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UNIFORM APPROXIMATION BY POLYNOMIALS WITH INTEGER COEFFICIENTS

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Abstract. Let r, n be positive integers with $n \geq 6r$. Let P be a polynomial of degree at most n on $[0, 1]$ with real coefficients, such that $P^{(k)}(0)/k!$ and $P^{(k)}(1)/k!$ are integers for $k = 0, \ldots, r - 1$. It is proved that there is a polynomial Q of degree at most n with integer coefficients such that $|P(x) - Q(x)| \leq C_1 C_2^r r^{2r+1/2} n^{-2r}$ for $x \in [0, 1]$, where C_1, C_2 are some numerical constants. The result is the best possible up to the constants.

Keywords: approximation by polynomials with integer coefficients, lattice, covering radius.

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1. INTRODUCTION

Approximation by polynomials with integer coefficients has penetrated far beyond the original area of applications in analysis. Many connections and generalizations were found in various areas of approximation theory and real analysis. It is well known that a continuous function $f : [0, 1] \to \mathbb{R}$ can be uniformly approximated by polynomials with integer coefficients if and only if $f(0), f(1) \in \mathbb{Z}$ (see for instance [3, 5]). This leads to the following question: let P be a polynomial of degree at most n, with $P(0), P(1) \in \mathbb{Z}$; how well can P be approximated by polynomials of degree at most n with integer coefficients? This question (in the L_2 norm) appeared in Aparicio [1] and (in the L_p norm) in Trigub [6]. Moreover, the paper [2] deals with the more general question of estimating $\gamma_{r,n}, r \geq 1$ in the L_p norm, $p \geq 1$. Let $\boldsymbol{P}_n$ denote the space of polynomials of degree at most n with real coefficients. Let $\boldsymbol{P}_n^{\mathbb{Z}} \subset \boldsymbol{P}_n$ be the additive subgroup consisting of polynomials with integer coefficients. We will treat polynomials as elements of the real normed space $C[0, 1]$. By $\|\cdot\|$ we denote the uniform norm in $C[0, 1]$ and d is the corresponding metric. Let us also denote

$$\boldsymbol{H}_1 := \{f \in C[0, 1] : f(0), f(1) \in \mathbb{Z}\}.$$

The problem is to estimate the quantity

$$\gamma_{1,n} := \max_{P\in \boldsymbol{P}_n\cap \boldsymbol{H}_1} d(P, \boldsymbol{P}_n^{\mathbb{Z}}).$$

The standard argument based on Bernstein polynomials yields $\gamma_{1,n} = O(n^{-1})$ (see for instance [3, 4]). The bound $\gamma_{1,n} = O(n^{-2})$ appers in Trigub [6]. In fact, one has $\gamma_{1,n} \asymp n^{-2}$ as $n \to \infty$ (see the remarks following Proposition 1.2).

We have to introduce some notation. Let $\mathbb{N}$ denote the set $\{1, 2, \dots\}$ and let $r \in \mathbb{N}$. By $\boldsymbol{M}_r$ we denote the space of polynomials divisible by $x^r(1-x)^r$. In other words,

$$\boldsymbol{M}_r := \{P : P^{(k)}(0) = P^{(k)}(1) = 0 \text{ for } k = 0, \dots, r-1\}.$$

By $\boldsymbol{H}_r$ we denote the set of all polynomials P such that $P^{(k)}(0)/k!$ and $P^{(k)}(1)/k!$ are integers for $k = 0, \dots, r-1$. For $n \geq 2r$ we denote

$$\gamma_{r,n} := \max_{P\in \boldsymbol{P}_n\cap \boldsymbol{H}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}}).$$

The main result of the paper is the following:

Theorem 1.1. *Let $r \in \mathbb{N}$ and let $n \geq 6r$. Then*

$$\gamma_{r,n} < C_1 \cdot C_2^r \cdot \frac{r^{2r+1/2}}{n^{2r}}, \tag{1.1}$$

where C_1, C_2 are some numerical constants. One may take $C_1 = 2\sqrt{\pi} + 1$ and $C_2 = 16$.

