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MATRIX POLYNOMIALS ORTHOGONAL WITH RESPECT TO A NON-SYMMETRIC MATRIX OF MEASURES

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Abstract. The paper focuses on matrix-valued polynomials satisfying a three-term recurrence relation with constant matrix coefficients. It is shown that they form an orthogonal system with respect to a matrix of measures, not necessarily symmetric. Moreover, it is stated the condition on the coefficients of the recurrence formula for which the matrix measure is symmetric.

Keywords: matrix orthogonal polynomials, recurrence formula, matrix of measures, block Jacobi matrices.

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1. INTRODUCTION

The theory of matrix orthogonal polynomials (which will be denoted as MOP) began with the works of Berezanski [1] and Krein [9]. The turn of millennia saw a flurry of articles, among them for instance [2,4–7,11,13,14]. They consider only positive definite matrices of measures of orthogonality. MOPs – which are known to satisfy a three-term recurrence relation of type (1.3) – are investigated for example in the theory of random walks on graphs or in the theory of multiple birth-and-death processes (cf. [5,8]), where such thre term formulae appear. Many properties of random walk or birth-and-death processes can be obtained from the matrix of measures of orthogonality. Investigation of more general classes of such processes require application of matrices of measures which are non symmetric, see [15]. Theorem 1.2 shows conditions on coefficients in (1.3) to provide the positive definiteness.

The case of non symmetric matrices of measure of orthogonality was discussed for example already by Naimark [10], although not in an explicit form. But still there is very little on the subject in the literature. In this paper we show that for constant terms in a three-term recurrence relation of type (1.3) there exists a matrix of measures, which are the measures of orthogonality for polynomials generated by (1.3).

Let $\{P_n\}$ be a family of matrix-valued polynomials such that $\deg P_n = n$ and the leading coefficient of P_n is invertible. In the common sense the family $\{P_n\}$ is said to be orthogonal if there is a matrix of measures Σ supported on a bounded set Δ such that the integral

$$\int_{\Delta} P_n(x)\Sigma(dx)P_m(x)^* = 0 \tag{1.1}$$

for $n \neq m$. In the literature on the subject – as it was said above – they consider mostly positive definite matrices of measures of orthogonality. In [6] was shown that in this case the generalization of Favard's theorem holds which can be stated as follows (cf. [6, Theorem 3.1]).

Theorem 1.1. *Let matrix valued polynomials P_n satisfy the three-term recurrence formula*

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x), \tag{1.2}$$

where matrices A_n are invertible and B_n self-adjoint. Then there exists a positive definite matrix of measures Σ satisfying (1.1). *Moreover, those polynomials are orthonormal, i.e. the following equality holds*

$$\int_{\mathbb{R}} P_n(x)\Sigma(dx)P_n(x)^* = I,$$

where I stands for the identity matrix (*note that in this case the orthogonality matrix measure is supported on the real line* $\mathbb{R}$).

If the normalization is not required, polynomials P_n fulfill more general recurrence formula (cf. [4, 5, 9]):

$$xP_n(x) = A_nP_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x),$$

where all matrices A_n and C_n are invertible. But not for all sets of matrices A_n, B_n, C_n there exists a positive definite matrix of measures which satisfy the orthogonality relation (1.1). In fact the following theorem holds to be true (the proof is given in the next section).

Theorem 1.2. *Let P_n be the family of matrix polynomials satisfying the recurrence relation*

$$xP_n(x) = A_nP_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x), \tag{1.3}$$

with $P_0(x) = I$ and $P_{-1}(x) = 0$, where all matrices A_n and C_n are invertible. Then there exists a positive definite matrix of measure, with respect to which the polynomials P_n are orthogonal, if and only if matrices $A_0 \cdots A_nC_{n+1} \cdots C_1$ and $A_0 \cdots A_{n-1}B_nC_n \cdots C_1$ are self-adjoint for every n.

In this paper we will use another way of defining orthogonality, which is of course equivalent in the scalar case: we will say that polynomials P_n are orthogonal with respect to the matrix of measures Σ if

$$\int\limits_{\Delta} x^k \, P_n(x) \, \Sigma(dx) = 0 \tag{1.4}$$

for all $n > k \geq 0$. The condition above means that polynomial P_n is orthogonal to the subspace of all polynomials of degree less than or equal to $n-1$. The reader can verify that both ways to define orthogonality used for example in such textbooks on classical orthogonal polynomials as [3] and [12].

