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ON A LINEAR-QUADRATIC PROBLEM WITH CAPUTO DERIVATIVE

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Abstract. In this paper, we study a linear-quadratic optimal control problem with a fractional control system containing a Caputo derivative of unknown function. First, we derive the formulas for the differential and gradient of the cost functional under given constraints. Next, we prove an existence result and derive a maximum principle. Finally, we describe the gradient and projection of the gradient methods for the problem under consideration.

Keywords: fractional Caputo derivative, linear-quadratic problem, existence and uniqueness of a solution, maximum principle, gradient method, projection of the gradient method.

Mathematics Subject Classification: 26A33, 49J15, 49K15, 49M37.

1. INTRODUCTION

In this paper, we consider the following fractional linear control system

$$\begin{cases} {}^C D^{\alpha}_{a+} x(t) = Ax(t) + Bu(t), \quad t \in [a,b] \text{ a.e.}, \\ x(a) = 0 \end{cases} \tag{1.1}$$

with a quadratic performance index

$$\begin{aligned} J(u) &= J_1(u) + J_2(u) + J_3(u) \\ &= \frac{1}{2} \left| x_u(b) - c \right|^2_{\mathbb{R}^n} + \frac{1}{2} \left\| x_u(\cdot) - y(\cdot) \right\|^2_{L^2} + \frac{1}{2} \left\| u(\cdot) \right\|^2_{I^{1-\alpha}_{a+}(L^2)} , \end{aligned} \tag{1.2}$$

where $\alpha \in (\frac{1}{2}, 1)$ is a fixed number, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, n, $m \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$, c is a fixed vector in $\mathbb{R}^n$, $y : [a,b] \to \mathbb{R}^n$ – a fixed function belonging to $L^2 = L^2([a,b], \mathbb{R}^n)$, $I^{1-\alpha}_{a+}(L^2)$ – a space of controls defined below. By ${}^C D^{\alpha}_{a+} x$ we denote the left Caputo derivative of a function $x : [a,b] \to \mathbb{R}^n$ and by $x_u : [a,b] \to \mathbb{R}^n$ – a solution of problem (1.1), corresponding to a control $u : [a, b] \to \mathbb{R}^m$. We shall consider the above optimal control problem in the set

$$AC^2 = AC^2([a, b], \mathbb{R}^n) = \{x : [a, b] \to \mathbb{R}^n; x \text{ is absolutely continuous and } x' \in L^2\}$$

of trajectories.

Problems of such a type can be used in the examination of the pointwise and functional controllabilities of systems (1.1) with taking an energy cost into consideration.

The paper is organized as follows. First, we calculate the differential and the gradient of the functional J. Next, we prove existence of a unique optimal control and derive a maximum principle for problem (1.1)–(1.2). Finally, we use the formula for the gradient of J to describe the gradient and projection of the gradient methods for the approximative solving of this problem.

Results of such a type for systems containing the classical derivative of the first order can be found in [22] (cf. also [11] for the comprehensive study of the classical linear-quadratic problems). The case of the fractional Riemann-Liouville derivative is studied in [10] for J containing only the pointwise term, in the space $AC^{\alpha,2}_{a+}$ (defined below) of trajectories and in the space L^2 of controls. The presence of the Caputo derivative means that the problem has to be investigated in quite different (given above) spaces of trajectories and controls.

Optimal control problems for systems of fractional order are studied for over ten years. For the first time, fractional optimal control problems of type

$$\begin{cases} D^{\alpha}_{a+}x(t) = G(t, x(t), u(t)), \\ x(a) = x_0, \end{cases}$$

where $D^{\alpha}_{a+}x$ denotes the left Riemann-Liouville derivative, with cost functional of integral type

$$I(u) = \int_0^1 F(t, x(t), u(t))dt$$

and without control constraints, were investigated in [2]. The author formulated the necessary optimality conditions for such a problem and described a numerical scheme for finding an approximative solution to such systems in the case of linear control systems and quadratic cost functionals

