $t - h \in [0,a)$ we obtain that $\|x^n(t+h,\omega) - x^n(t,\omega)\| \overset{\mathbb{P}.a.e}{\to} 0$ as $h \to 0^+$. Therefore, for $n = 0,1,2,\ldots$ the function $x^n(\cdot,\omega) : [-\sigma,a]\times\Omega \to \mathbb{R}^d$ is continuous on $[0,a]$ for every $\omega \in \Omega$. Since $\varphi \in C_\sigma$ is a random variable and for $t \in [0,a]$, the mapping $\omega \mapsto \int_0^t f_\omega(s, x_s^{n-1})ds$ are measurable for $n = 0,1,2,\ldots$. From the assumptions (f1) and (f2), it follows that $x^n(\cdot,\cdot)$ are Carathéodory functions.

Now for any $n = 0,1,2,\ldots$ and $t \in [-\sigma,0]$ we shall show that the sequence $\{x^(t,\omega)\}$ is a Cauchy sequence uniformly on the variable t with $\mathbb{P} - a.e.$ and then $\{x^n(\cdot,\omega)\}$ is uniformly convergent for all $\omega \in \Omega$. For $n > m \geq 0$, from (3.6) we have

$$\begin{aligned}\sup_{t\in[0,a]}\|x^n(t,\omega) - x^m(t,\omega)\| &\leq \sum_{k=m}^n \sup_{t\in[0,a]}\left\|x^{k+1}(t,\omega) - x^k(t,\omega)\right\| \\ &\leq M\sum_{k=m}^n \frac{[\hat{L}(\omega)a]^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)}.\end{aligned} \tag{3.10}$$

On the other hand, by the inequality,

$$\Gamma((k+1)\alpha) \geq k!\alpha^{2k}\Gamma^{k+1}(\alpha), \quad k \geq 0,$$

it follows that

$$\frac{[\hat{L}(\omega)a]^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)} \leq \frac{[\hat{L}(\omega)a]^{(k+1)\alpha}}{(k+1)!\alpha^{2k+1}\Gamma^{k+1}(\alpha)} = \frac{\alpha}{(k+1)!}\Big[\frac{[\hat{L}(\omega)a]^\alpha}{\alpha\Gamma(1+\alpha)}\Big]^{k+1}. \tag{3.11}$$

From (3.10) and (3.11) we get

$$\sup_{t\in[0,a]}\|x^n(t,\omega) - x^m(t,\omega)\| \leq \alpha M\sum_{k=m}^n \frac{1}{(k+1)!}\Big[\frac{[\hat{L}(\omega)a]^\alpha}{\alpha\Gamma(1+\alpha)}\Big]^{k+1}.$$

The convergence of the series $\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{[\hat{L}(\omega)a]^{\alpha}}{\alpha\Gamma(1+\alpha)}\right]^{k}$ implies that for any $\epsilon > 0$ we find $n_0 \in \mathbb{N}$ large enough such that for $n, m \geq n_0$,

$$\sup_{t\in[0,a]} \|x^n(t,\omega) - x^m(t,\omega)\| \leq \epsilon. \tag{3.12}$$

Therefore, the sequence $x^n(\cdot,\omega)$ is uniformly convergent on $[0,a]$ for all $\omega \in \Omega$. For $\omega \in \Omega$ denote its limit by $\hat{x}(\cdot,\omega)$. Define $x : [-\sigma, a] \times \Omega \to \mathbb{R}^d$ by $x(t,\omega) = \varphi(t,\omega)$ for $t \in [-\sigma, 0]$ and $x(t,\omega) = \hat{x}(t,\omega)$ for $t \in [0,a]$. Since $\varphi \in C_\sigma$ is a random variable and $\|x^n(t,\omega) - x(t,\omega)\| \overset{\mathbb{P}.a.e}{\to} 0$ as $n \to \infty$ for $t \in [0,a]$, we see $t \in [-\sigma, a]$ and $x(\cdot,\omega)$ is a measurable function. Hence, the function $x : [-\sigma, a] \times \Omega \to \mathbb{R}^d$ is a Carathéodory function. We shall show that $x(\cdot,\cdot)$ is a solution of the fractional random integral equation (3.2). Let $n \in \mathbb{N}$. For any $\epsilon > 0$, there exits n_0 large enough such that for every $n \geq n_0$ we derive