This result cannot be essentially improved:

Proposition 1.2. *Let $r \in \mathbb{N}$ and let $n \geq 2r$. Then*

$$\gamma_{r,n} > c_1 \cdot c_2^r \cdot \frac{r^{2r+1/2}}{n^{2r}},$$

where c_1, c_2 are some numerical constants. One may take $c_1 = \sqrt{\pi}$ and $c_2 = e^{-2}$.

Fix $r \in \mathbb{N}$. From Theorem 1.1 and Proposition 1.2 one gets $\gamma_{r,n} \asymp n^{-2r}$ as $n \to \infty$. It follows from [6, Lemma 3] that $\gamma_{r,n} = O(n^{-r})$. The results of [6] allow one to obtain the bound $\gamma_{r,n} = O(n^{-2r})$, but do not give more precise estimates of the form (1.1). Theorem 1.1 can be found in [2] (in a slightly different form). But the method used in the present paper is simpler, more direct, does not use the L_2 norm. It is not hard to verify that for certain values of r and n it gives better values of the corresponding numerical constants.

2. THE PROOFS

By a *lattice* in $C[0, 1]$ we mean an additive subgroup generated by a finite number of linearly independent vectors. It is not to hard to see if $n \geq 2r$, then $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ is a lattice generated by the Bernstein polynomials

$$x^k(1-x)^r, \qquad k = r, \dots, n-r;$$

hence it follows that $\boldsymbol{P}_n \cap \boldsymbol{M}_r = \operatorname{span}\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right)$. Given a lattice L, by $\mu(L)$ we denote its *covering radius*:

$$\mu(L) := \max_{P \in \operatorname{span} L} d(P, L).$$

Lemma 2.1. *Let $r \in \mathbb{N}$ and let $n \geq 2r$. Then*

$$\gamma_{r,n} \leq \mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r).$$

Proof. Given $P \in \boldsymbol{P}_n \cap \boldsymbol{H}_r$, let us write

$$\widetilde{P}(x) = (1-x)^r \sum_{k=0}^{r-1} \frac{P^{(k)}(0)}{k!} x^k + x^r \sum_{k=0}^{r-1} \frac{P^{(k)}(1)}{k!} (x-1)^k.$$

Then $\widetilde{P} \in \boldsymbol{P}_{2r-1}^{\mathbb{Z}} \subset \boldsymbol{P}_n^{\mathbb{Z}}$ and

$$\widetilde{P}^{(k)}(0) = P^{(k)}(0), \qquad \widetilde{P}^{(k)}(1) = P^{(k)}(1), \qquad k = 0, \ldots, r-1,$$

which means that $P - \widetilde{P} \in \boldsymbol{M}_r$. Since $\widetilde{P} \in \boldsymbol{P}_n^{\mathbb{Z}}$, we have $\boldsymbol{P}_n^{\mathbb{Z}} - \widetilde{P} = \boldsymbol{P}_n^{\mathbb{Z}}$ and therefore

$$d(P, \boldsymbol{P}_n^{\mathbb{Z}}) = d(P - \widetilde{P}, \boldsymbol{P}_n^{\mathbb{Z}} - \widetilde{P}) = d(P - \widetilde{P}, \boldsymbol{P}_n^{\mathbb{Z}}).$$

Hence it follows that

$$\gamma_{r,n} = \max_{P \in \boldsymbol{P}_n \cap \boldsymbol{H}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}}) = \max_{P \in \boldsymbol{P}_n \cap \boldsymbol{H}_r} d(P - \widetilde{P}, \boldsymbol{P}_n^{\mathbb{Z}}) = \max_{P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}}).$$

Now it remains to observe that

$$\max_{P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}}) \leq \max_{P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r) = \mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r). \qquad \square$$