In the matrix case the family of matrix valued polynomials, given by the Gramm–Schmidt procedure from the system of monomials $\{I, x, x^2, \dots\}$ orthogonalized with respect to a positive definite matrix of measures, are not only orthogonal to the subspace of polynomials of lower degree, but such polynomials are orthogonal to each other, i.e. they satisfy (1.1). This is due to the fact that the matrix valued bilinear form in this case is symmetric in the sense that

$$\left(\int\limits_{\mathbb{R}} Q(x)\Sigma(dx)\, R(x) \right)^* = \int\limits_{\mathbb{R}} R(x)^*\Sigma(dx)\, Q(x)^*, \tag{1.5}$$

for any polynomials Q and R.

In the whole generality the matrix of measures does not need to be symmetric, so (1.5) does not hold any more. What we can have is only (1.1) for $n > m$. Indeed, if (1.4) holds, then (1.1) for $n > m$ holds too, as every polynomial ${P_m}^*$ has degree lower than n. And vice-versa, every monomial x^m is a linear combination of polynomials ${P_0}^*, {P_1}^*, \dots, {P_m}^*$, hence (1.4) holds if only (1.1) holds for $n > m$. So in their full generality the formulas (1.1) and (1.4) do not need to be equivalent.

2. PROOF OF THE THEOREM 1.2

Proof of the Theorem 1.2. Let P_n be MOP, orthogonal with respect to a positive definite matrix of measures Σ and let them satisfy the recurrence formula (1.3). The condition that all coefficients A_n and C_n are invertible provides the fact that integral $\int Q(x)\, \Sigma(dx)\, Q(x)^*$ is a positive (strictly) definite matrix for all non zero polynomials Q, which do not have a non trivial constant null subspace (cf. §2 of [4], in particular Lemma 2.1).

In this case the corresponding orthonormal matrix polynomials (which we denote here by P_n^o) satisfy $P_n(x) = V_n P_n^o(x)$, where

$$V_n {V_n}^* = \left(\int\limits_{\Delta} P_n(x)\Sigma(dx)P_n(x)^* \right)^{-1}.$$

If we require $P_0 = I$, then we need the condition $V_0 = I$. But if $P_0 \neq I$, then we can consider polynomials $\hat{P}^o_n = P^o_n P_0$ which are orthonormal with respect to the measure ${P_0}^{-1} \Sigma {P_0^*}^{-1}$. Hence we can assume that $V_0 = I$.

Orthonormal polynomials P^o_n fulfill the recurrence formula

$$xP^o_n(x) = A^o_n P^o_{n+1}(x) + B^o_n P^o_n(x) + (A^o_{n-1})^* P^o_{n-1}(x), \text{ with } P^o_0(x) = I, \ P^o_{-1}(x) = 0,$$

with suitable coefficients A^o_n and B^o_n. Putting $V_n P^o_n(x)$ into (1.2) leads to the formula

$$xP^o_n(x) = {V_n}^{-1} A_n V_{n+1} P^o_{n+1}(x) + {V_n}^{-1} B_n V_n P^o_n(x) + {V_n}^{-1} C_n V_{n-1} P^o_{n-1}(x),$$

so

$$A^o_n = {V_{n-1}}^{-1} A_{n-1} V_n = \left({V_n}^{-1} C_n V_{n-1}\right)^*$$

and

$$B^o_n = {V_n}^{-1} B_n V_n = \left({V_n}^{-1} B_n V_n\right)^* . \tag{2.1}$$

Hence

$$V_n {V_n}^* = {A_{n-1}}^{-1} V_{n-1} V^*_{n-1} {C_n}^* = \cdots = {A_{n-1}}^{-1} \cdots {A_0}^{-1} {C_1}^* \cdots {C_n}^*.$$

But the left-hand side is a hermitian matrix, so the right-hand side is hermitian as well. Thus

$${A_{n-1}}^{-1} \cdots {A_0}^{-1} {C_1}^* \cdots {C_n}^* = C_n \cdots C_1 \left({A_0}^{-1}\right)^* \cdots \left({A_{n-1}}^{-1}\right)^*,$$

which leads to

$$A_0 \cdots A_{n-1} C_n \cdots C_1 = \Big(A_0 \cdots A_{n-1} C_n \cdots C_1\Big)^*.$$

Now (2.1) gives

$$B_n\left(V_n {V_n}^*\right) = \left(V_n {V_n}^*\right) {B_n}^*.$$

Combining it with the previous one shows that

$$A_0 \cdots A_{n-1} B_n C_n \cdots C_1 = \Big(A_0 \cdots A_{n-1} B_n C_n \cdots C_1\Big)^*.$$

