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INTEGRAL AND FRACTIONAL EQUATIONS, POSITIVE SOLUTIONS, AND SCHAEFER'S FIXED POINT THEOREM

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Abstract. This is the continuation of four earlier studies of a scalar fractional differential equation of Riemann-Liouville type

$$D^q x(t) = -f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \in \Re \quad (0 < q < 1), \tag{a}$$

in which we first invert it as a Volterra integral equation

$$x(t) = x^0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{b}$$

and then transform it into

$$x(t) = x^0 t^{q-1} - \int_0^t R(t-s) x^0 s^{q-1} ds + \int_0^t R(t-s) \left[x(s) - \frac{f(s, x(s))}{J} \right] ds, \tag{c}$$

where R is completely monotone with $\int_0^\infty R(s)\, ds = 1$ and J is an arbitrary positive constant. Notice that when x is restricted to a bounded set, then by choosing J large enough, we can frequently change the sign of the integrand in going from (b) to (c). Moreover, the same kind of transformation will produce a similar effect in a wide variety of integral equations from applied mathematics. Because of that change in sign, we can obtain an *a priori* upper bound on solutions of (b) with a parameter $\lambda \in (0, 1]$ and then obtain an *a priori* lower bound on solutions of (c). Using this property and Schaefer's fixed point theorem, we obtain positive solutions of an array of fractional differential equations of both Caputo and Riemann-Liouville type as well as problems from turbulence, heat transfer, and equations of logistic growth. Very simple results establishing global existence and uniqueness of solutions are also obtained in the same way.

Keywords: fixed points, fractional differential equations, integral equations, Riemann-Liouville operators.

Mathematics Subject Classification: 34A08, 34A12, 45D05, 45G05, 47H09, 47H10.

1. INTRODUCTION

Positive solutions of differential and integral equations is a central property in many areas of applied mathematics. There are entire books devoted to the subject such as [1] and [2]. Population problems are frequently meaningless unless the discussion is restricted to positive populations. Positive solutions of heat conduction problems provide a foundation for much of resolvent theory. This is seen in a good portion of Chapter IV of [21].

In this paper we will offer two keys to the search for positive solutions of a wide area of integral equations. These keys are Schaefer's fixed point theorem and a transformation in which the integrand of the integral equation changes sign as we transform from one equation to the next. The first form provides an upper bound for the solution on an interval $(0, E]$ of arbitrary length. At this point the first form does something very interesting. Having provided an upper bound for the solution, it then goes to work and helps the second form provide a lower bound for the solution. See, for example, items 1–3 in Example 5.1 found in Section 5. These two bounds offer the *a priori* bound needed for Schaefer's theorem which then provides the existence of a solution residing between those two bounds and being valid on an arbitrarily long interval, $(0, E]$. If solutions are unique that solution is then continued to $(0, \infty)$. In addition to the *a priori* bound, Schaefer's theorem also requires certain continuity and compactness conditions very much like those of Schauder's theorem. We offer several lemmas in Section 4 showing that these properties are quite automatic for a wide class of integral equations from applied mathematics. The result of this is that nothing except the *a priori* bound need be established.

We illustrate the theory with examples from applied mathematics including fractional differential equations of both Riemann-Liouville and Caputo type, both of which are used to model a myriad of real-world problems. It is relatively simple to put all of the equations considered into the form needed for this work, with one exception. That exception is the Riemann-Liouville type. Since such equations are of prime interest in applied mathematics, we start with them and devote a major part of this paper to showing the aforementioned transformation for them.

Thus, we start with a scalar fractional differential equation of Riemann-Liouville type and introduce a parameter $\lambda \in (0, 1]$ which is used in Schaefer's theorem. It is introduced early because we will take a transformation and we want to see where λ will appear in each equation. In the final conclusions, we will always say that there is a solution with given properties for $\lambda = 1$.

There is yet a third interesting part of this two-step process of finding the *a priori* bound. The reader may note as we go through Theorems 2.2 and 4.1, together with the six examples in Section 5, that the arguments for that upper and lower bound on all possible solutions for all $\lambda \in (0, 1]$ are all satisfied at $\lambda = 1$. This is very unusual in applications of Schaefer's theorem. In Example 5.6 of Section 5 we leave out the λ and invite the reader to supply it in order to emphasize this feature.

Schaefer's theorem [27, p. 29] will dictate what we do.

Theorem 1.1 (Schaefer)**.** *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda P x$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

Item (ii) of the theorem often causes the reader to stumble. But a study of the proof makes it clear that this is to be read as: The set of all such solutions x, *if any*, for $0 < \lambda < 1$, is unbounded.

The integral equation will define a natural mapping, P, and we see from (i) that a parameter λ is introduced. Our equation will go through a transformation and it will be important to see exactly where that parameter is at each stage. One way of doing so is illustrated in the way we set up the equation. Once we have established the bound discussed in (ii), then the value $\lambda = 1$ will be selected and we will proceed to establish the other conditions of the theorem for P. Looking at (1.1) and (1.2) with $\lambda = 1$ reveals the problems we have set out to solve.

For the presentation of fractional equations, we begin by writing

$$D^q x(t) = -\lambda f(t, x(t)), \quad 0 < q < 1, \quad \lim_{t \downarrow 0} t^{1-q} x(t) = \lambda x^0 \in \Re, \tag{1.1}$$

where $0 < \lambda \leq 1$, $x^0 \neq 0$ and $f\colon (0, \infty) \times \Re \to \Re$ with f continuous and $f(t, x) > 0$ if $x > 0$. The set of real numbers is designated in this paper by $\Re$. We later indicate that we can extend the problem to include

$$D^q x(t) = -\lambda f(t, x(t)) + p(t)$$

with $p(t) > 0$ and continuous, but it obscures the simplicity of the method presented here. That fractional derivative is defined by

$$D^q x(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds$$

where $\Gamma(q)$ is the Euler Gamma function. Equation (1.1) is formally inverted as the Volterra integral equation

$$x(t) = \lambda x^0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \lambda f(s, x(s))\, ds. \tag{1.2}$$

Substantial treatments of these equations are found in Diethelm [14], Kilbas *et al.* [18], Lakshmikantham *et al.* [19], and Podlubny [25]. An annotated bibliography is found in Oldham and Spanier [23]. That bibliography is very informative concerning both history and application.

An understanding of the equations begins with the definition of a solution of (1.2).

Definition 1.2. For a given $q \in (0,1)$ and a $T \in (0,\infty)$ a continuous function $\phi : (0,T] \to \Re$ is said to be a solution of (1.2) if ϕ satisfies (1.2) on $(0,T]$ and if

$$t^{1-q}\phi(t) \text{ is continuous on } [0,T] \text{ with } \lim_{t\to 0^+} t^{1-q}\phi(t) = \lambda x^0.$$

A basic result linking (1.1) and (1.2) is found in [5]. It states that if x is a continuous solution of (1.2) on $(0,T]$, then it is also a solution of (1.1) provided that

$$\int_0^T [|x(s)| + |f(s,x(s))|]\, ds < \infty. \tag{1.3}$$

Moreover, it is shown in Theorem 2.4 of [6] that for each $\epsilon \in (0,\lambda|x^0|)$, there is a $T^* \le T$ so that

$$(\lambda|x^0| - \epsilon)t^{q-1} < |x(t)| < (\lambda|x^0| + \epsilon)t^{q-1} < 2\lambda|x^0|t^{q-1} \tag{1.4}$$

for $0 < t \le T^*$ and that $x(t)$ has the sign of x^0 on this interval. If f satisfies polynomial growth, then we may use (1.4) in the integral appearing in (1.2) and use the Beta function to show when that integral exists. This is discussed in some detail in [6] and the example $f(t,x) = x^{2n+1}$ is featured, showing (in (2.18) of that paper) that the integral in (1.3) will exist if and only if $1 > q > 2n/(2n+1)$, a necessary and sufficient condition for (1.1) and (1.2) to share solutions for this f and for $\lambda = 1$.