$$\begin{aligned}
\|f_\omega(t,x_t^n) - f_\omega(t,x_t)\| &\leq L(t,\omega)\,\|x_t^n(\cdot,\omega) - x_t(\cdot,\omega)\|_{C_\sigma} \\
&= \hat{L}(\omega) \sup_{r\in[-\sigma,0]} \|x_t^n(r,\omega) - x_t(r,\omega)\| \\
&= \hat{L}(\omega) \sup_{r\in[-\sigma,0]} \|x^n(t+r,\omega) - x(t+r,\omega)\| \\
&= \hat{L}(\omega) \sup_{\gamma\in[t-\sigma,t]} \|x^n(\gamma,\omega) - x(\gamma,\omega)\| \\
&= \hat{L}(\omega) \sup_{\gamma\in[0,t]} \|x^n(\gamma,\omega) - x(\gamma,\omega)\| \leq \epsilon
\end{aligned}$$

for any $t \in [0,a]$, because the sequence $\{x^n(\cdot,\omega)\}$ is uniformly convergent on $[0,a]$ for all $\omega \in \Omega$. Thus, for any $t \in [0,a]$ we have

$$\begin{aligned}
\left\|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s^n)}{(t-s)^{1-\alpha}}ds - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds\right\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\|f_\omega(s,x_s^n) - f_\omega(s,x_s)\|}{(t-s)^{1-\alpha}}ds \\
&\leq \frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{\gamma\in[0,s]} \|x^n(\gamma,\omega) - x(\gamma,\omega)\|\,ds \\
&\leq \frac{\hat{L}(\omega)a^\alpha}{\Gamma(1+\alpha)} \sup_{\gamma\in[0,t]} \|x^n(\gamma,\omega) - x(\gamma,\omega)\|.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we infer that

$$\left\|\frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s^n)}{(t-s)^{1-\alpha}}ds - \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds\right\| \overset{\mathbb{P}.a.e}{\longrightarrow} 0,$$

as $n \to \infty$, for all $t \in [0,a]$ and $\omega \in \Omega$.

Consequently, we have

$$\begin{aligned}&\left\|x(t,\omega)-\varphi(0,\omega)-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds\right\|\\&\le\|x^n(t,\omega)-x(t,\omega)\|+\left\|x^n(t,\omega)-\varphi(0,\omega)-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}}ds\right\|\\&\quad+\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s^{n-1})}{(t-s)^{1-\alpha}}ds-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds\right\|.\end{aligned}$$

Thus, in view of the two previous convergences and the fact that the second term of the right-hand side is equal to zero, one obtains

$$x(t,\omega)=\varphi(0,\omega)+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds$$

for all $t\in[0,a]$ and $\omega\in\Omega$. Therefore, $x(t,\omega)$ is a solution of problem (3.1).

For the uniqueness of the solution, let us assume that $x,y:[-\sigma,a]\times\Omega\to\mathbb{R}^d$ are two Carathéodory functions which are solutions of the problem (3.1). Then we have

$$\begin{aligned}\|x(t,\omega)-y(t,\omega)\|&=\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}}ds-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f_\omega(s,y_s)}{(t-s)^{1-\alpha}}ds\right\|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\|f_\omega(s,x_s)-f_\omega(s,y_s)\|}{(t-s)^{1-\alpha}}ds\\&\le\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{\|x_s(\cdot,\omega)-y_s(\cdot,\omega)\|_{C_\sigma}}{(t-s)^{1-\alpha}}ds\\&=\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{1}{(t-s)^{1-\alpha}}\sup_{r\in[-\sigma,0]}\|x_s(r,\omega)-y_s(r,\omega)\|\,ds\\&=\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{1}{(t-s)^{1-\alpha}}\sup_{r\in[-\sigma,0]}\|x(s+r,\omega)-y(s+r,\omega)\|\,ds\\&=\frac{\hat{L}(\omega)}{\Gamma(\alpha)}\int_0^t\frac{1}{(t-s)^{1-\alpha}}\sup_{\gamma\in[s-\sigma,s]}\|x(\gamma,\omega)-y(\gamma,\omega)\|\,ds.\end{aligned}\tag{3.13}$$