To prove the implication in the opposite direction assume now that the matrices A_n, B_n and C_n fulfill the conditions in the thesis, i.e.

$$A_0 \cdots A_{n-1} C_n \cdots C_1 = \left(A_0 \cdots A_{n-1} C_n \cdots C_1\right)^* = {C_n}^* \cdots {C_1}^* {A_{n-1}}^* \ldots {A_0}^*$$

and

$$A_0 \cdots A_{n-1} B_n C_n \cdots C_1 = \left(A_0 \cdots A_{n-1} B_n C_n \cdots C_1\right)^* = {C_n}^* \cdots {C_1}^* {B_n}^* {A_{n-1}}^* \ldots {A_0}^*.$$

Then

$$C_n \cdots C_1 (A_0^{-1})^* \ldots ({A_{n-1}}^{-1})^* = {A_{n-1}}^{-1} \cdots {A_0}^{-1} {C_1}^* \cdots {C_n}^*$$

and

$$B_n C_n \cdots C_1 (A_0^{-1})^* \dots (A_{n-1}{}^{-1})^* = A_{n-1}{}^{-1} \cdots A_0{}^{-1} C_n{}^* \cdots C_1{}^* B_n{}^*,$$

which shows that the left-hand sides of both equations are self-adjoint and invertible matrices. Hence the first one can be rewritten as a product $W_n W_n{}^*$, where W_n is an invertible matrix satisfying $W_n W_n{}^* = A_{n-1}{}^{-1} W_{n-1} W_{n-1}{}^* C_n{}^*$ and the other as $B_n\big(W_n W_n{}^*\big) = \big(W_n W_n{}^*\big) B_n{}^*$. This shows that the polynomials $W_n{}^{-1} P_n$ fulfill the recurrence formula (1.2). By Theorem 1.1, polynomials P_n are orthonormal with respect to a positive definite matrix of measures. So polynomials P_n are also orthogonal with respect to the same matrix-valued measure. □

Now a question arises: In a more general case, is there any matrix-valued measure Σ supported on an appropriate set Δ such that

$$\int_\Delta x^m \, P_n(x) \, \Sigma(dx) = 0$$

for $n > m \geq 0$?

3. GENERAL CASE OF CONSTANT COEFFICIENTS

3.1. MATRIX OF MEASURES

Let P_n be polynomials satisfying the recurrence formula of type (1.3) with constant coefficients, i.e.

$$xP_n(x) = AP_{n+1}(x) + BP_n(x) + CP_{n-1}(x) \tag{3.1}$$

with $P_0 = I$ and $P_1(x) = A^{-1}(x - B)$, where A and C are invertible. Hence $\deg P_n = n$. We will extend the equality (3.1) to all $n \geq 0$ assuming that $P_{-1}(x) = 0$, i.e., a zero matrix.

In the case when $C = A^*$ and B is a symmetric matrix, we already know that polynomials P_n are MOP with respect to a positive definite matrix of measures supported on the real line $\mathbb{R}$. In the following we will show that there exists a matrix of measures – of course not necessarily symmetric or with its support outside real line $\mathbb{R}$ – which is an orthogonality measure for P_n in the sense of (1.4).

Let us now introduce new polynomials R_n which satisfy

$$xR_n(x) = R_{n+1}(x)C + R_n(x)B + R_{n-1}(x)A \tag{3.2}$$

for $n \geq 1$, where $R_0(x) = I$ and $R_{-1}(x) = 0$. One can put the initial condition $R_1(x) = (x - B)C^{-1}$ instead of R_{-1}.

Lemma 3.1. *Let matrix valued polynomials P_n and R_n satisfy* (3.1) *and* (3.2)*, respectively. Then we have*

$$\begin{aligned} &P_n(z)A^{-1}R_{m+1}(z) - P_{n+1}(z)C^{-1}R_m(z) \\ &= P_{n+k}(z)A^{-1}R_{m+k+1}(z) - P_{n+k+1}(z)C^{-1}R_{m+k}(z) \end{aligned} \tag{3.3}$$

for any $m, n, k \in \mathbb{N}$.