Lemma 2.2. *Let $r \in \mathbb{N}$ and let $n \geq 2r$. Then*

$$\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r) \leq \frac{1}{2} \binom{n}{r}^{-1}.$$

Proof. The argument is standard, but we give the proof to make the paper self-contained. Let $P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r$. We have to find some $Q \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ such that

$$\|P - Q\| \leq \frac{1}{2} \binom{n}{r}^{-1}.$$

We may write

$$P(x) = \sum_{k=r}^{n-r} a_k x^k (1-x)^{n-k},$$

where a_k are real coefficients. Let us write $a_k = [a_k] + \{a_k\}$, where $[a_k] \in \mathbb{Z}$ and

$$-1/2 < \{a_k\} \leq 1/2.$$

Consider the polynomial

$$Q(x) = \sum_{k=r}^{n-r} [a_k] x^k (1-x)^{n-k}.$$

For each $x \in [0,1]$ we have

$$|P(x) - Q(x)| = \left| \sum_{k=r}^{n-r} \{a_k\} x^k (1-x)^{n-k} \right| \leq \frac{1}{2} \sum_{k=r}^{n-r} x^k (1-x)^{n-k}.$$

It is now enough to observe that

$$\begin{aligned} \sum_{k=r}^{n-r} x^k (1-x)^{n-k} &\leq \binom{n}{r}^{-1} \sum_{k=r}^{n-r} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \binom{n}{r}^{-1} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \binom{n}{r}^{-1}. \end{aligned}$$

□

Remark 2.3. If $r = 0$, then problem of evaluating $\mu\left(\boldsymbol{P}_n^{\mathbb{Z}}\right)$ is trivial. For each $n \in \mathbb{N}$ we have

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}}\right) = \frac{1}{2}.$$

Let $P(x) = \frac{1}{2}$. For each $Q \in \boldsymbol{P}_n^{\mathbb{Z}}$ we have $\|P - Q\| \geq \frac{1}{2}|P(0) - Q(0)| \geq \frac{1}{2}$. This proves that

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}}\right) \geq \frac{1}{2}.$$

It is also not hard to see that

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}}\right) \leq \tfrac{1}{2}.$$

Let $U_1, U_2, \ldots$ be the sequence of polynomials given by

$$U_r(x) = x^r (1-x)^r, \qquad r = 1, 2, \ldots$$

Lemma 2.4. *Let $r \in \mathbb{N}$ and let $n \geq 2r + 3$. Then*

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) \leq \tfrac{1}{2} d\left(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}\right).$$

Proof. Let $P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r$. We have to prove that

$$d\left(P, \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) \leq \tfrac{1}{2} d\left(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}\right). \tag{2.1}$$

We may write

$$P(x) = x^r (1-x)^r [a + bx + x(1-x)Q(x)]$$

for some $a, b \in \mathbb{R}$ and some $Q \in \boldsymbol{P}_{n-2r-2}$. In other words, we may write

$$P = A \cdot U_r + R,$$

where $A \in \boldsymbol{P}_1$ and $R \in \boldsymbol{P}_n \cap \boldsymbol{M}_{r+1}$. It is not hard to see that the linear function A may be written in the form $A = B + C$, where $B \in \boldsymbol{P}_1^{\mathbb{Z}}$, $C \in \boldsymbol{P}_1$ and $\|C\| \le \frac{1}{2}$. Take $S \in \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}$ such that

$$\|U_r - S\| = d(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}).$$

Then $C \cdot S \in \boldsymbol{P}_n \cap \boldsymbol{M}_{r+1}$ and

$$\|C \cdot U_r - C \cdot S\| = \|C \cdot (U_r - S)\| \le \|C\| \cdot \|U_r - S\| \le \tfrac{1}{2}\|U_r - S\|. \tag{2.2}$$

