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FRACTIONAL EVOLUTION EQUATION NONLOCAL PROBLEMS WITH NONCOMPACT SEMIGROUPS

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Abstract. This paper is concerned with the existence results of mild solutions to the nonlocal problem of fractional semilinear integro-differential evolution equations. New existence theorems are obtained by means of the fixed point theorem for condensing maps. The results extend and improve some related results in this direction.

Keywords: fractional evolution equation, mild solution, nonlocal condition, C_0-semigroup, condensing maps, measure of noncompactness.

Mathematics Subject Classification: 34A12, 35F25, 35R11.

1. INTRODUCTION

Fractional calculus has been a mathematical topic more than 300 years. Indeed, the concept of the non-integer derivative and integral, as a generalization of the traditional integer order differential and integer calculus, was mentioned in 1695 by Leibniz and L'Hospital, but the first definition of the fractional derivative and integral was introduced at the end of nineteenth century by Liouville and Riemann. The most important advantage of fractional derivatives compared with integer derivatives is that it describes the property of memory and heredity of various materials and processes. In recent years, fractional differential calculus has attracted many physicists, mathematicians and engineers. Notable contributions have been made to both the theory and applications of fractional equations. It has been found that the differential equations involving fractional derivatives in time are more realistic when it comes to describing many phenomena in practical cases than those of integer order in time. For instance, the fractional calculus concepts have been used in the modeling of neurons [23] and viscoelastic materials [27]. Other examples from fractional order dynamics can be found in [1, 9, 12, 20, 25, 26] and the references therein.

In this paper, we use the fixed point theorem for condensing maps to discuss the existence of mild solutions for nonlocal problems of fractional integro-differential evolution equations in a Banach space E

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t), Gu(t)), & t \in J, \\ u(0) + g(u) = u_0, \end{cases} \tag{1.1}$$

where D^q is the Caputo fractional derivative of order q; $0 < q < 1$, $A : D(A) \subset E \to E$ is a closed linear operator, $-A$ generates an equicontinuous and uniformly bounded C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $J = [0, a]$, $a > 0$ is a constant, the term $Gu(t)$ which may be interpreted as a control on the system is defined by

$$Gu(t) := \int_0^t K(t, s)u(s)ds,$$

where $K \in C(D, \mathbb{R}^+)$ (the set of all positive continuous functions on $D := \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$), $f : J \times E \times E \to E$ and $g : C(J, E) \to E$ are given functions satisfying some assumptions, and u_0 is an element of the Banach space E.

The study of abstract nonlocal Cauchy problems was initiated by Byszewski and Lakshmikantham [8]. Since it is demonstrated that the nonlocal problems have are better in applications than the traditional Cauchy problems, differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [2,3,6–8,10,13,17,21,22,29,31,32] and the references therein.

To the best of the author's knowledge, no results exist for the fractional integro-differential evolution equation nonlocal problem (1.1) under noncompactness conditions. For some recent and deeper results on fractional differential equations under noncompactness conditions, see Bechohra *et al.* [5] and Fang and N'Guérékata [18].

We conclude this section by summarizing the contents of this paper. In Section 2, we recall briefly some basic definitions, lemmas and preliminary facts which are used throughout this article. The existence theorems of mild solutions for the fractional integro-differential evolution equation nonlocal problem (1.1) and their proofs are arranged in Section 3.

2. PRELIMINARIES

In this section, we review some notation, definitions and preliminary facts which are used throughout this paper.

Let E be a Banach space with the norm $\|\cdot\|$. we denote by $C(J, E)$ the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_c = \max_{t\in J} \|u(t)\|$. Let $L^p(J, E)$ $(1 \leq p < +\infty)$ be the Banach space of all E-value Bochner integrable functions defined on J with the norm $\|u\|_{L^p(J,E)} = (\int_0^1 \|u(t)\|^p dt)^{\frac{1}{p}}$. We set

$B_r = \{u \in C(J,E) \mid \|u\|_c < r\}$ ($r > 0$ is a constant), $\overline{B_r} = \{u \in C(J,E) \mid \|u\|_c \le r\}$ ($r > 0$ is a constant), and let

$$G^* = \sup_{t\in J} \int_0^t K(t,s)ds < \infty.$$

Definition 2.1 ([20])**.** The fractional integral of order $q > 0$ with the lower limit 0 for a function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([20])**.** The Caputo fractional derivative of order $q > 0$ with the lower limit 0 for a function f is defined as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \quad t > 0, \quad n-1 < q < n, \tag{2.2}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$.