Since the continuity and compactness required in Schaefer's theorem for (1.2) is automatic, half of the work required to establish a positive solution is contained in the following trivial observation.

Theorem 1.3. *Let f be continuous, $f(t,x) > 0$ for $x > 0$, and let $x^0 > 0$. Suppose that* (1.2) *has a positive solution on an interval* $(0,E]$ *for some* $E > 0$. *Then*

$$x(t) \le x^0 t^{1-q} \tag{1.5}$$

for $0 < t \le E$.

Proof. To set up the theorem, notice from (1.4) that it is immediate that if (1.2) has a solution with $x^0 > 0$ then the solution is positive on some interval $(0,T]$. Thus, existence and $x^0 > 0$ imply that there is an interval of the type described in the theorem. Once we see that, then the fact that f is positive for x positive tells us that the solution has the indicated upper bound. □

This idea is not new. In the classical treatment of resolvents, Miller [21, p. 210] gives the same argument for an equation with a continuous positive forcing function. However, Miller then presents a complex argument showing that the solution always remains non-negative ensuring the inequality

$$0 \le x(t) \le p(t)$$

with $p(t)$ the positive forcing function. For (1.2) we will show that there is a transformation which yields an equation for which that lower bound is obtained as simply as the upper bound. We finish by showing that the other requirements in Schaefer's theorem are automatically satisfied for equations in the general class in which (1.2) resides. A bit more detail will help the reader see the direction we are taking.

Having shown in [5] that (1.1) and (1.2) share solutions, we then showed in [9] (see also [6]) that (1.2) can be mapped into what will become (2.7) in the next section, namely,

$$x(t) = \lambda x^0 t^{q-1} - \int_0^t R(t-s)\lambda x^0 s^{q-1} ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right] ds,$$

where J denotes a positive constant and R a completely monotone function. Locally, R is much like t^{q-1}. In Theorem 2.2, a positivity and growth condition on f is given ensuring the integrand in the last term is positive if $x^0 > 0$ and if a solution does exist on a given interval. It follows from (2.5) and Lemma 2.1 that the sum of the first two terms on the right-hand side is positive. It is then immediate from this and (1.2) that

$$0 < x(t) \le x^0 t^{q-1} \tag{1.6}$$

and that this holds uniformly for every $x^0 > 0$ and $\lambda \in (0, 1]$.

We now go back to (1.2), set $\lambda = 1$, assume that there is a solution on a short interval $(0, T]$, translate by $y(t) = x(t+T)$, define the mapping P of Schaefer's theorem from the natural mapping of the y equation, prove that the mapping is compact, and walk away with a positive solution.

Next, we show that the same process works for six classical problems from applied mathematics. Finally, we quote two known theorems which would supply that solution on a short interval $(0, T]$ which we mentioned in the previous paragraph.

2. A SKETCH OF THE TRANSFORMATION

Here is a brief review of the steps transforming (1.2) to (2.7). Full details are found in [6]. This transformation has been used in [7, 9, 10], and [11]. We will be using a theorem on nonlinear variation of parameters found in Miller [21, pp. 191–193] and the properties of the resolvent, R, are found on pp. 212–213 and 224. The variation of parameters result requires an interchange in the order of integration which is valid using (1.3) in the Hobson-Tonelli theorem [22, p. 93].

It is crucial to begin by saying that $x^0 > 0$ and $0 < \lambda \le 1$ are fixed, but arbitrary numbers so that when we obtain our conclusion that (1.6) holds it will be clear that this is true for every such pair of numbers. In particular, the reader needs to understand that the resolvent R which we obtain depends on λ and J but the bounds in (2.6) do not. This ensures the uniformity of the lower bound on the solution.

Let J be an arbitrary positive constant and write (1.2) as

$$\begin{aligned} x(t) &= \lambda x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [-\lambda J x(s) + \lambda J x(s) - \lambda f(s, x(s))]\, ds \\ &= \lambda x^0 t^{q-1} - \frac{\lambda J}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\, ds \\ &\quad + \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\lambda x(s) - \frac{\lambda f(s, x(s))}{J} \right] ds. \end{aligned} \tag{2.1}$$

Define

$$C(t) = \frac{\lambda J t^{q-1}}{\Gamma(q)} \tag{2.2}$$

and write the linear part as

$$z(t) = \lambda x^0 t^{q-1} - \int_0^t C(t-s) z(s)\, ds \tag{2.3}$$

with resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s) R(s)\, ds \tag{2.4}$$

so that

$$z(t) = \lambda x^0 t^{q-1} - \int_0^t R(t-s) \lambda x^0 s^{q-1}\, ds. \tag{2.5}$$

Now R is completely monotone with

$$0 < R(t) \le \frac{\lambda J t^{q-1}}{\Gamma(q)} \le \frac{J t^{q-1}}{\Gamma(q)}, \quad \int_0^\infty R(s)\, ds = 1. \tag{2.6}$$

The nonlinear variation of parameters formula [21, pp. 190–193] yields

$$x(t) = z(t) + \int_0^t R(t-s) \left[x(s) - \frac{f(s, x(s))}{J} \right] ds. \tag{2.7}$$

Lemma 2.1. *The function z defined by* (2.5) *is positive and satisfies*

$$z(t) = \frac{x^0 \Gamma(q) R(t)}{J}.$$

Thus, $z(t)$ is a constant multiple of the completely monotone function $R(t)$; so it is decreasing on $(0,\infty)$. It follows that for each $\epsilon > 0$, $z(t)$ is bounded on $[\epsilon,\infty)$ and converges to zero. Moreover, for $\lambda \in (0,1]$ and $t > 0$ we have

$$|z(t)| \leq |x^0|t^{q-1}\left[1 - \int_0^t R(s)\,ds\right]. \tag{2.8}$$

Proof. To prove the first relation, multiply (2.4) by $x^0\Gamma(q)/J$ to see that the given function is the unique continuous solution of (2.5). Relation (2.8) follows from (2.5) and the fact that t^{q-1} is decreasing. □

This form of z in (2.7), coupled with (2.9) below, will show that any solution of (2.7) (equivalently of (1.2)) will satisfy $0 < x(t) \leq x^0 t^{q-1}$ and will be a main part in the use of Schaefer's fixed point theorem to show that (1.2) has a positive solution for $\lambda = 1$.

Theorem 2.2. *Let $f\colon (0,\infty) \times \Re \to \Re$ be continuous with $f(t,x) > 0$ for $x > 0$. Suppose that for a given $E > 0$ and $x^0 > 0$ there is an $L > 0$ such that*

$$0 < x \leq \lambda x^0 s^{q-1}, \quad 0 < s \leq E \implies \frac{f(s,x)}{x} \leq L. \tag{2.9}$$

If there is a solution x of (1.2) on $(0,E]$, then it satisfies

$$0 < x(t) \leq \lambda x^0 t^{q-1}, \quad 0 < t \leq E. \tag{2.10}$$

Proof. The solution $x(t)$ is initially positive since it has the same sign as x^0 (cf. (1.4)). Moreover, we see from (1.2) that so long as $x(t) > 0$ we have $x(t) \leq \lambda x^0 t^{q-1}$. To show that $x(t)$ remains positive on the entire interval $(0,E]$, suppose to the contrary that $x(t) > 0$ on $(0,t_1)$ and $x(t_1) = 0$ for some $t_1 \in (0,E]$. Then it follows from (2.7) with $J = 2L$ that

$$\int_0^{t_1} R(t_1 - s)\left[x(s) - \frac{f(s,x(s))}{2L}\right]ds = -z(t_1).$$

The left-hand side is positive because of (2.9) and $R(t) > 0$. However the right-hand side is negative by Lemma 2.1, a contradiction. □

We would be ready to use (1.2) to define the mapping of Schaefer's theorem, but there is a problem in that the forcing function is unbounded. To resolve this, we will use a translation and Lemma 2.1 to move (1.2) past that vertical asymptote and then be ready to define the mapping. We already have the *a priori* bound on any solution for any $\lambda \in (0,1]$ and continuity of the mapping will be an elementary exercise. We have the link between (1.1) and (1.2). Since a solution x satisfies (2.10) so that

$$0 < x(t) \leq \lambda x^0 t^{q-1}, \quad 0 < t \leq E,$$

then by (2.9) for any $T \in (0, E]$ we have

$$\int_0^T [|x(t)| + |f(t, x(t))|]\, dt \le \int_0^T [\lambda x^0 t^{q-1} + L\lambda x^0 t^{q-1}]\, dt = (1+L)\lambda x^0 \frac{T^q}{q}.$$

That is, (1.3) holds. It follows that (1.1) and (1.2) share solutions; thus (2.1) is also valid for solutions of (1.1).