Proof. Let

$$\mathcal{X}(x) = A^{-1}(x - B), \quad \mathcal{D} = A^{-1}C,$$

and

$$\hat{\mathcal{X}}(x) = (x - B)C^{-1}, \quad \hat{\mathcal{D}} = AC^{-1}.$$

The formulas (3.1) and (3.2) become as follows:

$$P_{n+1}(x) = \mathcal{X}(x)P_n(x) - \mathcal{D}P_{n-1}(x), \quad R_{n+1}(x) = R_n(x)\hat{\mathcal{X}}(x) - R_{n-1}(x)\hat{\mathcal{D}}.$$

But we have also

$$\begin{aligned} P_1(x) &= \mathcal{X}(x) = P_0(x)\,\mathcal{X}(x), \\ P_2(x) &= \mathcal{X}(x)^2 - \mathcal{D} = P_1(x)\,\mathcal{X}(x)^2 - P_0(x)\,\mathcal{D} \end{aligned}$$

and

$$\begin{aligned} R_1(x) &= \hat{\mathcal{X}}(x) = \hat{\mathcal{X}}(x)\,R_0(x), \\ R_2(x) &= \hat{\mathcal{X}}(x)^2 - \hat{\mathcal{D}} = \hat{\mathcal{X}}(x)\,R_1(x) - \hat{\mathcal{D}}\,R_0(x). \end{aligned}$$

By simply induction on the degree of polynomials one can see that

$$P_{n+1}(x) = P_n(x)\mathcal{X}(x) - P_{n-1}(x)\mathcal{D} \quad \text{and} \quad R_{n+1}(x) = \hat{\mathcal{X}}(x)R_n(x) - \hat{\mathcal{D}}R_{n-1}(x). \tag{3.4}$$

The equality (3.3) is a matter of applying (3.4), and simple calculations. □

Let us fix n for a while and let $\Sigma_{(n)}$ be a matrix of measures supported on zeros of P_n (denoted by $\Delta_n = \{\xi \in \mathbb{C} : \det(P_n(\xi)) = 0\}$) such that

$$\int_{\Delta_n} \frac{1}{z - \xi}\,\Sigma_{(n)}(d\xi) = P_n(z)^{-1}P_{n-1}(z)A^{-1}.$$

This means that

$$\Sigma_{(n)} = \sum_{\xi \in \Delta_n} W_\xi^{(n)}\,\delta_{\{\xi\}},$$

where $\delta_{\{\xi\}}$ is a well-known Dirac measure at ξ. The matrix coefficients $W_\xi^{(n)}$ can be calculated for instance from the Cauchy integral formula

$$W_{\xi_o}^{(n)} = \frac{1}{2\pi i} \int_{\Gamma_{\xi_o}} P_n(\zeta)^{-1}P_{n-1}(\zeta)A^{-1}\,d\zeta$$

provided that Γ_{ξ_o} is a simple closed curve enclosing ξ_o and leaving outside its contour other zeroes of P_n (note that ξ_o does not need to be a real number, as Examples 4.3 and 4.4 show). Indeed, as

$$P_n(z)^{-1}P_{n-1}(z)A^{-1} = \sum_{\xi \in \Delta_n} \frac{1}{z - \xi}\,W_\xi^{(n)},$$

we get

$$\frac{1}{2\pi i}\int_{\Gamma_{\xi_o}} P_n(\zeta)^{-1}P_{n-1}(\zeta)A^{-1}\,d\zeta = \sum_{\xi\in\Delta_n}\frac{1}{2\pi i}\int_{\Gamma_{\xi_o}}\frac{1}{\zeta-\xi}\,W_\xi^{(n)}\,d\zeta$$
$$= W_{\xi_o}^{(n)}\,\frac{1}{2\pi i}\int_{\Gamma_{\xi_o}}\frac{d\zeta}{\zeta-\xi_o} = W_{\xi_o}^{(n)}.$$

Let us now denote the moments of $\Sigma_{(n)}$ as

$$M_k^{(n)} = \int_\Delta \xi^k\,\Sigma_{(n)}(d\xi) = \sum_{\xi\in\Delta_n}\xi^k\,W_\xi^{(n)}$$

and let

$$F_{(n)}(z) = \int_\Delta \frac{1}{z-\xi}\,\Sigma_{(n)}(d\xi) = \sum_{\xi\in\Delta_n}\frac{1}{z-\xi}\,W_\xi^{(n)} = \sum_{k=0}^{+\infty}\frac{x^k}{z^{k+1}}\,M_k^{(n)}.$$

Lemma 3.2. *In the above notation*

$$F_{(n)}(z) = C^{-1}R_{n-1}(z)R_n(z)^{-1}$$

for all $n \geq 1$.