If f is an abstract function with values in E, then the integrals and derivatives appeared in Definitions 2.1 and 2.2 are taken in Bochner's sense.

For $u \in E$, define two operators $\mathscr{T}(t)$ ($t \ge 0$) and $\mathscr{S}(t)$ ($t \ge 0$) by

$$\mathscr{T}(t)u = \int_0^\infty \zeta_q(\theta) T(t^q\theta)u d\theta, \qquad \mathscr{S}(t)u = q\int_0^\infty \theta\zeta_q(\theta) T(t^q\theta)u d\theta, \tag{2.3}$$

where

$$\zeta_q(\theta) = \frac{1}{q}\theta^{-1-(1/q)}\rho_q(\theta^{-1/q}), \tag{2.4}$$

$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=0}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,+\infty),$$

$\zeta_q(\theta)$ is a probability density function on $(0,+\infty)$ satisfying

$$\zeta_q(\theta) \ge 0, \qquad \int_0^\infty \zeta_q(\theta)d\theta = 1, \qquad \int_0^\infty \theta\zeta_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}, \quad \theta \in (0,+\infty).$$

Let $M = \sup_{t\in[0,+\infty)} \|T(t)\|_{\mathscr{L}(E)}$, where $\mathscr{L}(E)$ stands for the Banach space of all linear and bounded operators in E. A C_0-semigroup $T(t)$ ($t \ge 0$) is called equicontinuous if the operator $T(t)$ is continuous on $(0,+\infty)$ by the operator norm. The following lemma follows from the results in [15, 16, 30].

Lemma 2.3. *The operators $\mathscr{T}(t)$ $(t \geq 0)$ and $\mathscr{S}(t)$ $(t \geq 0)$ have the following properties:*

(1) *for any fixed $t \geq 0$, $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$ are linear and bounded operators, i.e., for all $u \in E$,*

$$\|\mathscr{T}(t)u\| \leq M\|u\|, \quad \|\mathscr{S}u\| \leq \frac{M}{\Gamma(q)}\|u\|;$$

(2) *for every $u \in E$, $t \to \mathscr{T}(t)u$ and $t \to \mathscr{S}(t)u$ are continuous functions from $[0, +\infty)$ into E;*
(3) *if $T(t)(t \geq 0)$ is an equicontinuous semigroup, then $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$ are continuous in $(0, +\infty)$ by the operator norm, which means that for $0 < t' < t'' \leq a$, we have*

$$\|\mathscr{T}(t'') - \mathscr{T}(t')\| \to 0 \quad \textit{and} \quad \|\mathscr{S}(t'') - \mathscr{S}(t')\| \to 0 \quad \textit{as} \quad t'' \to t'.$$

Lemma 2.4 (Bochner's theorem)**.** *A measurable function $H : J \to E$ is Bochner's integrable if $\|H\|$ is Lebesgue.*

Lemma 2.4 is classical and it can be found in many books.

Definition 2.5. A function $u \in C(J, E)$ is said to be a mild solution of the fractional evolution equation nonlocal problem (1.1) if it satisfies

$$u(t) = Q_1 u(t) + Q_2 u(t), \tag{2.5}$$

where $Q_1 u(t) = \mathscr{T}(t)(u_0 - g(u))$, $Q_2 u(t) = \int\limits_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(s, u(s), Gu(s))ds$.

Next, we recall some properties of the measure of noncompactness that will be used in the proof of our main results. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, we refer to the monographs [4] and [14]. For any $B \subset C(J, E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J, E)$, then $B(t)$ is bounded in E and $\alpha(B(t)) \leq \alpha(B)$.