$$I(u) = \frac{1}{2}\int_0^1 (q(t)x^2(t) + r(t)u^2(t))dt.$$

Necessary optimality conditions are given in the form of a system of equations containing the Lagrange multipliers and do not contain any minimum condition. Unconstrained problems are considered also in [5] where second order optimality conditions are derived and in [19] where the final time is not fixed. A numerical scheme presented in [2] is adopted from the case of positive integer order problems, given in [1]. Numerical schemes for fractional optimal control problems, consisting in an approximation of a fractional derivative can be found in [3, 17, 19, 21]. These schemes lead to problems of positive integer order. In paper [15], system (1.1) with a non-zero initial condition and cost functional

$$I(u) = \int_0^1 (\langle \gamma, x(t)\rangle + g(t, u(t)))dt,$$

where $\gamma \in \mathbb{R}^n$, is considered. Authors obtained existence of an optimal solution and necessary optimality conditions in the form of a maximum principle.

To the best of our knowledge results presented in our paper has not been obtained by other authors.

2. PRELIMINARIES

In all the paper we consider linear spaces over $\mathbb{R}$.

Let $\alpha \in (0, 1)$. By the left Caputo derivative of order α of an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ we mean the left Riemann-Liouville derivative of the function $x(\cdot) - x(a)$. Of course,

$${}^C D^\alpha_{a+} x(t) = D^\alpha_{a+} x(t) - \frac{1}{\Gamma(1-\alpha)} \frac{x(a)}{(t-a)^\alpha}, \quad t \in [a, b] \text{ a.e.},$$

where $D^\alpha_{a+} x$ is the left Riemann-Liouville derivative of order α of x. It is easy to see that

$${}^C D^\alpha_{a+} x(t) = I^{1-\alpha}_{a+} x'(t), \quad t \in [a, b] \text{ a.e.},$$

where $I^{1-\alpha}_{a+}$ is the left Riemann-Liouville integral operator of order $1-\alpha$ defined on the space $L^1 = L^1([a, b], \mathbb{R}^m)$ of integrable functions. Indeed, if x is absolutely continuous on $[a, b]$, then

$$\begin{aligned} {}^C D^\alpha_{a+} x(t) = D^\alpha_{a+}(x(\cdot) - x(a))(t) &= \frac{d}{dt}(I^{1-\alpha}_{a+}(x(\cdot) - x(a))(t) \\ &= \frac{d}{dt}(I^{1-\alpha}_{a+}(I^1_{a+} x'))(t) = \frac{d}{dt}(I^1_{a+}(I^{1-\alpha}_{a+} x'))(t) = I^{1-\alpha}_{a+} x'(t) \end{aligned}$$

for $t \in [a, b]$ a.e. (we used here the fact that $I^{1-\alpha}_{a+}(I^1_{a+} x') = I^1_{a+}(I^{1-\alpha}_{a+} x')$ everywhere on $[a, b]$ (cf. [20, formula (2.21) and the subsequent comments])). More properties of the fractional integrals and derivatives can be found in [20] and [16].

By $AC^{\alpha,2}_{a+} = AC^{\alpha,2}_{a+}([a, b], \mathbb{R}^n)$ we denote the set of all functions $x : [a, b] \to \mathbb{R}^n$ of the form

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + I^\alpha_{a+} \varphi(t), \quad t \in [a, b] \text{ a.e.},$$

with $c \in \mathbb{R}^n$, $\varphi \in L^2$. One can show ([4]) that $x \in AC^{\alpha,2}_{a+}$ if and only if x possesses the left Riemann-Liouville derivative $D^\alpha_{a+} x \in L^2$. In such a case

$$I^{1-\alpha}_{a+} x(a) = c \quad \text{and} \quad D^\alpha_{a+} x = \varphi.$$

It is easy to show that $AC^{\alpha,2}_{a+}$ with natural operations and the scalar product

$$\langle x,y\rangle = I^{1-\alpha}_{a+}x(a)I^{1-\alpha}_{a+}y(a) + \int_a^b D^{\alpha}_{a+}x(t)D^{\alpha}_{a+}y(t)dt$$

is the Hilbert space.