There are three parts to be noted.

(a) Inequality (2.10) now establishes the bound needed in Schaefer's theorem and at this point λ has completed its purpose. We are now going to focus on other properties of solutions and λ will now be replaced by one.

(b) Condition (2.9) is stringent in that it must hold for arbitrarily large x. It is only stringent for Riemann-Liouville equations. None of the six examples in Section 5 require that condition.

(c) We view the result as new, simple, and useful, as will be seen in Section 4. While it may be most useful if it remains simple, it clearly can be generalized. If $u\colon [0, \infty) \to [0, \infty)$ is continuous, then we can extend (1.1) to

$$D^q x(t) = -\lambda f(t, x(s)) + \lambda u(t)$$

so that (1.2) becomes

$$x(t) = \lambda x^0 t^{q-1} + \frac{\lambda}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \lambda f(s, x(s))\, ds$$

and the sum of the first two terms on the right-hand side will provide an upper bound on $(0, E]$. Then in (2.7), $z(t)$ must be carefully checked for positivity. A gain is made in that the coefficient of $u(t)$ in the integral will now be $R(t-s)$ instead of $(t-s)^{q-1}/\Gamma(q)$ and that can be most helpful.

Interconnections

Condition (2.9) seems severe, but it does not stand alone in this regard. Equations (1.1) and (1.2) share solutions if and only if (1.3) holds. But we have shown in [6, (2.18)] that (1.3) holds for the function $f(t, x) = x^{2n+1}$, where n is a non-negative integer, if and only if $2n/(2n+1) < q < 1$. So, for (1.3) to hold for all $q \in (0, 1)$, we must have $n = 0$, in which case $f(t, x) = x$. Now, going back to (2.9), we see that the same is true there too, namely, it holds for $f(t, x) = x^{2n+1}$ if and only if $n = 0$. On the other hand, looking ahead to Theorem 6.1 and (6.1) in the last section, we notice that there is far more latitude for n in both (1.3) and (2.9) if we take $f(t, x) = t^{r_1} x^{2n+1}$. Far more important is the fact that we are working here only on the Riemann-Liouville problem having that singularity at $t = 0$ in the forcing function. The difficulties mentioned here vanish for the variety of problems considered in Section 5.

3. BACK TO (1.2) AND A TRANSLATION

We have established that any solution of (1.2) for any $x^0 > 0$ and $\lambda \in (0,1]$ has a bound given by (2.10). Now, let us return to (1.2), change to $\lambda = 1$, assume there is a solution on a short interval $(0, 2T]$, and translate to a completely equivalent equation which will have a continuous forcing function, but the same kernel. We will be able to show that the natural mapping defined by this equation will satisfy the conditions of Schaefer's theorem and still retain that bound of (2.10). From (1.2) we have

$$\begin{aligned} x(t+T) &= \lambda x^0 (t+T)^{q-1} - \frac{1}{\Gamma(q)} \int_0^{t+T} (t+T-s)^{q-1} \lambda f(s, x(s))\, ds \\ &= \lambda x^0 (t+T)^{q-1} - \frac{\lambda}{\Gamma(q)} \int_0^{T} (t+T-s)^{q-1} f(s, x(s))\, ds \\ &\quad - \frac{\lambda}{\Gamma(q)} \int_T^{t+T} (t+T-s)^{q-1} f(s, x(s))\, ds \\ &= \lambda x^0 (t+T)^{q-1} - \frac{\lambda}{\Gamma(q)} \int_0^{T} (t+T-s)^{q-1} f(s, x(s))\, ds \\ &\quad - \frac{\lambda}{\Gamma(q)} \int_0^{t} (t+T-s-T)^{q-1} f(s+T, x(s+T))\, ds \\ &= \lambda \Bigg[x^0 (t+T)^{q-1} - \frac{1}{\Gamma(q)} \int_0^{T} (t+T-s)^{q-1} f(s, x(s))\, ds \\ &\quad - \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} f(s+T, x(s+T))\, ds \Bigg]. \end{aligned}$$

Now let

$$y(t) := x(t+T), \quad t \geq 0, \tag{3.1}$$

to write this as

$$y(t) = \lambda \left[F(t) - \frac{1}{\Gamma(q)} \int_0^{t} (t-s)^{q-1} f(s+T, y(s))\, ds \right], \tag{3.2}$$

where

$$F(t) := x^0 (t+T)^{q-1} - \frac{1}{\Gamma(q)} \int_0^{T} (t+T-s)^{q-1} f(s, x(s))\, ds. \tag{3.3}$$

Notice that if $f(t,x)$ satisfies the conditions of Theorem 2.2, which includes (2.9) on an interval $(0,E]$, and if $x(t)$ is a solution of (1.2) on $(0,E]$, then the bounds for $y(t)$ on $[0,E-T]$ for a given $T\in(0,E)$ will be the same as those given for $x(t)$ in (2.10).

The function F that is defined above will play a large role in the work here. The following theorem lists some of its properties.

Theorem 3.1. *If $x(t)$ is a solution of (1.2) on $(0,2T]$ and if $f(t,x(t))$ is absolutely integrable on $(0,2T]$, then the function $F\colon[0,\infty)\to\Re$ defined by (3.3) is uniformly continuous on $[0,\infty)$ and converges to zero as $t\to\infty$.*

A proof of this can be readily patterned after one found in Theorem 4.2 in [6, p. 266] where the kernel here is replaced with $R(t-s)$. If the reader consults that reference, note that we have not concluded here that $F\in L^1[0,\infty)$. Now, that is the main property which separates $R(t-s)$ and $(t-s)^{q-1}$ and if we follow through the proof of Theorem 4.2 we see that the finite integral of R is used in exactly one place, namely (4.13) in [6, p. 268], which is the part of the proof showing that $F\in L^1[0,\infty)$. In any case, we now leave (3.2) and begin with a new F, namely H below, which is continuous by hypothesis and is the form for all of our examples in Section 5.

We call (3.2) a member of the *standard form.* But to indicate that it is only one of many, we will designate the standard form by

$$x(t)=H(t)-\int_0^t (t-s)^{q-1}h(s,x(s))\,ds, \tag{3.4}$$

where $H\colon[0,\infty)\to(0,\infty)$ and $h:[0,\infty)\times\Re\to\Re$ are both continuous, while $h(t,x)>0$ if $x>0$. Referring back to Schaefer's theorem we will define a mapping on a certain Banach space of continuous functions $\phi\colon[0,E]\to\Re$ with the supremum norm so that for ϕ in the space then

$$(P\phi)(t)=H(t)-\int_0^t (t-s)^{q-1}h(s,\phi(s))\,ds. \tag{3.5}$$

Our task now will be to show that the conditions of Schaefer's theorem are satisfied.

With (3.2) and (3.4) formulated together here it is appropriate to offer a main result concerning mappings of the type given in (3.5). A form of this was given in [10].

Theorem 3.2. *Let $E>0$ and S be a set of uniformly bounded and continuous functions $\phi\colon[0,E]\to\Re$. Then the set QS of bounded continuous functions $\psi\colon[0,E]\to\Re$ defined by $\phi\in S$ implies that*

$$\psi(t)=(Q\phi)(t)=\int_0^t (t-s)^{q-1}\phi(s)\,ds,\quad t\in[0,E],$$

is equicontinuous.