Proof. From Lemma 3.1 we have

$$P_{n-1}(z)A^{-1}R_n(z) - P_n(z)C^{-1}R_{n-1}(z) = P_0(z)A^{-1}R_1(z) - P_1(z)C^{-1}R_0(z) = 0.$$

Hence

$$P_n(z)F_{(n)}(z)R_n(z) = P_{n-1}(z)A^{-1}R_n(z) = P_n(z)C^{-1}R_{n-1}(z),$$

which leads to the conclusion. □

Let now $\Delta_{n,\infty} = \overline{\bigcup_{k\geq n}\Delta_k}$. It is clear that

$$\int_{\Delta_{n,\infty}} \xi^k\,P_n(\xi)\,\Sigma_{(n)}(d\xi) = 0.$$

We shall show the following result.

Theorem 3.3. *Let P_n and $\Sigma_{(n)}$ be as above. Then*

$$\int_{\Delta_{n,\infty}} \xi^k\,P_m(\xi)\,\Sigma_{(n)}(d\xi) = 0$$

for all $0 \leq k < m \leq n$.

Proof. Fix $n \geq 1$. Now

$$\begin{aligned} F_{(n+1)}(z) - F_{(n)}(z) &= P_{n+1}(z)^{-1} P_n(z) A^{-1} - C^{-1} R_{n-1}(z) R_n(z)^{-1} \\ &= P_{n+1}(z)^{-1} \Big(P_n(z) A^{-1} R_n(z) - P_{n+1}(z) C^{-1} R_{n-1}(z) \Big) R_n(z)^{-1} \end{aligned}$$

and by Lemma 3.1

$$= P_{n+1}(z)^{-1} A^{-1} R_n(z)^{-1}.$$

Hence

$$\lim_{z \to \infty} z^k \big(F_{(n+1)}(z) - F_{(n)}(z) \big) = 0$$

for $0 \leq k \leq 2n$. This means that $M_k^{(n+1)} = M_k^{(n)}$ for such k. Now the thesis of the Theorem follows from a simple induction argument and the fact that $\Delta_{n+1,\infty} \subset \Delta_{n,\infty}$. □

Now we are able to show the following result.

Theorem 3.4. *Let P_n be polynomials satisfying the recurrence formula* (3.1)*. Then there exists a matrix of measures Σ such that*

$$\int_\Delta \xi^k \, P_n(\xi) \, \Sigma(d\xi) = 0$$

for all integers $n > k \geq 0$, where

$$\Delta = \bigcap_{n=1}^{\infty} \Delta_{n,\infty} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} \Big\{ \xi \in \mathbb{C} \, : \, \det \big(P_m(\xi) \big) = 0 \Big\}}.$$

Proof. Let us fix $m \in \mathbb{N}$. Every matrix-valued polynomial P of degree m has the unique decomposition

$$P = \sum_{k=0}^{m} \mathcal{C}_{k,P} \, P_k.$$

Hence, by Theorem 3.3, we have

$$\int_{\Delta_{n,\infty}} P(\xi) \, \Sigma_{(n)}(d\xi) = \mathcal{C}_{0,P}$$

for all $n > m$. So the sequence of measures $\Sigma_{(n)}$ is weakly convergent on the space of all matrix-valued polynomials. Thus, by the Banach–Steinhaus theorem, it has the weak limit Σ such that we have

$$\int_{\Delta_{n,\infty}} P(\xi) \, \Sigma(d\xi) = \mathcal{C}_{0,P}$$

for all $n > m$. It is clear that $\mathrm{supp}\Sigma \subset \Delta_{n,\infty}$ for all $n \in \mathbb{N}$, hence

$$\mathrm{supp}\Sigma \subset \bigcap_{n=1}^{\infty} \Delta_{n,\infty}.$$

Now let us fix $n, k \in \mathbb{N}$, $n > k \geq 0$. Then

$$\int_{\Delta} \xi^k \, P_n(\xi) \, \Sigma(d\xi) = \lim_{m \to \infty} \int_{\Delta_{m,\infty}} \xi^k \, P_n(\xi) \, \Sigma_{(m)}(d\xi) = 0,$$

where the last equality holds by Theorem 3.3. □

3.2. CAUCHY TRANSFORM OF THE MATRIX OF MEASURES OF ORTHOGONALITY

Let us now define

$$F(z) = \int_{\Delta} \frac{1}{z - \xi} \, \Sigma(d\xi).$$

Function F is often called the *Cauchy Transform* of Σ. Knowing the function F allows us to find an explicit formula for the orthogonality measure in many cases.