We have $C \cdot S + R \in \boldsymbol{P}_n \cap \boldsymbol{M}_{r+1}$, so that there is some $T \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}$ such that

$$\|C \cdot S + R - T\| \le \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}\right). \tag{2.3}$$

Then we may write

$$P = (B + C) \cdot U_r + R = B \cdot U_r + (C \cdot U_r + R - T) + T.$$

Since $B \cdot U_r \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ and $T \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1} \subset \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$, it follows that $B \cdot U_r + T \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$. Thus

$$\begin{aligned} d(P, \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r) &\le \|P - B \cdot U_r - T\| = \|C \cdot U_r - C \cdot S + C \cdot S + R - T\| \\ &\le \|C \cdot U_r - C \cdot S\| + \|C \cdot S + R - T\|. \end{aligned}$$

Hence, by (2.2) and (2.3), we obtain (2.1). □

Remark 2.5. Let $r = 1$. It is easy to check that

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1\right) \le \frac{4}{n^2}.$$

Moreover, we have

$$d\left(U_1, \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1\right) \ge \frac{1}{4n^2}$$

(see the proof of Proposition 1.2 below) and

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1\right) \ge d\left(U_1, \boldsymbol{P}_n \cap \boldsymbol{M}_1\right).$$

Thus

$$\frac{1}{4n^2} \le \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1\right) \le \frac{4}{n^2}.$$

Lemma 2.6. *Let $r \in \mathbb{N}$ and let $m \ge 2$. There exists a polynomial $\widetilde{T}_{r,m} \in \boldsymbol{P}_{mr} \cap \boldsymbol{M}_r$ such that*

$$(\widetilde{T}_{r,m})^{(r)}(0) = (-1)^r (\widetilde{T}_{r,m})^{(r)}(1) = r!, \tag{2.4}$$

and

$$\|\widetilde{T}_{r,m}\| \le \frac{1}{m^{2r}}.$$

Proof. Let T_m be the Chebyshev polynomial given by

$$T_m(x) = \cos(m \arccos x), \quad -1 \le x \le 1.$$

Let $x_m = \cos(\pi/2m)$ be the greatest zero of T_m. Consider the polynomial

$$S_m(x) = -T_m((2x-1)x), \quad 0 \le x \le 1.$$

We have $S_m(0) = S_m(1) = 0$ and it is easy to check that

$$S'_m(0) = -S'_m(1) = 2x_m T'_m(x_m) = 2m \cdot \cot \frac{\pi}{2m} \ge m^2.$$

Let $\widetilde{T}_{1,m} \in \boldsymbol{P}_m \cap \boldsymbol{M}_1$ be given by

$$\widetilde{T}_{1,m}(x) = \frac{S_m(x)}{S'_m(0)}, \quad 0 \le x \le 1.$$

Then $\widetilde{T}'_{1,m}(0) = -\widetilde{T}'_{1,m}(1) = 1$ and

$$\|\widetilde{T}_{1,m}\| \le \frac{1}{m^2}.$$

For $r > 1$, we define

$$\widetilde{T}_{r,m} = (\widetilde{T}_{1,m})^r.$$

Then $\|\widetilde{T}_{r,m}\| = \|\widetilde{T}_{1,m}\|^r \le 1/m^{2r}$. We may write

$$\widetilde{T}_{1,m}(x) = x(1-x)W(x)$$

for some $W \in \boldsymbol{P}_{m-2}$, therefore

$$\widetilde{T}_{r,m}(x) = x^r(1-x)^r(W(x))^r.$$

Hence it follows that $\widetilde{T}_{r,m} \in \boldsymbol{M}_r$ and (2.4) is satisfied. □

Corollary 2.7. *Let $r \in \mathbb{N}$ and let $n \ge 4r+2$. There exists a polynomial $T \in \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_r$ such that*

$$T^{(r)}(0) = (-1)^r T^{(r)}(1) = r! \tag{2.5}$$

and

$$\|T\| \le 2^{2r} \cdot r^{2r} \cdot \frac{1}{n^{2r}}.$$

Consequently,

$$d(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}) < 2^{2r} \cdot r^{2r} \cdot \frac{1}{n^{2r}}. \tag{2.6}$$