Similarly, by $AC^{\alpha,2}_{b-} = AC^{\alpha,2}_{b-}([a,b],\mathbb{R}^n)$ we mean the set of all functions $x:[a,b]\to\mathbb{R}^n$ of the form

$$x(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{(b-t)^{1-\alpha}} + I^{\alpha}_{b-}\psi(t), \quad t\in[a,b] \text{ a.e.},$$

with $d\in\mathbb{R}^n$, $\psi\in L^2$, where I^{α}_{b-} is the right Riemann-Liouville integral operator of order $1-\alpha$ defined on the space L^1. As in the "left" case $x\in AC^{\alpha,2}_{b-}$ if and only if x possesses the right Riemann-Liouville derivative $D^{\alpha}_{b-}x\in L^2$ and, in such a case,

$$I^{1-\alpha}_{b-}x(b) = d \quad \text{and} \quad D^{\alpha}_{b-}x = \psi.$$

$AC^{\alpha,2}_{b-}$ with the scalar product

$$\langle x,y\rangle = I^{1-\alpha}_{b-}x(b)I^{1-\alpha}_{b-}y(b) + \int_a^b D^{\alpha}_{b-}x(t)D^{\alpha}_{b-}y(t)dt$$

is complete.

Of course, $AC^{1,2}_{a+} = AC^{1,2}_{b-} = AC^2$.

By $I^{\alpha}_{a+}(L^2)$ we denote the set $\{I^{\alpha}_{a+}(\varphi);\ \varphi\in L^2\}$ which is contained in $AC^{\alpha,2}_{a+}$. It is clear that $I^{\alpha}_{a+}(L^2)$ with the scalar product

$$\langle x,y\rangle = \int_a^b D^{\alpha}_{a+}x(t)D^{\alpha}_{a+}y(t)dt$$

is the Hilbert space.

In what follows, we shall use the following variant of a fractional theorem on integration by parts.

Theorem 2.1. *Let $\alpha\in(0,1)$. If $f\in AC^2_0 = \{x\in AC^2;\ x(a)=0\}$, $g\in AC^{\alpha,2}_{b-}\cap L^2$, then*

$$\int_a^b f(t)D^{\alpha}_{b-}g(t)dt = -f(b)I^{1-\alpha}_{b-}g(b) + \int_a^b D^{\alpha}_{a+}f(t)g(t)dt.$$

Proof. Since $f\in AC^2_0$, therefore (cf. [9, Theorem 6]) $f\in AC^{\alpha,1}_{a+}$ and

$$D^{\alpha}_{a+}f(t) = I^{1-\alpha}_{a+}(f').$$

So,

$$\int_a^b D^\alpha_{a+} f(t)g(t)dt = \int_a^b I^{1-\alpha}_{a+}(f')(t)g(t)dt = \int_a^b f'(t)I^{1-\alpha}_{b-}g(t)dt$$

and

$$\begin{aligned}\int_a^b f(t)D^\alpha_{b-}g(t)dt &= \int_a^b f(t)\Big(-\frac{d}{dt}I^{1-\alpha}_{b-}g\Big)(t)dt \\ &= -f(b)I^{1-\alpha}_{b-}g(b) + \int_a^b f'(t)I^{1-\alpha}_{b-}g(t)dt\end{aligned}$$

and the proof is complete. □

Using the method applied in [7] one can obtain (cf. [12]) the following result.

Theorem 2.2. *If $\alpha \in (\frac{1}{2}, 1)$, $c \in \mathbb{R}^n$, $v \in L^2$, then the problem*

$$\begin{cases} D^\alpha_{a+}x(t) = Ax(t) + v(t), \quad t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{a+}x(a) = c \end{cases}$$

has a unique solution x_v in $AC^{\alpha,2}_{a+}$. It is given by

$$x_v(t) = \Phi^A_{\alpha,\alpha}(t-a)c + \int_a^t \Phi^A_{\alpha,\alpha}(t-s)v(s)ds, \quad t \in [a,b] \text{ a.e.},$$

where $\Phi^A_{\alpha,\beta}(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}$.