Theorem 3.5. *Matrix valued function F satisfies the equation*

$$F(z) = \big(z - B - AF(z)C\big)^{-1}.$$

Proof. We have

$$\lim_{n \to \infty} F_{(n)}(z) = F(z)$$

for all z in a neighborhood of infinity. By analytic continuation the previous equality holds for all $z \in \mathbb{C} \setminus \Delta$.

Now, by (3.4), we have

$$P_n(z)^{-1} P_{n+1}(z) = \mathcal{X}(z) - P_n(z)^{-1} P_{n-1}(z) \mathcal{D},$$

hence

$$F_{(n+1)}(z)^{-1} = z - B - AF_{(n)}(z)C.$$

Taking the limit as n tends to the infinity leads to the thesis. □

Corollary 3.6. *We have*

$$F(z) = \cfrac{1}{z - B - A \cfrac{1}{z - B - A \frac{1}{z - B - \ddots} C} C}.$$

4. INSTRUCTIVE EXAMPLES

What follows are four examples which show really unexpected behaviour and properties of the orthogonality measure. We will always assume that $P_0(x) = I$, $P_{-1}(x) = 0$.

The first two examples are taken from [13].

Example 4.1. Let P_n be polynomials satisfying the recurrence formula

$$xP_n(x) = AP_{n+1}(x) + BP_n(x) + CP_{n-1}(x)$$

with

$$A = C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.$$

In this case it is not difficult to compute the corresponding Cauchy transform[1)], i.e.

$$F(z) = \frac{1}{2}\left(1 - \sqrt{\frac{z^2 - \gamma^2 - 4}{z^2 - \gamma^2}}\right) \begin{pmatrix} z & \gamma \\ \gamma & z \end{pmatrix}.$$

It is shown in [13] that polynomials R_n are orthogonal with respect to the matrix of measures

$$\Sigma_1(dx) = \frac{1}{2\pi} \sqrt{\frac{\gamma^2 + 4 - x^2}{x^2 - \gamma^2}} \begin{pmatrix} |x| & \gamma \\ \gamma & |x| \end{pmatrix} dx,$$

which is supported on the set

$$\Delta = \left[-\sqrt{\gamma^2 + 4}, -\gamma\right] \cup \left[\gamma, \sqrt{\gamma^2 + 4}\right].$$

This result can be achieved also by the Stieltjes-Perron's inversion formula from function F.

Example 4.2. Let P_n be polynomials satisfying the recurrence formula

$$xP_n(x) = AP_{n+1}(x) + CP_{n-1}(x)$$

with

$$A = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \qquad \text{where } \alpha > \beta > 0.$$

In this case the midterm coefficient B is equal to the zero matrix.

Direct calculation shows that

$$F(z) = \begin{pmatrix} \frac{\alpha^2-\beta^2}{2z\alpha^2} & 0 \\ 0 & \frac{\beta^2-\alpha^2}{2z\beta^2} \end{pmatrix} + \frac{z^2 - \sqrt{(z-\alpha-\beta)(z+\alpha-\beta)(z-\alpha+\beta)(z+\alpha+\beta)}}{2z} \begin{pmatrix} \frac{1}{\alpha^2} & 0 \\ 0 & \frac{1}{\beta^2} \end{pmatrix}.$$

[1)] This and other formulas for $F(z)$ in this section were calculated in *Mathematica* v.10.0, licence L4887-2121

Author in [13] shows that polynomials P_n are orthogonal with respect to the matrix of measures $\Sigma = \Sigma_0 + \Sigma_1$, where

$$\Sigma_0 = \begin{pmatrix} 1-\left(\frac{\beta}{\alpha}\right)^2 & 0 \\ 0 & 0 \end{pmatrix} \delta_0,$$

where δ_0 is Dirac measure at 0, and

$$\Sigma_1(dx) = \frac{\sqrt{((\alpha+\beta)^2 - x^2)\,(x^2-(\alpha-\beta)^2)}}{4\pi|x|} \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} dx.$$

The measure Σ is supported on the set

$$\Delta = [-(\alpha+\beta), -(\alpha-\beta)] \cup \{0\} \cup [\alpha-\beta, \alpha+\beta]\,.$$

Hence, the orthogonality measure can contain a non-empty atomic part.