Proof. Suppose that m is odd such that

$$\frac{n-1}{r} - 2 < m \le \frac{n-1}{r}.$$

According to Lemma 2.6, there exists some polynomial $T \in \boldsymbol{P}_{mr} \cap \boldsymbol{M}_r$ satisfying (2.5) and such that $\|T\| \le 1/m^{2r}$. It follows that $mr \le n-1$, whence $T \in \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_r$. Condition $n \ge 4r+2$ means that

$$\frac{n-1}{r} - 2 \ge \frac{n}{2r},$$

and therefore

$$\|T\| \le \frac{1}{m^{2r}} < \left(\frac{n-1}{r} - 2\right)^{-2r} \le \frac{2^{2r} r^{2r}}{n^{2r}}.$$

To obtain (2.6) it is enough to observe that $U_r - T \in \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}$. The proof for m even is similar. □

Proof of Theorem 1.1. In view of Lemma 2.1 it is enough to prove that

$$\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r) < C_1 \cdot C_2^r \cdot \frac{r^{2r+1/2}}{n^{2r}}.$$

According to Lemma 2.4, we may write

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) \le \tfrac{1}{2} d\left(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}\right),$$

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}\right) \le \tfrac{1}{2} d\left(U_{r+1}, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+2}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+2}\right),$$

$$\dots$$

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r-1}\right) \le \tfrac{1}{2} d\left(U_{2r-1}, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{2r}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r}\right).$$

By adding these inequalities we obtain

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) \le \frac{1}{2} \sum_{k=0}^{r-1} d\left(U_{r+k}, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+k+1}\right) + \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r}\right). \tag{2.7}$$

By Corollary 2.7, for $k = 0, 1, \dots, r-1$ we have

$$d\left(U_{r+k}, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+k+1}\right) \le 2^{2(r+k)} \cdot (r+k)^{2(r+k)} \cdot \frac{1}{n^{2(r+k)}}.$$

As $k \le r-1$, we have $2(r+k) \le 4r-2$. Consequently, we may write

$$\sum_{k=0}^{r-1} d\left(U_{r+k}, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+k+1}\right) \le 2^{4r-2} \cdot s_{n,r} \tag{2.8}$$

where

$$s_{n,r} = \frac{r^{2r}}{n^{2r}} + \frac{(r+1)^{2r+2}}{n^{2r+2}} + \frac{(r+2)^{2r+4}}{n^{2r+4}} + \ldots + \frac{(2r-1)^{4r-2}}{n^{4r-2}}.$$

Let

$$a_k = \frac{(r+k)^{2r+2k}}{n^{2r+2k}}, \quad k = 0, 1, \ldots, r-1$$

It is easy to check that

$$a_k \leq a_0 \cdot \left(4e^2 r^2 \cdot \frac{1}{n^2}\right)^k, \quad k = 1, 2, \ldots, r-1.$$

Consequently,

$$s_{n,r} = \sum_{k=0}^{r-1} a_k < a_0 \sum_{k=0}^{\infty} \left(4e^2 r^2 \cdot \frac{1}{n^2}\right)^k = a_0 \left(1 - \frac{4e^2 r^2}{n^2}\right)^{-1}.$$

Since, by assumption, $n \geq 6r$, it follows that

$$1 - \frac{4e^2 r^2}{n^2} \geq 1 - \frac{4e^2}{36} > \frac{1}{6}.$$

Thus

$$s_{n,r} < 6a_0 = 6 \cdot \frac{r^{2r}}{n^{2r}}. \tag{2.9}$$

From Lemma 2.2 it follows that

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r}\right) \leq \frac{1}{2} \cdot \binom{n}{2r}^{-1}.$$