If we take

$$\xi(s,\omega)=\sup_{\gamma\in[s-\sigma,s]}\|x(\gamma,\omega)-y(\gamma,\omega)\|$$

for all $s \in [0,t]$ and $\omega \in \Omega$, then from (3.13) we see that

$$\xi(t,\omega) \leq \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \xi(s,\omega) ds.$$

Applying Lemma 2.1 we obtain that

$$\|x(t,\omega) - y(t,\omega)\| = 0,$$

for all $t \in [0,a]$ and $\omega \in \Omega$, which completes the proof. □

The next two theorems present boundedness type results for the solutions of (3.1).

Theorem 3.3. *Suppose that the function* $f : \Omega \times [0,a] \times C_\sigma \to \mathbb{R}^d$ *and* $\varphi, \hat{\varphi} \in C_\sigma$ *satisfy the assumptions as in Theorem* 3.2. *Then the solution* x *to the problem* (3.1) *satisfies*

$$\|x(t,\omega)\| \leq \|\varphi(0,\omega)\| \, E_\alpha\Big(\frac{Ma^\alpha}{\Gamma(\alpha)}\Big)$$

for all $s \in [0,t]$ *and* $\omega \in \Omega$.

Proof. Since for all $t \in [0,a]$ and $\omega \in \Omega$,

$$\begin{aligned}
\|x(t,\omega)\| &= \left\| \varphi(0,\omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds \right\| \\
&\leq \|\varphi(0,\omega)\| + \frac{M}{\Gamma(\alpha)} \int_0^t \frac{\|x_s(\cdot,\omega)\|_{C_\sigma}}{(t-s)^{1-\alpha}} ds \\
&\leq \|\varphi(0,\omega)\| + \frac{M}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{r\in[-\sigma,0]} \|x(s+r,\omega)\| \, ds \\
&\leq \|\varphi(0,\omega)\| + \frac{M}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{\gamma\in[s-\sigma,s]} \|x(\gamma,\omega)\| \, ds.
\end{aligned}$$

Applying Lemma 2.1, for all $s \in [0,t]$ and $\omega \in \Omega$, we obtain that

$$\|x(t,\omega)\| \leq \|\varphi(0,\omega)\| \, E_\alpha\Big(\frac{Ma^\alpha}{\Gamma(\alpha)}\Big),$$

where $E_\alpha(z)$ is the Mittag- Leffler function. This proof is completed. □

Theorem 3.4. *Suppose that the function* $f : \Omega \times [0,a] \times C_\sigma \to \mathbb{R}^d$ *and* $\varphi \in C_\sigma$ *satisfy the assumptions as in Theorem* 3.2. *Then the solution* x *and* y *are solutions of the problem* (3.1) *with* $x(0,\omega) = \varphi(0,\omega)$ *and* $y(0,\omega) = \hat{\varphi}(0,\omega)$ *for* $t \in [-\sigma,0]$, *satisfies*

$$\|x(t,\omega) - y(t,\omega)\| \leq \|\varphi - \hat{\varphi}\| \, E_\alpha\Big(\frac{\hat{L}(\omega)a^\alpha}{\Gamma(\alpha)}\Big)$$

for all $s \in [0,t]$ *and* $\omega \in \Omega$.