Example 4.3. Let now P_n be polynomials satisfying the recurrence formula

$$xP_n(x) = AP_{n+1}(x) + CP_{n-1}(x)$$

with

$$A = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It can be easily seen that multiplying (4.3) on the left-hand side by matrix Λ and on the right-hand side by Λ^{-1}, where

$$\Lambda = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix},$$

leads to the formula

$$x\tilde{P}_n(x) = \tilde{A}\tilde{P}_{n+1}(x) + \tilde{C}\tilde{P}_{n-1}(x)$$

with

$$\tilde{A} = \tilde{C} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \tilde{P}_n(x) = \Lambda\, P_n(x)\, \Lambda^{-1}.$$

Hence $\tilde{P}_n$ has non-zero terms only on the diagonal, i.e. $\tilde{P}_n(x) = \begin{pmatrix} \tilde{p}_n^{1,1}(x) & 0 \\ 0 & \tilde{p}_n^{2,2}(x) \end{pmatrix}$, and those satisfy

$$-\frac{1}{2}ix\,\tilde{p}_n^{1,1}(x) = \frac{1}{2}\tilde{p}_{n+1}^{1,1}(x) + \frac{1}{2}\tilde{p}_{n-1}^{1,1}(x), \quad \frac{1}{2}ix\,\tilde{p}_n^{2,2}(x) = \frac{1}{2}\tilde{p}_{n+1}^{2,2}(x) + \frac{1}{2}\tilde{p}_{n-1}^{2,2}(x).$$

So

$$\tilde{p}_n^{1,1}(x) = U_n\Big(-\frac{1}{2}ix\Big) \quad \text{and} \quad \tilde{p}_n^{2,2}(x) = U_n\Big(-\frac{1}{2}ix\Big),$$

where scalar polynomials $U_n(x)$ are Chebyshev polynomials of the second kind and are known to be orthogonal with respect to the measure $\mu_u(dx) = \frac{2}{\pi}\sqrt{1-x^2}\,dx$ supported on the interval $[-1,1]$ (ie. polynomials $U_n(\frac{1}{2}ix)$ are orthogonal with respect to the measure $\mu_+(dx) = \frac{\sqrt{4+x^2}}{2\pi}\,dx$ supported on the segment of the line between points $-2i$ and $2i$ on complex plane $\mathbb{C}$, i.e. the set $\{z\in\mathbb{C} : z = -2i\lambda + 2i(1-\lambda),\ \text{for } \lambda\in[0,1]\}$, which we call later the interval $[-2i, 2i]$). So the polynomials P_n are equal to

$$P_n(x) = \begin{pmatrix} \frac{1}{2}\big(U_n(\frac{1}{2}ix) + U_n(-\frac{1}{2}ix)\big) & -\frac{i}{2}\big(U_n(\frac{1}{2}ix) - U_n(-\frac{1}{2}ix)\big) \\ \frac{i}{2}\big(U_n(\frac{1}{2}ix) - U_n(-\frac{1}{2}ix)\big) & \frac{1}{2}\big(U_n(\frac{1}{2}ix) + U_n(-\frac{1}{2}ix)\big) \end{pmatrix}$$

and they are orthogonal with respect to the matrix of measures $\Sigma = I\,\tilde{\mu}$, where $\tilde{\mu}(dx) = \frac{1}{2\pi}\sqrt{4+x^2}\,dx$. In this case the Cauchy transform equals to

$$F(z) = \frac{z - \sqrt{z^2+4}}{2}\, I.$$

Example 4.4. Let P_n be polynomials satisfying the recurrence formula

$$xP_n(x) = AP_{n+1}(x) + BP_n(x) + CP_{n-1}(x) \tag{4.1}$$

with

$$A = C = I \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As in Example 4.3 multiplying (4.1) by matrices Λ and Λ^{-1} leads to

$$x\hat{P}_n(x) = \hat{P}_{n+1}(x) + \hat{B}\hat{P}_n(x) + \hat{P}_{n-1}(x),$$

with $\hat{B} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Hence the solutions is

$$\hat{P}_n(x) = \begin{pmatrix} U_n(\frac{x-i}{2}) & 0 \\ 0 & U_n(\frac{x+i}{2}) \end{pmatrix}.$$