Remark 2.3. If $c \neq 0$, then condition $\alpha > \frac{1}{2}$ can not be omitted. Indeed, let us consider problem

$$\begin{cases} D^\alpha_{a+}x(t) = x(t), \quad t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{a+}x(a) = c \end{cases}$$

If

$$x(t) = \frac{1}{\Gamma(\alpha)}\frac{c}{(t-a)^{1-\alpha}} + I^\alpha_{a+}\varphi(t), \quad t \in [a,b] \text{ a.e.},$$

where $\varphi \in L^2$, is a solution to the above system, belonging to $AC^{\alpha,2}_{a+}$, then $x = D^\alpha_{a+}x \in L^2$, $I^\alpha_{a+}\varphi \in L^2$, and, consequently, $\frac{c}{(t-a)^{1-\alpha}} \in L^2$. It means that $\alpha > \frac{1}{2}$.

Similarly, we have the following theorem.

Theorem 2.4. *If $\alpha \in (\frac{1}{2}, 1)$, $c \in \mathbb{R}^n$, $v \in L^2$, then problem*

$$\begin{cases} D^\alpha_{b-}y(t) = Ay(t) + v(t), \quad t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{b-}y(b) = c \end{cases} \tag{2.1}$$

has a unique solution $y_v \in AC^{\alpha,2}_{b-}$ *and it is given by*

$$y_v(t) = \Phi^A_{\alpha,\alpha}(b-t)c + \int_t^b \Phi^A_{\alpha,\alpha}(s-t)v(s)ds, \quad t \in [a,b] \text{ a.e.} \tag{2.2}$$

In particular, a unique solution y_0 *corresponding to control* $v(\cdot) \equiv 0$ *is given by*

$$y_0(t) = \Phi^A_{\alpha,\alpha}(b-t)c, \quad t \in [a,b] \text{ a.e.}$$

Remark 2.5. If $c = 0$, then the assumption $\alpha \in (\frac{1}{2}, 1)$ in Theorems 2.2, 2.4 can be replaced by $\alpha \in (0,1)$.

Theorem 2.4 implies the following corollary.

Corollary 2.6. *If* $\alpha \in (\frac{1}{2}, 1)$, *then the function*

$$[a,b] \ni t \longmapsto \Phi^A_{\alpha,\alpha}(b-t) \in \mathbb{R}^{n\times n}$$

belongs to $L^2([a,b], \mathbb{R}^{n\times n})$.

3. REMARKS ON SYSTEM (1.1)

Let $\alpha \in (0,1)$. From the results contained in [8, Theorem 4] it follows that if $u \in I^{1-\alpha}_{a+}(L^2)$, then there exists a unique in

$$AC^1 = AC^1([a,b], \mathbb{R}^n) = \{x : [a,b] \to \mathbb{R}^n; x \text{ is absolutely continuous and } x' \in L^1\}$$

solution x_u of problem (1.1). Moreover (cf. [8, Theorem 6]), the following result holds.

Lemma 3.1. *If* $u_j \longrightarrow u_0$ *in* $I^{1-\alpha}_{a+}(L^2)$, *then* $x_j \rightrightarrows x_0$ *uniformly on* $[a,b]$ (*here* x_j *is a unique solution to problem* (1.1), *corresponding to control* u_j).

Of course, a solution x_u of problem (1.1) , corresponding to $u \in I^{1-\alpha}_{a+}(L^2)$, is a solution of system

$$D^\alpha_{a+}x(t) = Ax(t) + Bu(t), \quad t \in [a,b] \text{ a.e.} \tag{3.1}$$

Since $Ax(\cdot) + Bu(\cdot)$ belongs to L^2, $D^\alpha_{a+}x_u \in L^2$, too. Using this fact and [8, Proof of Theorem 4, formula (16)] we deduce that $x_u \in AC^2$. Consequently, $x_u \in AC^2_0$. Thus, if $u \in I^{1-\alpha}_{a+}(L^2)$ and $x_u \in AC^2$ satisfies (1.1), then x_u belongs to AC^2_0 and satisfies (3.1). Conversely, if $u \in I^{1-\alpha}_{a+}(L^2)$ and $x_u \in AC^2_0$ satisfies (3.1), then x_u belongs to AC^2 and satisfies (1.1).