Lemma 2.6 ([11])**.** *Let E be a Banach space, $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$ such that*

$$\alpha(D) \leq 2\alpha(D_0).$$

Lemma 2.7 ([4])**.** *Let E be a Banach space, $D \subset C(J, E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on J, and*

$$\alpha(D) = \max_{t \in J} \alpha(D(t)) = \alpha(D(J)).$$

Lemma 2.8 ([19])**.** *Let E be a Banach space, $D = \{u_n\} \subset C(J, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is a Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int\limits_J u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int\limits_J \alpha(D(t))dt.$$

Lemma 2.9 ([14])**.** *Let E be a Banach space, $D \subset E$ be a bounded closed and convex set in E, $Q : D \to D$ be condensing which means that $\alpha(Q(D)) < \alpha(D)$. Then Q has a fixed point in D.*

Lemma 2.10 ([14])**.** *Let E be a Banach space, Ω be a bounded open subset of E and $\theta \in \Omega$. Suppose that $Q : \overline{\Omega} \to E$ is condensing and assume that*

$$u \neq \lambda Q(u) \quad \text{for} \quad u \in \partial\Omega \quad \text{and} \quad \lambda \in (0, 1),$$

hold. Then Q has a fixed point in $\overline{\Omega}$.

3. MAIN RESULTS

Theorem 3.1. *Let E be a Banach space, $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generates an equicontinuous and uniformly bounded C_0-semigroup $T(t)(t \geq 0)$ in E, $f : J \times E \times E \to E$, $g : C(J, E) \to E$. Suppose that the following conditions hold:*

- (H1) *for each $t \in J$, the function $f(t, \cdot, \cdot) : E \times E \to E$ is continuous and for each $(x, y) \in E \times E$, the function $f(\cdot, x, y) : J \to E$ is Lebesgue measurable;*
- (H2) *there exist a constant $q_1 \in (0, q)$ and a positive function $m \in L^{\frac{1}{q_1}}(J, \mathbb{R})$ such that*
$$\|f(t, u, v)\| \leq m(t) \quad \text{for}\, u, v \in E \text{ and}\, t \in J;$$
- (H3) *$g : C(J, E) \to E$ is continuous and there exist constants $K > 0$ such that for any $R > 0$*
$$\|g(u) - g(v)\| \leq K\|u - v\|_c \quad \text{for all}\, u, v \in \overline{B_R};$$
- (H4) *there exist constants $L_1, L_2 > 0$ such that*
$$\alpha(f(t, D_1, D_2)) \leq L_1\alpha(D_1) + L_2\alpha(D_2) \quad \text{for all}\, t \in J \text{ and } D_1, D_2 \in E.$$

Then the fractional integro-differential evolution equation nonlocal problem (1.1) *has at least one mild solution in $C(J, E)$ provided that*

$$2M\Big(K + \frac{2(L_1 + G^* L_2)a^q}{\Gamma(q + 1)}\Big) < 1. \tag{3.1}$$

Proof. For any positive constant R and $u \in \overline{B_R}$, since $u(t)$ and $Gu(t)$ are continuous in t, $f(t, u(t), Gu(t))$ is a measurable function on J according to condition (H1). Let

$$q_2 = \frac{q - 1}{1 - q_1} \in (-1, 0), \quad M_1 = \|m\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})}. \tag{3.2}$$

By using Hölder's inequality and condition (H2), we obtain

$$\begin{aligned}\int_0^t \|(t - s)^{q-1} f(s, u(s), Gu(s))\| ds &\leq \Big(\int_0^t (t - s)^{\frac{q-1}{1-q_1}} ds\Big)^{1-q_1} \|m\|_{L^{\frac{1}{q_1}}(J,\mathbb{R})} \\ &\leq \frac{M_1}{(1 + q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}.\end{aligned}$$

By Lemma 2.3 (1), we have

$$\int_0^t \|(t-s)^{q-1}\mathscr{S}(t-s)f(s,u(s),Gu(s))\|ds \le \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}. \quad (3.3)$$

Thus, $\|(t-s)^{q-1}\mathscr{S}(t-s)f(s,u(s),Gu(s))\|$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$. From Lemma 2.4 it follows that $(t-s)^{q-1}\mathscr{S}(t-s)f(s,u(s),Gu(s))$ is Bochner's integrable with respect to $t \in J$.

We now consider the operator $Q : C(J,E) \to C(J,E)$ defined by (2.5). It is easy to see that the fixed point of Q is the mild solution of the fractional integro-differential evolution equation nonlocal problem (1.1). Therefore, the existence of a mild solution of (1.1) is equivalent to determining a positive constant R_0 such that Q has at least one fixed point on $\overline{B_{R_0}}$.