Proof. Note that for all $t \in [0,a]$ and $\omega \in \Omega$,

$$\begin{aligned}
&\|x(t,\omega) - y(t,\omega)\| \\
&= \left\| \varphi(0,\omega) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,x_s)}{(t-s)^{1-\alpha}} ds - \hat{\varphi}(0,\omega) - \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f_\omega(s,y_s)}{(t-s)^{1-\alpha}} ds \right\| \\
&\le \|\varphi(0,\omega) - \hat{\varphi}(0,\omega)\|_{C_\sigma} + \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{\|x_s(\cdot,\omega) - y_s(\cdot,\omega)\|_{C_\sigma}}{(t-s)^{1-\alpha}} ds \\
&\le \|\varphi(0,\omega) - \hat{\varphi}(0,\omega)\|_{C_\sigma} + \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{r\in[-\sigma,0]} \|x(s+r,\omega) - y(s+r,\omega)\| \, ds \\
&\le \|\varphi - \hat{\varphi}\| + \frac{\hat{L}(\omega)}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \sup_{\gamma\in[s-\sigma,s]} \|x(\gamma,\omega) - y(\gamma,\omega)\| \, ds.
\end{aligned}$$

Applying Lemma 2.1, for all $s \in [0,t]$ and $\omega \in \Omega$, we obtain that

$$\|x(t,\omega) - y(t,\omega)\| \le \|\varphi - \hat{\varphi}\| \, E_\alpha\Big(\frac{\hat{L}(\omega)a^\alpha}{\Gamma(\alpha)}\Big),$$

where $E_\alpha(z)$ is the Mittag- Leffler function. This proof is complete. ☐

4. ILLUSTRATIVE EXAMPLES

In this section we give some examples to illustrate the usefulness of our main results.

Example 4.1. Let us consider the class of delay random fractional differential equations with distributed delay. For $m \in \mathcal{N}$ and $0 < \sigma_1 < \sigma_2 < \ldots < \sigma_m < \sigma$. Consider the random fractional differential equations with delay as follows:

$$\begin{aligned}
&x(t,\omega) = \varphi(t,\omega) \text{ for } t \in [-\sigma, 0], \\
&\mathcal{D}^\alpha x(t,\omega) = \int_{-\sigma}^0 g_{0,\omega}(s, x(t+s,\omega))ds + \sum_{i=1}^m g_{i,\omega}(t, x(t-\sigma_i,\omega)) \text{ for } t \in [0,a],
\end{aligned} \tag{4.1}$$

where $g_i : \Omega \times [0,a] \times C_\sigma \to \mathbb{R}^d$, $i = 0,1,2,\ldots,m$ are some random mapping, Ω is a complete probability space and C_σ is the space $C([-\sigma,0] \times \Omega, \mathbb{R}^d)$.

Assume that $g_{i,\omega} : [0,a] \times C_\sigma \to \mathbb{R}^d$ satisfy the following assumptions:

(g1) The mapping $g_{i,\omega}(\cdot,\cdot) : [0,a] \times C_\sigma \to \mathbb{R}^d$ are measurable and continuous, for every $(t_0,\varphi_0) \in [0,a] \times C_\sigma$, each $\omega \in \Omega$;

(g2) There exists a Carathéodory function $L_i : [0, a] \times \Omega \to \mathbb{R}^d$ such that

$$\|g_{i,\omega}(t, \Phi) - g_{i,\omega}(t, \Psi)\| \leq L_i(t, \omega) \|\Phi - \Psi\|_{C_\sigma}$$

for $i = 0, 1, 2, \ldots, m$, every $t \in [0, a]$, $\Phi, \Psi \in C_\sigma$ and $\mathbb{P} - a.e.\ \omega \in \Omega$;

(g3) There exists some non-negative constants M_i such that

$$\|g_{i,\omega}(t, \Phi)\| \leq M_i$$

for $i = 0, 1, 2, \ldots, m$, every $t \in [0, a]$, $\Phi \in C_\sigma$ and $\mathbb{P} - a.e.\ \omega \in \Omega$.

Observe that, if $g_{i,\omega} : [0, a] \times C_\sigma \to \mathbb{R}^d$ satisfy the following assumptions (g1)–(g3), then the problem (4.1) has a unique solution. Indeed, let the mapping $g : \Omega \times [0, b] \times C_\sigma \to \mathbb{R}^d$ given by

$$g_\omega(t, \varphi) = \int_{-\sigma}^{0} g_{0,\omega}(\tau, \varphi(\tau)) d\tau + \sum_{i=1}^{m} g_{i,\omega}(t, \varphi(-\sigma_i))$$