Proof. Let M denote a uniform bound for the set S, i.e., $|\phi(s)| \le M$ for all $s \in [0, E]$ and $\phi \in S$. Note that $(Q\phi)(0) = 0$ for all $\phi \in S$. Then, for $0 \le t_1 \le t_2 \le E$ and $\phi \in S$,

$$|\psi(t_1) - \psi(t_2)| = \left| \int_0^{t_1} (t_1 - s)^{q-1}\phi(s)ds - \int_0^{t_2} (t_2 - s)^{q-1}\phi(s)\, ds \right|$$

$$\le \int_0^{t_1} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}||\phi(s)|ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1}|\phi(s)|\, ds$$

$$\le M\left\{ \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]\, ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1}\, ds \right\}$$

$$= M\left[\frac{t_1^q}{q} - \frac{t_2^q}{q} + 2\frac{(t_2 - t_1)^q}{q} \right],$$

which goes to zero for $|t_1 - t_2| \to 0$ independently of $\phi \in S$. □

This is the result which will show that the mapping in Schaefer's theorem is compact. It will map bounded sets into closed, bounded, equicontinuous sets on a finite interval.

4. SCHAEFER'S THEOREM

Our equations for (i) and (ii) of Schaefer's theorem come from

$$x(t) = H(t) - \int_0^t (t - s)^{q-1}h(s, x(s))\, ds$$

and

$$(P\phi)(t) = H(t) - \int_0^t (t - s)^{q-1}h(s, \phi(s))\, ds.$$

We will see that equations of this general type always satisfy the continuity and compactness conditions of Schaefer's theorem on bounded intervals.

Notation.

(a) $(B, \|\cdot\|)$, or just B, denotes the Banach space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm.

(b) The closed L-ball in B is

$$M^L := \{\phi \in B\colon \|\phi\| \le L\}. \tag{4.1}$$

(c) For $E > 0$, B_E denotes the Banach space of continuous functions $\phi\colon [0, E] \to \Re$ with the supremum norm $\|\cdot\|_E$.

(d) The closed L-ball in B_E is

$$M_E^L := \{\phi \in B_E \colon \|\phi\|_E \le L\}. \tag{4.2}$$

While (1.2) led the way to the standard form (3.4) which satisfies all conditions of Schaefer's theorem except for the *a priori* bound, it is a counterpart of Theorem 2.2 which now leads us to that bound in all six examples to be considered in Section 5. To obtain that counterpart, we introduce λ in (3.4) writing

$$x(t) = \lambda\Big[H(t) - \int_0^t (t-s)^{q-1} h(s, x(s))\, ds\Big]. \tag{3.4λ}$$

Theorem 4.1. *Let $H\colon [0,\infty) \to (0,\infty)$ and $h\colon [0,\infty)\times\Re \to \Re$ be continuous functions where $h(t,x) > 0$ if $x > 0$. Suppose that for a given $E > 0$ there is a constant k such that*

$$\frac{h(t,x)}{x} \le k \tag{4.3}$$

for $0 \le t \le E$, $0 < x \le H(t)$. Moreover, suppose

$$H(t) - \int_0^t R(t-s)H(s)\, ds > 0 \quad \text{for} \quad 0 \le t \le E, \tag{4.4}$$

where R arises as in (2.4) with $C(t) = \lambda J t^{q-1}$ and $J \ge k$. If there is a solution $x(t)$ of (3.4λ) on $[0,E]$, then

$$0 < x(t) \le H(t)$$

for $0 \le t \le E$.

Proof. If $x(t)$ is a solution of (3.4λ) on $[0,E]$, then it is either positive over the entire interval $[0,E]$ or over some subinterval $[0,t_1) \subset E$ since $x(0) = \lambda H(0) > 0$. The transformation of Section 2 applied to (3.4λ) and the application of the nonlinear variation of parameters formula yield

$$x(t) = \lambda\Big\{H(t) - \int_0^t R(t-s)H(s)\, ds\Big\} + \int_0^t R(t-s)\Big[x(s) - \frac{h(s,x(s))}{J}\Big] ds, \tag{4.5}$$

where $R(t)$ denotes the resolvent for the kernel $C(t) = \lambda J t^{q-1}$. We see from (3.4λ) and the positivity condition for h that $x(t) \le \lambda H(t) \le H(t)$ as long as $x(t) > 0$. This, $x(t)$ starting off positive, and (4.4) imply that $x(t)$ can never become zero at a point in $[0,E]$ for that would imply that the second integral in (4.5) is negative at that point. But because of (4.3) that is clearly not the case. Thus $x(t) > 0$ for all $t \in [0,E]$. $\square$

Theorem 4.2. *Let*

(i) $0 < \lambda \leq 1$, E *be a positive constant,*
(ii) $h\colon [0,\infty) \times \Re \to \Re$ *and* $H\colon [0,\infty) \to (0,\infty)$ *both be continuous.*

If there is a continuous function $b\colon [0,\infty) \to (0,\infty)$ *so that any solution of* $x = \lambda Px$ *satisfies*

$$0 < x(t) \leq b(t) \text{ for } 0 \leq t \leq E,$$

then there is a $\phi \in B_E$ *with* $0 < \phi(t) \leq b(t)$ *and* $P\phi = \phi$. *If these conditions hold for every* $E > 0$ *and if solutions of* (3.4) *are unique, then there is a* $\phi \in B$ *solving* (3.4) *on* $[0,\infty)$ *and it satisfies* $0 < \phi(t) \leq b(t)$.

Corollary 4.3. *If the conditions of Theorem 2.2 hold, then* (3.2) *has a positive solution on any interval* $(0, T+E]$ *for* $\lambda = 1$. *Moreover, if solutions of* (3.2) *are unique, then the solution exists on* $(0,\infty)$.

The proof of both this theorem and its corollary consists of proving the following three lemmas and then applying Schaefer's theorem.

Proof. First, with Lemmas 4.4 and 4.5, we show P maps B_E into itself and that it is continuous.

Lemma 4.4. *If* $\psi \in B_E$, *then* $(P\psi)(t)$ *is a continuous function of* t *and hence* $P\colon B_E \to B_E$.

This is an immediate consequence of Theorem 3.2. That is, $\phi(t) = h(t,\psi(t))$ is continuous, resides in a bounded subset of B_E, and by Theorem 3.2 it is in an equicontinuous subset of B_E.

The next lemma asserts that the assumption of continuity in Schaefer's theorem will always hold for (3.5).

Lemma 4.5. *The mapping* $P\colon B_E \to B_E$ *is continuous.*

Proof. Continuity of this mapping follows from the hypothesis that the function $h\colon [0,\infty) \times \Re \to \Re$ is continuous and the convergence of

$$\int_0^t (t-s)^{q-1}\, ds = \frac{t^q}{q}.$$

Here are the details.

Choose any $\phi \in B_E$. Let $m := 2\|\phi\|_E$ unless $\phi \equiv 0$, in which case let $m = 1$. Now let $\epsilon > 0$ be given. We will prove P is continuous on B_E by showing that a $\delta > 0$ exists such that $\psi \in B_E$ and $\|\psi - \phi\|_E < \delta$ imply $\|P\psi - P\phi\|_E < \epsilon$.