Indeed, by choosing

$$R_0 = \frac{M(\|u_0\| + \|g(\theta)\|)}{1-MK} + \frac{M_1 M a^{(1+q_2)(1-q_1)}}{(1-MK)\Gamma(q)(1+q_2)^{1-q_1}},$$

we can prove that Q has at least one fixed point on $\overline{B_{R_0}}$. Our proof will be divided into three steps.

Step 1. The operator $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is continuous.

For any $u \in B_{R_0}$ and $t \in J$, by using (3.3) we have

$$\begin{aligned}
\|Qu(t)\| &\le \|\mathscr{T}(t)(u_0 - g(u))\| + \int_0^t \|t-s)^{\alpha-1}\mathscr{S}(t-s)f(s,u(s),Gu(s))\|ds \\
&\le M(\|u_0\| + K\|u-\theta\|_c + \|g(\theta)\|) + \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)} \\
&\le M(\|u_0\| + KR_0 + \|g(\theta)\|) + \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)} \\
&= R_0.
\end{aligned}$$

Hence, $\|Qu\|_c \le R_0$ for every $u \in \overline{B_{R_0}}$.

For all $u_n,\ u \subset \overline{B_{R_0}}$, $n = 1, 2, \dots$ with $\lim\limits_{n\to+\infty} \|u_n - u\|_c = 0$, we get that

$$\lim_{n\to+\infty} u_n(t) = u(t), \quad \text{for all } t \in J.$$

Thus, by condition (H1) we have that

$$\lim_{n\to+\infty} f(t,u_n(t),Gu_n(t)) = f(t,u(t),Gu(t)) \quad \text{for all } t \in J.$$

So, we can conclude that

$$\sup_{t\in J} \|f(t,u_n(t),Gu_n(t)) - f(t,u(t),Gu(t))\| \to 0 \ \text{ as } \ n \to +\infty.$$

On the other hand, for $t \in J$

$$
\begin{aligned}
&\|Qu_n(t) - Qu(t)\| \\
&\quad \leq M\|g(u_n) - g(u)\| \\
&\qquad + \int_0^t \|(t-s)^{q-1}\mathscr{S}(t-s)[f(s,u_n(s),Gu_n(s)) - f(s,u(s),Gu(s))]\|ds \\
&\quad \leq MK\|u_n - u\|_c + \frac{Ma^q}{\Gamma(1+q)} \sup_{t\in J} \|f(t,u_n(t),Gu_n(t)) - f(t,u(t),Gu(t))\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|Qu_n - Qu\|_c \leq{}& MK\|u_n - u\|_c \\
&+ \frac{Ma^q}{\Gamma(1+q)} \sup_{t\in J} \|f(t,u_n(t),Gu_n(t)) - f(t,u(t),Gu(t))\|.
\end{aligned}
$$

Hence,

$$\|Qu_n - Qu\|_c \to 0 \quad \text{as} \quad n \to +\infty.$$

This means that the operator $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is continuous.

Step 2. The operator $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is equicontinuous, which means that $\|Qu(t_2) - Qu(t_1)\|$ tends to 0 as $t_2 \to t_1$ for any $u \in \overline{B_{R_0}}$.

For $0 \leq t_1 < t_2 \leq a$, we can get that

$$
\begin{aligned}
&\|(Qu)(t_2) - (Qu)(t_1)\| \\
&\leq \|\mathscr{T}(t_2)(u_0 - g(u)) - \mathscr{T}(t_1)(u_0 - g(u))\| \\
&\quad + \Big\| \int_0^{t_2} (t_2-s)^{q-1}\mathscr{S}(t_2-s)f(s,u(s),Gu(s))ds \\
&\quad - \int_0^{t_1} (t_1-s)^{q-1}\mathscr{S}(t_1-s)f(s,u(s),Gu(s))ds \Big\| \\
&\leq \|\mathscr{T}(t_2)(u_0 - g(u)) - \mathscr{T}(t_1)(u_0 - g(u))\| \\
&\quad + \Big\| \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathscr{S}(t_2-s)f(s,u(s),Gu(s))ds \Big\| \\
&\quad + \Big\| \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}]\mathscr{S}(t_2-s)f(s,u(s),Gu(s))ds \Big\| \\
&\quad + \Big\| \int_0^{t_1} (t_1-s)^{q-1}[\mathscr{S}(t_2-s) - \mathscr{S}(t_1-s)]f(s,u(s),Gu(s))ds \Big\| \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$I_1 = \|\mathscr{T}(t_2)(u_0 - g(u)) - \mathscr{T}(t_1)(u_0 - g(u))\|,$$