Now, let us consider the operator

$$F : AC^2_0 \to I^{1-\alpha}_{a+}(L^2),$$
$$F(x) = D^\alpha_{a+}x - Ax.$$

The above operator is well defined because

$$D^\alpha_{a+}x = I^{1-\alpha}_{a+}x' \in I^{1-\alpha}_{a+}(L^2)$$

and

$$x = I^1_{a+}x' = I^{1-\alpha}_{a+}I^{\alpha}_{a+}x' \in I^{1-\alpha}_{a+}(I^{\alpha}_{a+}(L^2)) \subset I^{1-\alpha}_{a+}(L^2)$$

for any $x \in AC^2_0$. Of course, it is linear. Moreover, since

$$\left\| D^{\alpha}_{a+}x \right\|_{I^{1-\alpha}_{a+}(L^2)} = \left\| I^{1-\alpha}_{a+}x' \right\|_{I^{1-\alpha}_{a+}(L^2)} = \|x'\|_{L^2} = \|x\|_{AC^2_0},$$

$$\|x\|_{I^{1-\alpha}_{a+}(L^2)} = \left\| I^{1-\alpha}_{a+}I^{\alpha}_{a+}x' \right\|_{I^{1-\alpha}_{a+}(L^2)} = \left\| I^{\alpha}_{a+}x' \right\|_{L^2} \le d\,\|x'\|_{L^2} = d\,\|x\|_{AC^2_0}$$

for any $x \in AC^2_0$ and some $d > 0$, where $\|x\|_{I^{1-\alpha}_{a+}(L^2)} = \left\| D^{1-\alpha}_{a+}x \right\|_{L^2}$, $\|x\|_{AC^2_0} = \|x'\|_{L^2}$, therefore F is bounded. From its bijectivity and from the Banach inverse mapping theorem it follows that F is a homeomorphism.

4. GRADIENT OF THE POINTWISE TERM

Let us assume that $\alpha \in (\frac{1}{2}, 1)$ and consider the pointwise term

$$J_1(u) = \frac{1}{2}\left| x_u(b) - c \right|^2_{\mathbb{R}^n}, \quad u \in I^{1-\alpha}_{a+}(L^2),$$

of functional (1.2). J_1 can be written as the following superposition

$$u \in I^{1-\alpha}_{a+}(L^2) \longmapsto x_u \in AC^2_0 \longmapsto x_u(b) - c \in \mathbb{R}^n \longmapsto \frac{1}{2}\left| x_u(b) - c \right|^2_{\mathbb{R}^n} \in \mathbb{R}. \tag{4.1}$$

The interior mapping

$$\lambda : u \in I^{1-\alpha}_{a+}(L^2) \longmapsto x_u \in AC^2_0$$

is linear and continuous. It follows from the fact that λ is the following superposition

$$u \in I^{1-\alpha}_{a+}(L^2) \longmapsto Bu \in I^{1-\alpha}_{a+}(L^2) \longmapsto F^{-1}(Bu) \in AC^2_0.$$

So, the differential $\lambda'(u)$ of λ at a point $u \in I^{1-\alpha}_{a+}(L^2)$ is the following mapping

$$\lambda'(u) : v \in I^{1-\alpha}_{a+}(L^2) \longmapsto h_v \in AC^2_0,$$

where $h_v \in AC^2_0$ is such that

$$D^{\alpha}_{a+}h_v(t) = Ah_v(t) + Bv(t), \quad t \in [a, b] \text{ a.e.}$$

Consequently, the differential $J_1'(u)$ of J_1 at a point $u \in I^{1-\alpha}_{a+}(L^2)$ is the mapping

$$J_1'(u) : v \in I^{1-\alpha}_{a+}(L^2) \longmapsto (x_u(b) - c)h_v(b) \in \mathbb{R}.$$

Now (cf. Theorem 2.4), let $\Psi^1_u \in AC^{\alpha,2}_{b-}$ be a unique solution of problem

$$\begin{cases} D^{\alpha}_{b-}\Psi(t) = A^T\Psi(t), & t \in [a, b] \text{ a.e.}, \\ I^{1-\alpha}_{b-}\Psi(b) = x_u(b) - c. \end{cases} \tag{4.2}$$