Since h is continuous on the closed, bounded set

$$\mathcal{S} := \{(t,x)\colon 0 \leq t \leq E, -m \leq x \leq m\},$$

a $\delta \in (0, m/2)$ exists for the given $\epsilon > 0$ such that

$$|h(t,x_1) - h(t,x_2)| < \frac{\epsilon q}{2E^q}$$

if x_1, x_2 are a pair of numbers satisfying $|x_1| \le m/2$ and $|x_1 - x_2| < \delta$. Note that $(t, x_2) \in \mathcal{S}$ since

$$|x_2| \le |x_2 - x_1| + |x_1| < \delta + \frac{m}{2} < m.$$

For $\psi \in B_E$ with $\|\psi - \phi\|_E < \delta$,

$$\|\psi\|_E \le \|\psi - \phi\|_E + \|\phi\|_E < \delta + \frac{m}{2} < m.$$

From (3.5), we have

$$|(P\psi)(t) - (P\phi)(t)| \le \int_0^t (t-s)^{q-1} |h(s, \psi(s)) - h(s, \phi(s))| \, ds$$

for $t \in [0, E]$. Since $|\phi(s)| \le m/2$ and $|\psi(s) - \phi(s)| < \delta$ for $0 \le s \le t$, it follows that

$$|(P\phi)(t) - (P\psi)(t)| \le \frac{\epsilon q}{2E^q} \int_0^t (t-s)^{q-1} \, ds = \frac{\epsilon q}{2E^q} \cdot \frac{t^q}{q} \le \frac{\epsilon q}{2E^q} \cdot \frac{E^q}{q} = \frac{\epsilon}{2}$$

for $t \in [0, E]$. Therefore,

$$\|P\phi - P\psi\|_E < \epsilon. \qquad \square$$

The next lemma asserts that P is always a compact map on bounded intervals and, hence, the first sentence of Schaefer's theorem is satisfied.

Lemma 4.6. *The mapping P maps every bounded subset of B_E into a compact subset of B_E.*

Proof. Let G be an arbitrary bounded subset of B_E and find an $L > 0$ so that $G \subset M_E^L$. By Theorem 3.2 the integral in P maps M_E^L into an equicontinuous subset, say Z, of B_E. Adding the uniformly continuous function H to each element of $-Z$ results in an equicontinuous subset of B_E. The closure of this last equicontinuous subset is compact. This completes the proof. $\square$

To finish the proof of existence on $[0, E]$, notice that the second sentence of Theorem 4.2, as well as Theorem 2.2, asserts that there is an *a priori* bound on all possible solutions of $x = \lambda P x$. Conditions of Schaefer's theorem are now satisfied so there is a solution of (3.4) on any interval $[0, E]$.

If solutions of (3.4) are unique we construct a sequence of solutions x_n of (3.4) each of which is defined on $[0, n]$ with $n = 1, 2, \ldots$. By the uniqueness, for each positive integer p we see that x_{n+p} coincides with x_n on $[0, n]$. Now, obtain a sequence of functions X_n on $[0, \infty)$ where $X_n = x_n$ on $[0, n]$ and $X_n(t) = x_n(n)$ for $t \ge n$. This sequence converges uniformly on compact sets to a continuous function $X(t)$ on $[0, \infty)$ which solves (3.4) at any $t \in [0, \infty)$. $\square$

Uniqueness results can be cumbersome, as seen in [21, p. 91], but there is a very simple one for (3.4).

Theorem 4.7. *Let h, H, and b be as in Theorem 4.2. Suppose that x_1 and x_2 are two solutions of* (3.4) *on an interval* $[0, E]$ *residing in the strip of functions*

$$S := \{\, x : 0 < x(t) \le b(t),\ 0 \le t \le E \,\}.$$

If there are constants $J > 0$ and $K < 1$ such that

$$0 \le \frac{h(s, z(s)) - h(s, w(s))}{J(z(s) - w(s))} \le K \tag{4.6}$$

whenever $z, w \in S$ and $z(s) \ne w(s)$, then $x_1 \equiv x_2$ on $[0, E]$.

Proof. The idea is to introduce the constant J here in order to have the flexibility of choosing its value so as to satisfy condition (4.6) for the given interval $[0, E]$. This is accomplished with (4.5) (with $\lambda = 1$), which is the result of applying the transformation of Section 2 to (3.4). Thus we begin with

$$\begin{aligned} x(t) = H(t) - \int_0^t R(t-s)H(s)\,ds \\ + \int_0^t R(t-s)\Big[x(s) - \frac{h(s, x(s))}{J}\Big]\,ds. \end{aligned} \tag{4.7}$$

Note that since H is continuous, the Hobson-Tonelli conditions are satisfied; so this equation is completely equivalent to (3.4).

Suppose that x_1 and x_2 are distinct solutions of (3.4) – and so of (4.7) as just noted. Then

$$\begin{aligned} &|x_1(t) - x_2(t)| \\ &\le \int_0^t R(t-s)\Big|x_1(s) - x_2(s) - \frac{h(s, x_1(s)) - h(s, x_2(s))}{J}\Big|\,ds. \end{aligned} \tag{4.8}$$

For those $s \in [0, E]$ with $x_1(s) = x_2(s)$,

$$\Big|x_1(s) - x_2(s) - \frac{h(s, x_1(s)) - h(s, x_2(s))}{J}\Big| = 0,$$

which is less than $\|x_1 - x_2\|_E = \sup_{s\in[0,E]} |x_1(s) - x_2(s)| > 0$.

Now consider those $s \in [0, E]$ with $x_1(s) \ne x_2(s)$. Then condition (4.6) implies that

$$\begin{aligned} &\Big|x_1(s) - x_2(s) - \frac{h(s, x_1(s)) - h(s, x_2(s))}{J}\Big| \\ &= |x_1(s) - x_2(s)|\Big|1 - \frac{h(s, x_1(s)) - h(s, x_2(s))}{J(x_1(s) - x_2(s))}\Big| \\ &\le |x_1(s) - x_2(s)| \le \|x_1 - x_2\|_E. \end{aligned}$$

Consequently, it follows from (4.8) that

$$|x_1(t) - x_2(t)| \le \int_0^t R(t-s)\|x_1 - x_2\|_E \, ds \le \|x_1 - x_2\|_E \int_0^E R(s)\, ds.$$

However, this implies

$$\|x_1 - x_2\|_E \le \|x_1 - x_2\|_E \int_0^E R(s)\, ds,$$

a contradiction since $\int_0^E R(s)\, ds < 1$. Therefore, $x_1 \equiv x_2$. □

5. THE TRANSFORMATION AND SCHAEFER'S THEOREM

A main purpose of these examples is to illustrate how counterparts of (1.2) yield an upper bound on all possible solutions, while the transformation of Section 2 producing a counterpart of (3.2) yields an integrand with sign changed which immediately supplies the lower bound on all possible solutions. In particular, once the upper bound is established then there is an upper bound for x and we then use J to secure that lower bound. It is most rewarding to see how this is done in Example 5.4; we could not have made x dominate cx^4/J had it not been for the upper bound established for x.

One of the pleasant properties of both of the theorems of Section 4 is that the intervals $[0, E]$ are arbitrarily large. This is a wonderful property promoted by Schaefer's theorem. So often other methods require that we get a solution on a short interval and then extend it over and over again to get a solution on a desired interval. Miller [21, pp. 93–98] describes the process.

We are going to present some classical examples in which that is the traditional case, but the transformation and Schaefer's theorem give simple and clean results.

Example 5.1. In the study of turbulence, Consiglio [13] (see also Miller [21, p. 72]) considered the scalar equation

$$x(t) = \frac{\lambda}{L}\left(1 - \int_0^t (t-s)^{-1/2} x^2(s)\, ds\right) \tag{5.1}$$

when $\lambda = 1$ and L is any positive constant.

Theorem 5.2. *A solution $x(t)$ of* (5.1) *when $\lambda = 1$ exists on every interval* $[0, E]$ *and it satisfies* $0 < x(t) \le 1/L$.

Proof. Consider (5.1) for any fixed $\lambda \in (0, 1]$.