$$I_2 = \Big\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathscr{S}(t_2 - s) f(s, u(s), Gu(s)) ds \Big\|,$$

$$I_3 = \Big\| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \mathscr{S}(t_2 - s) f(s, u(s), Gu(s)) ds \Big\|,$$

$$I_4 = \Big\| \int_0^{t_1} (t_1 - s)^{q-1} [\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s)] f(s, u(s), Gu(s)) ds \Big\|.$$

We now only need to check that I_i tends to 0 independently of $u \in \overline{B_{R_0}}$ when $t_2 \to t_1$, $i = 1, 2, 3, 4$. For I_1, by Lemma 2.3 (3), we have that

$$I_1 \leq \|(\mathscr{T}(t_2) - \mathscr{T}(t_1))u_0\| + \|(\mathscr{T}(t_2) - \mathscr{T}(t_1))g(u)\| \to 0 \quad \text{as} \quad t_2 \to t_1.$$

For I_2,

$$I_2 \leq \frac{M_1 M}{\Gamma(q)(1 + q_2)^{1-q_1}} (t_2 - t_1)^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \quad t_2 \to t_1.$$

For I_3, by Lemma 2.3 (1), we have that

$$\begin{aligned}
I_3 &\leq \frac{M}{\Gamma(q)} \Big(\int_0^{t_1} \big[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\big]^{\frac{1}{1-q_1}} ds \Big)^{1-q_1} \|m\|_{L^{\frac{1}{q_1}}(J,E)} \\
&\leq \frac{M_1 M}{\Gamma(q)} \Big(\int_0^{t_1} ((t_1 - s)^{q_2} - (t_2 - s)^{q_2}) ds \Big)^{1-q_1} \\
&= \frac{M_1 M}{\Gamma(q)(1 + q_2)^{1-q_1}} (t_1^{1+q_2} - t_2^{1+q_2} + (t_2 - t_1)^{1+q_2})^{1-q_1} \\
&\leq \frac{M_1 M}{\Gamma(q)(1 + q_2)^{1-q_1}} (t_2 - t_1)^{(1+q_2)(1-q_1)} \to 0 \quad \text{as} \ \ t_2 \to t_1.
\end{aligned}$$

For $t_1 = 0$, $0 < t_2 \le a$, it is easy to see that $I_4 = 0$. For $t_1 > 0$ and $\epsilon > 0$ small enough, by Lemma 2.3 (3), we know that

$$\begin{aligned}
I_4 \le & \Big\| \int_0^{t_1-\epsilon} (t_1 - s)^{q-1}[\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s)]f(s, u(s), Gu(s))ds \Big\| \\
& + \Big\| \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1}[\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s)]f(s, u(s), Gu(s))ds \Big\| \\
\le & \int_0^{t_1-\epsilon} \|(t_1 - s)^{q-1} f(s, u(s), Gu(s))\| ds \sup_{s\in[0,t_1-\epsilon]} \|\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s)\|_{\mathscr{L}(E)} \\
& + \frac{2M}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} \|(t_1 - s)^{q-1} f(s, u(s), Gu(s))\| ds \\
\le & \frac{M_1(t^{(1+q_2)} - \epsilon^{(1+q_2)})^{1-q_1}}{(1 + q_2)^{1-q_1}} \sup_{s\in[0,t_1-\epsilon]} \|\mathscr{S}(t_2 - s) - \mathscr{S}(t_1 - s)\|_{\mathscr{L}(E)} \\
& + \frac{2M_1 M}{\Gamma(q)(1 + q_2)^{1-q_1}} \epsilon^{(1+q_2)(1-q_1)} \quad \to 0 \quad \text{as} \quad t_2 \to t_1.
\end{aligned}$$

As a result, $\|Qu(t_2) - Qu(t_1)\|$ tends to 0 independently of $u \in \overline{B_{R_0}}$ as $t_2 \to t_1$. Therefore, $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is equicontinuous.