This shows that polynomials P_n are equal to

$$P_n(x) = \begin{pmatrix} \frac{1}{2}\big(U_n(\frac{x+i}{2}) + U_n(\frac{x-i}{2})\big) & -\frac{i}{2}\big(U_n(\frac{x+i}{2}) - U_n(\frac{x-i}{2})\big) \\ \frac{i}{2}\big(U_n(\frac{x+i}{2}) - U_n(\frac{x-i}{2})\big) & \frac{1}{2}\big(U_n(\frac{x+i}{2}) + U_n(\frac{x-i}{2})\big) \end{pmatrix}$$

and they are orthogonal with

$$\Sigma = \begin{pmatrix} \frac{1}{2}(\hat{\mu}_+ + \hat{\mu}_-) & \frac{i}{2}(\hat{\mu}_+ - \hat{\mu}_-) \\ -\frac{i}{2}(\hat{\mu}_+ - \hat{\mu}_-) & \frac{1}{2}(\hat{\mu}_+ + \hat{\mu}_-) \end{pmatrix}$$

supported on the set $\Delta = [-2+i, 2+i] \cup [-2-i, -2+i]$ on complex plane $\mathbb{C}$, where

$$\hat{\mu}_+(dx) = \frac{1}{2\pi}\sqrt{5+2ix-x^2}\,dx \quad \text{and} \quad \hat{\mu}_-(dx) = \frac{1}{2\pi}\sqrt{5-2ix-x^2}\,dx$$

are supported respectively on intervals $[-2+i, 2+i]$ and $[-2-i, -2+i]$. In this example the Cauchy transform equals to

$$F(z) = \begin{pmatrix} \frac{2z-\sqrt{z^2+2iz-5}-\sqrt{z^2-2iz-5}}{4} & \frac{-2-i\sqrt{z^2+2iz-5}+i\sqrt{z^2-2iz-5}}{4} \\ \frac{2+i\sqrt{z^2+2iz-5}-i\sqrt{z^2-2iz-5}}{4} & \frac{2z-\sqrt{z^2+2iz-5}-\sqrt{z^2-2iz-5}}{4} \end{pmatrix}.$$

Example 4.5. Let P_n be polynomials satisfying the recurrence formula

$$xP_n(x) = AP_{n+1}(x) + BP_n(x) + CP_{n-1}(x)$$

with

$$A = C = I \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Straightforward computation of the Cauchy transform in this example shows that

$$F(z) = \begin{pmatrix} \frac{z-\sqrt{z^2-4}}{2} & \frac{1}{2}\left(\frac{z}{\sqrt{z^2-4}} - 1\right) \\ 0 & \frac{z-\sqrt{z^2-4}}{2} \end{pmatrix}.$$

This shows that the solution of the recurrence formula is an upper triangular matrix of polynomials

$$P_n(x) = \begin{pmatrix} p_n^{1,1}(x) & p_n^{1,2}(x) \\ 0 & p_n^{2,2}(x) \end{pmatrix}.$$

Scalar polynomials $p_n^{1,1}, p_n^{1,2}, p_n^{2,2}$, satisfy the following recurrences:

$$\begin{aligned} x\,p_n^{1,1}(x) &= p_{n+1}^{1,1}(x) + p_{n-1}^{1,1}(x) && \text{with } p_0^{1,1}(x) = 1, p_{-1}^{1,1}(x) = 0, \\ x\,p_n^{1,2}(x) &= p_{n+1}^{1,2}(x) + p_n^{2,2}(x) + p_{n-1}^{1,2}(x) && \text{with } p_0^{1,2}(x) = 0, p_{-1}^{1,2}(x) = 0, \\ x\,p_n^{2,2}(x) &= p_{n+1}^{2,2}(x) + p_{n-1}^{2,2}(x) && \text{with } p_0^{2,2}(x) = 1, p_{-1}^{2,2}(x) = 0. \end{aligned}$$

This shows that

$$p_n^{1,1}(x) = p_n^{2,2}(x) = U_n\Big(\frac{1}{2}x\Big).$$

Differentiation of the third equation above gives

$$x\,\tfrac{d}{dx}p_n^{2,2}(x) = \tfrac{d}{dx}p_{n+1}^{2,2}(x) + \tfrac{d}{dx}p_{n-1}^{2,2}(x) - p_n^{2,2}(x),$$

so

$$p_n^{1,2}(x) = -\frac{d}{dx}p_n^{2,2}(x) = -\frac{1}{2}U'_n\Big(\frac{1}{2}x\Big).$$

Now simple calculation proves that polynomials P_n are orthogonal with respect to the matrix of measures

$$\Sigma = \begin{pmatrix} \mu_U & \mu'_U \\ 0 & \mu_U \end{pmatrix},$$

where $\mu'_U(dx) = \frac{x\,dx}{2\pi\sqrt{4-x^2}}$ and all measures are supported on the interval $[-2, 2]$ on the real line $\mathbb{R}$. This shows that in this case Σ cannot be transformed neither to a positive definite matrix of measures, nor to a symmetric one.

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