1. First note that if a solution of (5.1) exists, then it is bounded above by $1/L$.

2. But notice in (5.1) that nothing appears to be stopping x from becoming negative. To counter that possibility, apply the transformation of Section 2 to (5.1). Observing that (5.1) can be obtained from (3.4λ) by letting $H(t) = 1/L$, $q = 1/2$, and $h(t,x) = x^2/L$, we see that this work was actually done in Section 4 when we transformed (3.4λ) into (4.5). Using the above H and h in (4.5), we obtain the transformed equation

$$x(t) = \frac{\lambda}{L}\left(1 - \int_0^t R(s)\,ds\right) + \int_0^t R(t-s)\left(x(s) - \frac{x^2(s)}{JL}\right) ds,$$

where $R(t)$ denotes the resolvent for the kernel $C(t) = \lambda J t^{-1/2}$.

3. Let $J = 2/L^2$. Then, for $0 < x(s) \le 1/L$, we have

$$x(s) - \frac{x^2(s)}{JL} = x(s)\left[1 - \frac{x(s)}{2/L}\right] > 0,$$

since

$$\frac{x(s)}{2/L} \le \frac{1/L}{2/L} = 1/2.$$

It follows that should a solution $x(t)$ of (5.1) exist, then the right-hand side of the above transformed equation must be positive on some interval to the right of $t = 0$ as $x(0) = \lambda/L > 0$. Consequently, we can argue as in the proofs of Theorems 2.2 and 4.1 that

$$0 < x(t) \le \frac{1}{L}$$

for $t \ge 0$ so long as this solution exists. In other words, we have obtained an *a priori* bound on possible solutions of (5.1).

This bound also follows from Theorem 4.1 since $h(t,x) = x^2/L$ satisfies the positivity condition and (4.3) is satisfied on any interval $[0,E]$ with $k = 1/L^2$. Clearly $H(t) = 1/L$ satisfies condition (4.4).

We are now in a position to apply Schaefer's fixed point theorem via Theorem 4.2. Since the conditions of this theorem are satisfied with the functions $H(t) = 1/L$, $h(t,x) = x^2/L$, and $b(t) = 1/L$, it follows that (5.1) with $\lambda = 1$ has a solution in the strip of functions $\{x(t) : 0 < x(t) \le L,\ t \in [0,E]\}$ for any $E > 0$. □

We will now show that solutions are unique and that will then allow us to say that there is a solution on $[0,\infty)$ satisfying $0 < x(t) \le 1/L$.

Theorem 5.3. *There is at most one solution of* (5.1) *when* $\lambda = 1$ *on any interval* $[0,E]$.

Proof. Let $\lambda = 1$ in (5.1). By way of contradiction, suppose that there are distinct solutions x_1 and x_2 of (5.1) on an interval $[0, E]$. Then from the transformed equation in the previous proof, we obtain

$$\begin{aligned} |x_1(t) - x_2(t)| &= \left| \int_0^t R(t-s)\left(x_1(s) - \frac{x_1^2(s)}{JL} - x_2(s) + \frac{x_2^2(s)}{JL}\right) ds \right| \\ &\le \int_0^t R(t-s)\left| x_1(s) - x_2(s) - \frac{x_1^2(s) - x_2^2(s)}{JL} \right| ds, \end{aligned}$$

where

$$\begin{aligned} &\left| x_1(s) - x_2(s) - \frac{x_1^2(s) - x_2^2(s)}{JL} \right| \\ &= \left| \big(x_1(s) - x_2(s)\big)\left(1 - \frac{x_1(s) + x_2(s)}{JL}\right) \right| \\ &= |x_1(s) - x_2(s)| \left[1 - \frac{x_1(s) + x_2(s)}{JL}\right] \le |x_1(s) - x_2(s)| \end{aligned}$$

if J is chosen large enough so that

$$\frac{x_1(s) + x_2(s)}{JL} < 1.$$

This yields

$$|x_1(t) - x_2(t)| \le \int_0^t R(t-s)|x_1(s) - x_2(s)|\, ds$$

for $0 \le t \le E$. Now the supremum of the left side is achieved at some $t_1 \in [0, E]$ so that

$$\|x_1 - x_2\|_E \le \|x_1 - x_2\|_E \int_0^E R(s)\, ds,$$

a contradiction. □

Theorem 5.3 also follows from Theorem 4.7. Since a solution $x(t)$ of (5.1) must satisfy $0 < x(t) \le 1/L$, the reader can check that condition (4.6) is satisfied with $J = 4/L^2$ and $K = 1/2$.

Example 5.4. This example is adapted from a heat transfer problem studied by Miller [21, pp. 207–209] which we write as

$$x(t) = -c \int_0^t (t-s)^{-1/2}(x^4(s) - r^4)\, ds, \tag{5.2}$$

where c and r are positive constants. It will illustrate further interplay between the two forms analogous to (1.2) and (2.7). This problem is different than Example 5.1 because $x(0) = 0$ and it seems unclear if $x(t)$ increases or decreases. It will require the counterpart of (2.7) to show that $x(t)$ increases. Once we get that information, then we will go back to (5.2) to get the upper bound. This upper bound will then restrict x and enable us to use J to make x dominate cx^4/J, thereby obtaining a lower bound from the counterpart of (3.2).

Inserting the constant $\lambda \in (0, 1]$ in (5.2) and integrating the second term, we obtain

$$x(t) = \lambda\left[2cr^4\sqrt{t} - c\int_0^t (t-s)^{-1/2}x^4(s)\,ds\right]. \tag{5.3}$$

In the notation of (3.4λ), $H(t) = 2cr^4\sqrt{t}$ and $h(s, x(s)) = cx^4(s)$. Accordingly, we obtain from (4.5) the transformed equation

$$x(t) =\lambda\left[2cr^4\sqrt{t} - 2cr^4\int_0^t R(t-s)\sqrt{s}\,ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{cx^4(s)}{J}\right] ds, \tag{5.4}$$

where $R(t)$ is the resolvent for the kernel $C(t) = \lambda J t^{-1/2}$. As $\sqrt{t}$ is increasing and $\int_0^\infty R(s)\,ds = 1$, we see that

$$2cr^4\left[\sqrt{t} - \int_0^t R(t-s)\sqrt{s}\,ds\right] > 0$$

for $t > 0$.

Theorem 5.5. *Equation* (5.2) *has a unique solution on* $[0, \infty)$ *and for any* $E > 0$ *the solution on* $[0, E]$ *satisfies* $0 \le x(t) \le 2cr^4\sqrt{E}$. *The solution is positive on* $(0, \infty)$.

Proof.

1. Suppose there is a solution $x(t)$ of (5.2). Recall by a solution we mean a continuous function satisfying the equation. Hence, as $x(0) = 0$ and $r > 0$, $x^4(s) - r^4 < 0$ on an interval $(0, T]$ for some $T > 0$. It follows that the right-hand side of (5.2) is positive on this interval.

2. Since the solution $x(t)$ is initially positive, we see from (5.3) that it is bounded above by $2cr^4\sqrt{t}$ so long as $x(t) > 0$.

3. Now let $[0, E]$ be given and take J sufficiently large so that $x > cx^4/J$ for $0 < x \le 2cr^4\sqrt{E}$. This implies that the integrand in the last term of (5.4) is positive so long as $x(t)$ is positive. Thus, by adapting the proof of Theorem 4.1 to the interval $(0, E]$, we see that $x(t)$ is always positive and $0 < x(t) \le 2cr^4\sqrt{t}$ for $0 < t \le E$. Hence, $0 \le x(t) \le 2cr^4\sqrt{t}$ is an *a priori* bound for any solution of (5.2) on $[0, E]$.

4. Notice that the proof of Theorem 4.2 remains valid if the strict inequalities at $t = 0$ are replaced with equalities. Consequently, with $b(t) = 2cr^4\sqrt{t}$, we conclude

from the theorem that there is a solution $x(t)$ of (5.2) on every interval $[0, E]$ and that $0 \le x(t) \le 2cr^4\sqrt{E}$. Uniqueness of this solution follows from the proof of Theorem 4.7 with $h(t,x) = cx^4$ (cf. (5.4)). □

Example 5.6. Equation (1.1) inverts as (1.2) and it is both a surprise and a source of frustration that the initial condition does not yield $x(0) = x^0$, as we expect throughout the entire theory of differential equations.