Step 3. $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is a condensing operator.

For all $B \subset \overline{B_{R_0}}$, $Q(B)$ is bounded and equicontinuous. Hence, by Lemma 2.6, there exists a countable set $B_1 = \{u_n\}_{n=1}^{\infty} \subset B$ such that

$$\alpha(Q(B)) \le 2\alpha(Q(B_1)). \tag{3.4}$$

Since $Q(B_1) \subset Q(\overline{B_{R_0}})$ is equicontinuous, Lemma 2.7 implies

$$\alpha(Q(B_1)) = \max_{t\in J} \alpha(Q(B_1)(t)). \tag{3.5}$$

Moreover, Q_1 is Lipschitz continuous with constant MK by condition (H3). Indeed, for all $x, y \in B_1$, we know

$$\begin{aligned}
\|Q_1 x - Q_1 y\| &= \sup_{t\in J} \|\mathscr{T}(t)(u_0 - g(x)) - \mathscr{T}(t)(u_0 - g(y))\| \\
&\le M\|g(x) - g(y)\| \le MK\|x - y\|_c,
\end{aligned}$$

from this inequality and the definition of the measure of noncompactness, it follows that

$$\alpha(Q_1(B_1)) \le MK\alpha(B_1). \tag{3.6}$$

For $t \in J$, according to Lemma 2.3 (1), Lemma 2.8, and the conditions (H3) and (H4), we have

$$
\begin{aligned}
\alpha(Q(B_1)(t)) &= \alpha(Q_1(B_1)(t)) + \alpha(Q_2(B_1)(t)) \\
&\le \alpha(Q(B_1) + \alpha\Big(\Big\{\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)f(s,u_n(s),Gu_n(s))ds\Big\}_{n=1}^{\infty}\Big) \\
&\le MK\alpha(B_1) + 2\int_0^t \alpha(\{t-s)^{q-1}\mathscr{S}(t-s)f(s,u_n(s),Gu_n(s))ds\}_{n=1}^{\infty}) \\
&\le MK\alpha(B) + \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}\alpha(\{f(s,u_n(s),Gu_n(s))\}_{n=1}^{\infty})ds \\
&\le MK\alpha(B) + \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}(L_1\alpha(B_1(s) + L_2\alpha(GB_1(s)))ds \\
&\le MK\alpha(B) + \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}(L_1\alpha(B_1) + L_2\alpha(GB_1(s)))ds.
\end{aligned} \tag{3.7}
$$

Meanwhile, we have

$$
\alpha(GB_1(s)) \le \alpha(GB_1) \le \|G\|\alpha(B_1) \le G^*\alpha(B_1) \le G^*\alpha(B). \tag{3.8}
$$

Therefore, we know

$$
\begin{aligned}
\alpha(Q(B_1)(t)) &\le MK\alpha(B) + \frac{2M}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}(L_1\alpha(B) + L_2G^*\alpha(B))ds \\
&\le MK\alpha(B) + \frac{2M(L_1 + G^*L_2)a^q}{\Gamma(q+1)}\alpha(B) \\
&\le M\Big(K + \frac{2(L_1 + G^*L_2)a^q}{\Gamma(q+1)}\Big)\alpha(B).
\end{aligned}
$$

Form this inequality, (3.4) and (3.5), it follows that

$$
\alpha(Q(B)) \le 2M\Big(K + \frac{2(L_1 + G^*L_2)a^q}{\Gamma(q+1)}\Big)\alpha(B).
$$

Thus, from (3.1), we find that $Q : \overline{B_{R_0}} \to \overline{B_{R_0}}$ is a condensing operator.

Finally, Lemma 2.9 guarantees that Q has at least one fixed point in $\overline{B_{R_0}}$. Therefore, the fractional integro-differential evolution equation nonlocal problem (1.1) has at least one mild solution. This completes the proof. □

In the following, we give an existence result in the case that condition (H3) is not satisfied. We need the following condition:

(H5) $g : C(J, E) \to E$ is completely continuous, and there exist positive constants $b < 1/M$ and d such that for any $r > 0$

$$\|g(u)\| \leq b\|u\|_c + d \quad \text{for all } u \in \overline{B_r}.$$

The following existence result for the fractional integro-differential evolution equation nonlocal problem (1.1) based on Lemma 2.10.