satisfies assumptions (f1), (f2) and (f3). From assumption (g1) one can infer that assumption (f1) is satisfied. And by assumption (g2), we have

$$\begin{aligned}\|g_\omega(t, \varphi) - g_\omega(t, \psi)\| &\leq \int_{-\sigma}^{0} \|g_{0,\omega}(\tau, \varphi(\tau)) - g_{0,\omega}(\tau, \psi(\tau))\| \, d\tau \\ &\quad + \sum_{i=1}^{m} \|g_{i,\omega}(t, \varphi(-\sigma_i)) - g_{i,\omega}(t, \psi(-\sigma_i))\| \\ &\leq L(t, \omega) \|\varphi - \psi\|_{C_\alpha},\end{aligned}$$

where

$$L(t, \omega) = \sigma \sup_{\tau \in [-\sigma, 0]} L_0(\tau, \omega) + \sum_{i=1}^{m} L_i(t, \omega).$$

Hence, g satisfies (f2). Moreover, by assumption ($g3$), we get

$$\|g_\omega(t, \varphi)\| \leq \int_{-\sigma}^{0} \|g_{0,\omega}(\tau, \varphi(\tau))\| \, d\tau + \sum_{i=1}^{m} \|g_{i,\omega}(t, \varphi(-\sigma_i)\| \leq M,$$

where $M = \sigma M_0 + \sum_{i=1}^{m} M_i$. So, g satisfies (f3).

Example 4.2. In the classical population models, it is considered that the birth rate changes immediately as soon as a change in the number of individuals is produced. However, the members of the population must reach a certain degree of development to give birth to new individuals and this suggests introducing a delay term into the system. Now, let $(\Omega, \mathcal{A}, \mathbb{P})$ be complete probability space measure space, we consider a random time-delay Malthusian model as follows:

$$\begin{aligned} x(t,\omega) &= \varphi(t,\omega) = (1+\omega^2)t, \quad -1 \le t \le 0, \\ \mathcal{D}^\alpha x(t,\omega) &= \frac{t\omega^2}{1+\omega^2} x(t-1,\omega), \quad t \ge 0, \end{aligned} \tag{4.2}$$

where ω symbolizes a random factor and $x : [-1,\infty) \times \Omega \to \mathbb{R}$ is the population at time t.

Let us the mapping $f_\omega(\cdot,\cdot) : [0,\infty) \times C_\sigma \to \mathbb{R}$ given by

$$f_\omega(t,x_t) = \frac{t\omega^2}{1+\omega^2} x(t-1,\omega) \quad \text{for all } \omega \in \Omega.$$

It is easy to check that $f_\omega(t,x_t)$ satisfies the assumption (f2)–(f3). Indeed, we have

(i) for all $(t,\omega) \in [0,a] \times \Omega$,

$$\begin{aligned} \left|f_\omega(t,x_t) - f_\omega(t,y_t)\right| &= \left|\frac{t\omega^2}{1+\omega^2} x(t-1,\omega)\right| \le \left|\frac{t\omega^2}{1+\omega^2}\right| \left|x(t-1,\omega) - y(t-1,\omega)\right| \\ &= t \sup_{r\in[-1,0]} \left|x(r,\omega) - y(r,\omega)\right| = t\left|x-y\right|_{C_\sigma} \end{aligned}$$

i.e. $f_\omega(\cdot,\cdot)$ satisfies the assumption (f2), where $L(t,\omega) = t$.

(ii) for all $(t,\omega) \in [0,a] \times \Omega$,

$$\begin{aligned} \left|f_\omega(t,x_t)\right| &= \left|\frac{t\omega^2}{1+\omega^2} x(t-1,\omega)\right| \le \left|\frac{t\omega^2}{1+\omega^2}\right| \left|x(t-1,\omega)\right| \\ &\le t \sup_{r\in[-1,0]} \left|x(r,\omega)\right| \le a^2(1+\omega^2). \end{aligned}$$

i.e. $f_\omega(\cdot,\cdot)$ satisfies the assumption (f3), where $M = a^2(1+\omega^2)$.

Hence, by Theorem 3.2, the problem (4.2) has a random solution defined on $[-1,\infty)$.

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