In order to rectify this, Caputo offered a different definition of the fractional derivative so that his equation

$$^cD^q x(t) = -h(t, x(t)), \quad x(0) \in \Re, \quad 0 < q < 1, \tag{5.5}$$

inverts for all continuous h as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s, x(s))\, ds, \tag{5.6}$$

where Caputo's fractional derivative of order q is defined by

$$^cD^q x(t) := D^q[x - x(0)](t).$$

Recall from Section 1 that D^q denotes the Riemann-Liouville fractional differential operator of order q; thus

$$^cD^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q}[x(s) - x(0)]\, ds.$$

See, for example, Diethelm [14, pp. 50, 86].

It turns out that much is gained and much is lost by this change but today in applied mathematics it is difficult to say which of (1.1) and (5.5) is more useful in mathematical models. The transformation of Section 2 was actually developed for (5.6) and details are found in [9].

As we mentioned earlier in Section 1, it is interesting to note that in the two-step process the arguments are independent of λ. The reader is invited to insert the constant λ in both (5.6) and in

$$x(t) = x(0)\left[1 - \int_0^t R(s)\, ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{h(s, x(s))}{J}\right] ds, \tag{5.7}$$

which is the result of applying the transformation of Section 2 to (5.6) and then using the nonlinear variation of parameters formula. Here $R(t)$ denotes the resolvent for the kernel $C(t) = Jt^{q-1}/\Gamma(q)$.

Theorem 5.7. *Suppose there are positive constants $x(0)$, J, K, and E such that the relation*

$$0 < \frac{h(t,x)}{Jx} \le K < 1$$

holds for $0 < x \le x(0)$ *and* $0 \le t \le E$. *Then* (5.6) *has a positive solution on* $[0, E]$. *If the relation holds for all* $E > 0$ *and if each solution on* $[0, E]$ *is unique, then there is a positive solution on* $[0, \infty)$ *and it resides in the strip* $0 < x \le x(0)$.

Proof. Should a solution $x(t)$ of equation (5.6) exist, then it follows from the relation that $x(t)$ is bounded above by $x(0)$ so long as it remains positive. Also, we see from the relation that

$$x(s) - \frac{h(s, x(s))}{J} = x(s)\left[1 - \frac{h(s, x(s))}{Jx(s)}\right] > 0$$

when $0 < x(s) \le x(0)$. Thus, we can argue as in Example 5.1 that if $x(t)$ is a solution of (5.6), then $0 < x(t) \le x(0)$ throughout the interval of its existence.

The remainder of the proof follows the same line of reasoning as in Example 5.1. □

Example 5.8. One of the much sought properties in applied mathematics is the existence of a positive periodic solution. It is known ([12,16]) that neither Riemann-Liouville nor Caputo equations can have a periodic solution, but asymptotically periodic solutions occur in a natural way. Consider the equation

$$ {}^{c}D^{q}x(t) = -a(t)x(t) + p(t), \quad 0 < q < 1, \quad x(0) \in \Re \tag{5.8}$$

with $a, p \colon \Re \to \Re$ both continuous and suppose there is a $T > 0$ with $a(t + T) = a(t)$ and $p(t + T) = p(t)$.

Theorem 5.9. *Let the conditions with* (5.8) *hold. Additionally, suppose that* a *and* p *are positive. If* $x(0) > 0$, *then* (5.8) *has a unique solution* x *and it is positive. Furthermore, there is a* T*-periodic function* $g \colon \Re \to \Re$ *such that* $|x(t) - g(t)| \to 0$ *as* $t \to \infty$.

Proof. Burton and Zhang [12, Thm. 6.1] show that for every $x(0) \in \Re$ a unique solution x of (5.8) exists and that there is a T-periodic function g with $|x(t) - g(t)| \to 0$ as $t \to \infty$. So all we must show is that $x(t)$ is positive when $x(0) > 0$. Inverting equation (5.8), we obtain

$$x(t) = x(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)\,ds - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}a(s)x(s)\,ds. \tag{5.9}$$

From (5.9) we see that any solution $x(t)$ is bounded above by the continuous function

$$x(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)\,ds$$

so long as it remains positive (recall $x(0) > 0$ by hypothesis).

From (5.6) and (5.7) we readily see that the result of applying the transformation developed in Section 2 to (5.9) is

$$x(t) = x(0)\left[1 - \int_0^t R(s)\,ds\right] + \int_0^t R(t-s)\,\frac{p(s)}{J}\,ds + \int_0^t R(t-s)\left[x(s) - \frac{a(s)}{J}x(s)\right]ds.$$

So if we choose $J > \|a\|$, then the last integral in this display remains positive for $x > 0$. Hence, $x(t)$ is always positive. □

We illustrate this theorem by letting $q = 1/2$, $x(0) = 1$, $a(t) \equiv 1$, and $p(t) = 1 + 0.5\,\sin(4t)$ in (5.8). For these values and functions, we find in [4, (7.8)] that the solution of (5.8) is given by

$$x(t) = E_q(-t^q)\,x(0) + q\int_0^t p(t-s)s^{q-1}E_q'(-s^q)\,ds,$$

where E_q denotes the Mittag-Leffler function of order $q = 1/2$ and E_q' its derivative. Figure 1 shows the graph of this solution, which was drawn with the computer algebra system *Maple*. Similar graphs can be drawn using other values of $q \in (0,1)$.

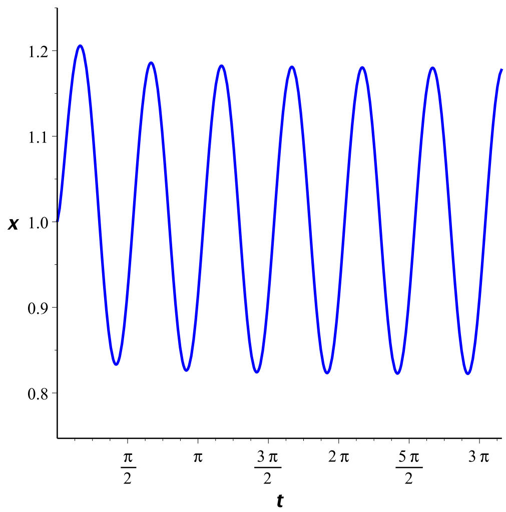

Fig. 1. A solution of (5.8)

Example 5.10. Logistic type problems are very common in applied mathematics. The classical version is an ordinary differential equation

$$x' = ax - bx^2$$

with a and b positive constants. There are two constant solutions: $x = 0$, the empty population, and $x = a/b$, the so-called carrying capacity of the medium. A population with $x(0) > a/b$ decreases and approaches a/b, while a population with $0 < x(0) < a/b$ increases and approaches a/b.

Recently several authors have studied a logistic equation of Caputo type. All of those results involve some kind of special functions and we think of them as quantitative. Thus, we offer a qualitative result by means of Schaefer's theorem as both a companion and a contrast. Khader and Babatin [17] study logistic equations as fractional differential equations of Caputo type obtaining approximate solutions using Laguerre polynomials. El-Sayed *et al.* [15] use series to obtain several different properties. West [28] offers an exact solution, arousing some controversy in Area *et al.* [3].

Here we consider the following logistic equation of Caputo type:

$$^{c}D^{q}x(t) = ax(t) - bx^2(t), \quad 0 < q < 1, \quad x(0) > 0, \tag{5.10}$$

where $a > 0$ and $b > 0$. This inverts as

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[ax(s) - bx^2(s)]\, ds. \tag{5.11}$$

Observe that the constant solution $x(t) \equiv x(0)$ with $x(0) = a/b$ is a solution of (5.10) and (5.11), just as it is of the classical logistic equation. Our goal is to show that there is a unique solution corresponding to every initial value $x(0) > a/b$ and that it remains above $x = a/b$ for all $t \geq 0$. That is, a solution starting above the carrying capacity a/b will never drop below that value. If (5.10) were an ordinary differential equation, then uniqueness would tell us that the solution starting above the carrying capacity would never cross it. But the same can not be said for a general integral equation; so the conclusion of the next theorem does not seem obvious.