Theorem 3.2. *Let E be a Banach space, $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generates an equicontinuous and uniformly bounded C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $f : J \times E \times E \to E$, and $g : C(J, E) \to E$. If the conditions* (H1), (H2), (H4), (H5) *and the following inequality hold*

$$\frac{4M(L_1 + G^* L_2)a^q}{\Gamma(q+1)} < 1. \tag{3.9}$$

Then the fractional integro-differential evolution equation nonlocal problem (1.1) *has at least one mild solution in $\overline{B_r}$ with r satisfying*

$$\frac{(1 - bM)r}{M\Big(\|u_0\| + d + \frac{M_1}{(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}\Big)} > 1, \tag{3.10}$$

where q_2 and M_1 are defined by (3.2).

Proof. From the proof of Theorem 3.1, we know that $(t - s)^{q-1}\mathscr{S}(t - s)f(s, u(s), Gu(s))$ is Bochner's integrable with respect to $t \in J$, and the fixed point of operator Q defined by (2.5) is the mild solution of the fractional integro-differential evolution equation nonlocal problem (1.1). Therefore, the existence of a mild solution of (1.1) is equivalent to determining a positive constant r such that Q has at least one fixed point on $\overline{B_r}$. Indeed, by choosing

$$r > \frac{M\Big(\|u_0\| + d + \frac{M_1}{(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}\Big)}{1 - bM},$$

we can prove that Q has at least one fixed point on $\overline{B_r}$.

By using the similar method with the proof of Theorem 3.1, we can prove that the operator $Q : \overline{B_r} \to C(J, E)$ is equicontinuous. Next, we prove that $Q : \overline{B_r} \to C(J, E)$ is a condensing operator. For any $D \subset \overline{B_r}$, $Q(D)$ is bounded and equicontinuous, by Lemma 2.7 there exists a countable set $D_1 = \{u_n\}_{n=1}^{\infty} \subset D$ such that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_1)). \tag{3.11}$$

Since $Q(D_1) \subset Q(\overline{B_r})$ is equicontinuous, Lemma 2.7 implies

$$\alpha(Q(D_1)) = \max_{t \in J} \alpha(Q(D_1)(t)). \tag{3.12}$$

By the conditions (H4), (H5) and a similar method with the proof of Theorem 3.1, we get that

$$\alpha(Q(D)) \le \frac{4M(L_1 + G^* L_2)a^q}{\Gamma(q+1)} \alpha(D).$$

From this inequality and (3.9), we can conclude that $Q : \overline{B_r} \to C(J,E)$ is a condensing operator.

Finally, let $\lambda \in (0,1)$ and $u = \lambda Q(u)$. Then, for $t \in J$

$$u(t) = \lambda \mathscr{T}(t)(u_0 - g(u)) + \lambda \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) f(s, u(s), Gu(s)) ds,$$

and one has

$$\|u(t)\| \le M(\|u_0\| + br + d) + \frac{M_1 M}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}.$$

Consequently,

$$\frac{(1-bM)r}{M\Big(\|u_0\| + d + \frac{M_1}{\Gamma(q)(1+q_2)^{1-q_1}} a^{(1+q_2)(1-q_1)}\Big)} \le 1.$$

Thus, by (3.10), there exists some constant r such that $\|u\| \neq r$. By the choice of B_r, there dose not exist $u \in \partial B_r$ such that $u = \lambda Q(u)$ for some $\lambda \in (0,1)$. Thus, we get a fixed point in $\overline{B_r}$ by Lemma 2.10, which is the mild solution of the fractional integro-differential evolution equation nonlocal problem (1.1). This completes the proof. □

Remark 3.3. From [24] we know that analytic semigroup and differentiable semigroup are an equicontinuous semigroup. In the application of partial differential equations, such as parabolic equations and strongly damped wave equations, the corresponding solution semigroup is analytic semigroup. Therefore, Theorem 3.1 and Theorem 3.2 in this paper have broad applicability.

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