Standard estimates based on Stirling's formula yield

$$\binom{n}{2r}^{-1} < 2\sqrt{2\pi} \cdot (2/e)^{2r} \cdot (2r)^{2r+1/2} \cdot \frac{1}{n^{2r}}.$$

Hence

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r}\right) \leq 2\sqrt{\pi} \cdot (4/e)^{2r} \cdot (r)^{2r+1/2} \cdot \frac{1}{n^{2r}}. \tag{2.10}$$

From (2.7)–(2.10) we obtain

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) < \frac{1}{2} \cdot 2^{4r-2} \cdot 6\frac{r^{2r}}{n^{2r}} + 2\sqrt{\pi} \cdot (4/e)^{2r} \cdot (r)^{2r+1/2} \cdot \frac{1}{n^{2r}} < (2\sqrt{\pi}+1) \cdot 16^r \cdot \frac{r^{2r+1/2}}{n^2 r}.$$

□

Proof of Proposition 1.2. As $\frac{1}{2}U_r \in \boldsymbol{P}_n \cap \boldsymbol{M}_r \subset \boldsymbol{P}_n \cap \boldsymbol{H}_r$, we have

$$\gamma_{r,n} \geq d(\tfrac{1}{2}U_r, \boldsymbol{P}_n^{\mathbb{Z}}). \tag{2.11}$$

Let f be the linear functional on $\boldsymbol{P}_n$ given by

$$f(P) = (-1)^r \frac{1}{r!} \cdot P^{(r)}(0).$$

Then $f_{|\boldsymbol{M}_{r+1}} \equiv 0$, $f(U_r) = 1$ and $f(\boldsymbol{P}_n^{\mathbb{Z}}) = \mathbb{Z}$. If $P \in \boldsymbol{P}_n$, then

$$\left|P^{(r)}(0)\right| \le 2^r \cdot \frac{n^2(n^2-1)\dots(n^2-(r-1)^2)}{1\cdot 3 \dots (2r-1)} \|P\|, \qquad r = 0, 1, 2, \dots, n. \tag{2.12}$$

according to the Markov inequality. Equality in (2.12) holds for the Chebyshev polynomial of degree n on $[0, 1]$. This means that

$$\|f\| = \frac{1}{r!} \cdot 2^r \cdot \frac{n^2(n^2-1)\dots(n^2-(r-1)^2)}{1\cdot 3 \dots (2r-1)}.$$

We may write

$$n^2(n^2-1)\dots(n^2-(r-1)^2) \le n^{2r}$$

and

$$1 \cdot 3 \dots (2r-1) = \frac{(2r)!}{2^r \cdot r!}.$$

Using standard estimates based on Stirling's formula, we obtain

$$\|f\| < \frac{e^{2r}}{2\sqrt{\pi} \cdot r^{2r+1/2}} \cdot n^{2r}. \tag{2.13}$$

Take any $P \in \boldsymbol{P}_n^{\mathbb{Z}}$. We have $f(P) \in \mathbb{Z}$ and $f(2^{-1}U_r) = \frac{1}{2}$, therefore

$$\left|f(P) - f(2^{-1}U_r)\right| \ge 2^{-1}.$$

On the other hand, we have

$$\left|f(P) - f(2^{-1}U_r)\right| \le \|f\| \cdot \|P - 2^{-1}U_r\|.$$

Thus

$$\|2^{-1}U_r - P\| \ge \frac{1}{2} \cdot \|f\|^{-1}.$$

Since $P \in \boldsymbol{P}_n^{\mathbb{Z}}$ was arbitrary, it follows that

$$d(\tfrac{1}{2}U_r, \boldsymbol{P}_n^{\mathbb{Z}}) \ge \tfrac{1}{2}\|f\|^{-1}. \tag{2.14}$$

From (2.11), (2.13) and (2.14) we obtain

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) > \sqrt{\pi} e^{-2r} \cdot r^{2r+1/2} \cdot \frac{1}{n^{2r}}. \qquad \square$$

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