Theorem 5.11. *If $x(0) > a/b$, then there exists a unique solution $x(t)$ of* (5.10) *and*

$$\frac{a}{b} < x(t) \leq x(0).$$

for all $t \geq 0$.

Proof. Assume $x(0) > a/b$. Expressing (5.11) in terms of

$$w(t) := x(t) - \frac{a}{b},$$

we obtain

$$w(t) + \frac{a}{b} = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\left[a\left(w(s) + \frac{a}{b}\right) - b\left(w(s) + \frac{a}{b}\right)^2\right] ds,$$

which simplifies to

$$w(t) = x(0) - \frac{a}{b} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[aw(s) + bw^2(s)]\, ds. \tag{5.12}$$

Since $a, b > 0$ and $x(0) > a/b$, it is clear that $w(t) \leq x(0) - a/b$ so long as $w(t) > 0$. In other words, $x(t) \leq x(0)$ so long as $x(t) - a/b > 0$.

Inserting a constant $\lambda \in (0,1]$ in (5.12) and then applying the transformation of Section 2, we obtain

$$\begin{aligned} w(t) = \lambda\Big[x(0) - \frac{a}{b}\Big]\left[1 - \int_0^t R(s)\,ds\right] \\ + \int_0^t R(t-s)\left[w(s) - \frac{aw(s)+bw^2(s)}{J}\right] ds. \end{aligned} \tag{5.13}$$

Now suppose that $w(t)$ is a solution of (5.13) on an interval $[0,E]$. Choose J large enough so that the integrand of the second term is positive on $[0,E]$. As a result, the right-hand side is positive on $[0,E]$. And so the solution $w(t)$ is positive throughout $[0,E]$.

Therefore the above work shows that any solution $w(t)$ on an interval $[0,E]$ must lie in the strip of functions that are bounded above by $x(0) - a/b$ and strictly below by 0. Thus we have obtained an *a priori* bound for any solution on $[0,E]$. Schaefer's theorem or a variant of Theorem 4.2 leads to the conclusion that (5.13) has a solution on $[0,E]$. Furthermore the solution is unique because (4.6) holds in the strip. Finally, following previous arguments we conclude that (5.10) has a unique solution $x(t)$ on $[0,\infty)$ and it resides in the strip bounded above by $x(0)$ and strictly below by a/b. □

We are also interested in the existence and behavior of solutions when the initial value $x(0)$ is located in the strip $0 < x < a/b$. Our analysis begins by setting $J = a$ in (5.13). Then we have

$$w(t) = \lambda\Big[x(0) - \frac{a}{b}\Big]\left[1 - \int_0^t R(s)\,ds\right] - \frac{b}{a}\int_0^t R(t-s)w^2(s)\,ds.$$

Since $x(0) < a/b$, we see from the right-hand side that if this equation has a solution, then $w(t) < 0$ throughout its interval of existence. In other words, if a solution $x(t)$ of (5.11) with a positive initial value $x(0) < a/b$ exists, then $x(t) < a/b$. Consequently, with (5.11) rewritten as

$$x(t) = x(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} b x(s)\left[\frac{a}{b} - x(s)\right] ds,$$

we can see that this implies $x(t) > 0$ for as long as this solution exists.

Incidentally, we have once again obtained a strip in which solutions must reside should they exist. Consequently, we can now establish existence as in previous examples with Schaefer's theorem and thereby conclude that corresponding to an initial value $x(0) \in (0, a/b)$ there is a solution on $[0, \infty)$ residing in that strip.

Example 5.12. In classic papers, Mann and Wolf [20], Padmavally [24], and Roberts and Mann [26] studied the temperature $u(x,t)$ in a semi-infinite rod by means of the integral equation

$$u(0,t) = \frac{1}{\pi^{1/2}} \int_0^t (t-s)^{-1/2} G(u(0,s))\, ds, \tag{5.14}$$

where $G(u)$ is continuous and strictly decreasing with $G(1) = 0$. Miller's work in Example 5.4 was related to it, although he was seeking conditions for the resolvent kernel.

Theorem 5.13. *Let $v(y) := -G(y+1)$ and suppose there is a $J > 0$ so that for $-1 \le y < 0$ we have*

$$y - \frac{v(y)}{J} \le 0. \tag{5.15}$$

Then for each $E > 0$ there is a solution of (5.14) on $[0, E]$ and

$$0 \le u(0,t) < 1$$

for all $t \in [0, E]$.

Proof. We change notation to put (5.14) into the form of this paper. Let $u(0,t) := x(t)$ and $y(t) := x(t) - 1$ so that for $\lambda = 1$ we have

$$y(t) = \lambda \left[-1 + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} G(y(s)+1)\, ds \right].$$

Then from $v(y) = -G(y+1)$ we see that $v(0) = 0$, $v(y)$ is strictly increasing, $yv(y) > 0$ for $y \neq 0$, and

$$y(t) = \lambda \left[-1 - \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} v(y(s))\, ds \right].$$

Notice that $y(0) = -\lambda$, while the integrand is negative so long as $y(s) < 0$. Hence, $y = -\lambda$ is a lower bound of any possible solution so long as it remains negative. The transformation of Section 2 yields

$$y(t) = -\lambda \left[1 - \int_0^t R(s)\, ds \right] + \int_0^t R(t-s) \left[y(s) - \frac{v(y(s))}{J} \right] ds.$$

From (5.15) it is now clear that $y(t)$ does not have a zero because $1 - \int_0^t R(s)\, ds > 0$ and the integrand is always negative when $y(s) < 0$.

All of this establishes that if there is a solution on any interval $[0, E]$ then

$$-\lambda \le y(t) < 0.$$

This means that $-\lambda \le u(0,t) - 1 < 0$; so $0 \le u(0,t) < 1$ for all $t \in [0, E]$. This is the *a priori* bound needed in Schaefer's theorem. □

6. APPENDIX

We have assumed that there is a solution of (1.2) on a short interval and that is a cornerstone of one of our main results in this paper, namely Theorem 3.1. We have referred to several papers offering existence theorems. Here is one which can be found in [8, Thm. 3.1].

Theorem 6.1. *Let $q \in (0,1)$ and $x^0 \in \Re$ with $x_0 \neq 0$. Let $r_1 > -1$ and $r_2 \ge 0$ be constants that satisfy the inequality*

$$r_1 - r_2 + q\,(r_2 + 1) > 0. \tag{6.1}$$

Let $f\colon (0,\infty) \times \Re \to \Re$ be continuous. Suppose there are nonnegative constants K_1 and K_2 such that

$$|f(t,x)| \le K_1 + K_2\, t^{r_1} |x|^{r_2} \tag{6.2}$$

for $x \in \Re$ and $0 < t < T_0$, where $T_0 \in (0,\infty]$. Then for some $T \in (0, T_0)$ there is a continuous function $x\colon (0,T] \to \Re$ that satisfies the integral equation

$$x(t) = x^0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{6.3}$$

on $(0,T]$. Furthermore, $|x(t)| \le 2|x^0| t^{q-1}$ for $t \in (0,T]$.

The next result is found in [6] and is offered here for reference as an exact statement of existence.

Theorem 6.2. *Let $f\colon [0,T] \times \Re \to \Re$ be continuous and satisfy the Lipschitz condition*

$$|f(t,x) - f(t,y)| \le K_2 |x - y|$$

for some $K_2 > 0$. Then, for each $q \in (0,1)$, there is a $T_0 \in (0,T]$ such that (1.2) has a unique continuous solution ϕ on $(0, T_0]$ with

$$\lim_{t \to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} f(s, \phi(s))\, ds = 0, \qquad \lim_{t \to 0^+} t^{1-q} \phi(t) = x^0.$$

Finally, both $\phi(t)$ and $f(t, \phi(t))$ are absolutely integrable.

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