We have (cf. Theorem 2.1 and Corollary 2.6)

$$
\begin{aligned}
(x_u(b)-c)h_v(b) &= I_{b-}^{1-\alpha}\Psi_u^1(b)h_v(b) \\
&= \int_a^b D_{a+}^{\alpha}h_v(t)\Psi_u^1(t)dt - \int_a^b h_v(t)D_{b-}^{\alpha}\Psi_u^1(t)dt \\
&= \int_a^b \left(Ah_v(t)+Bv(t)\right)\Psi_u^1(t)dt - \int_a^b h_v(t)A^T\Psi_u^1(t)dt \\
&= \int_a^b \Psi_u^1(t)Bv(t)dt
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
\int_a^b \Psi_u^1(t)Bv(t)dt &= \int_a^b I_{a+}^{1-\alpha}D_{a+}^{1-\alpha}(Bv)(t)\Psi_u^1(t)dt \\
&= \int_a^b D_{a+}^{1-\alpha}v(t)I_{b-}^{1-\alpha}(B^T\Psi_u^1)(t)dt \\
&= \int_a^b D_{a+}^{1-\alpha}v(t)D_{a+}^{1-\alpha}I_{a+}^{1-\alpha}I_{b-}^{1-\alpha}(B^T\Psi_u^1)(t)dt \\
&= \left\langle v, I_{a+}^{1-\alpha}I_{b-}^{1-\alpha}(B^T\Psi_u^1)\right\rangle_{I_{a+}^{1-\alpha}(L^2)}.
\end{aligned}
$$

So,

$$
J_1'(u)v = \int_a^b \Psi_u^1(t)Bv(t)dt = \left\langle v, I_{a+}^{1-\alpha}(B^T I_{b-}^{1-\alpha}\Psi_u^1)\right\rangle_{I_{a+}^{1-\alpha}(L^2)} \tag{4.4}
$$

for $v \in I_{a+}^{1-\alpha}(L^2)$ and, consequently,

$$
\nabla J_1(u) = I_{a+}^{1-\alpha}(B^T I_{b-}^{1-\alpha}\Psi_u^1), \tag{4.5}
$$

where

$$
\Psi_u^1(t) = \Phi_{\alpha,\alpha}^{A^T}(b-t)(x_u(b)-c) \tag{4.6}
$$

for $t \in [a,b]$ a.e.

Now, we shall show that the gradient ∇J_1 of J_1 is Lipschitzian. In the proof of this fact we shall use the integrability of the function

$$
[a,b] \ni t \longmapsto \int_a^t \left|\Phi_{\alpha,\alpha}^{A^T}(t-s)\right|^2 ds \in \mathbb{R}_0^+, \tag{4.7}
$$

proved in [10].

We have the following result.

Theorem 4.1. *Gradient ∇J_1 satisfies the Lipschitz condition, i.e.*

$$\|\nabla J_1(u) - \nabla J_1(w)\|_{I_{a+}^{1-\alpha}(L^2)} \leq \frac{(b-a)^{1-\alpha} L}{\Gamma(2-\alpha)} \|u - w\|_{I_{a+}^{1-\alpha}(L^2)}$$

for u, $w \in I_{a+}^{1-\alpha}(L^2)$, where

$$\begin{aligned} L = (2\,|A|^2 \Big(\int_a^b \Big(\int_a^t \left|\Phi_{\alpha,\alpha}^A(t-s)\right|^2 ds\Big)^{\frac{1}{2}} dt\Big)^2 \\ + 2\,|B|^2 (b-a))^{\frac{1}{2}} \,|B|^2 \left\| I_{b-}^{1-\alpha} \left|\Phi_{\alpha,\alpha}^A(b-\cdot)\right|\right\|_{L^2}. \end{aligned} \tag{4.8}$$

Proof. Let us fix u, $w \in I_{a+}^{1-\alpha}(L^2)$. We have

$$\|\nabla J_1(u) - \nabla J_1(w)\|_{I_{a+}^{1-\alpha}(L^2)} = \left\|B^T I_{b-}^{1-\alpha}\Psi_u - B^T I_{b-}^{1-\alpha}\Psi_w\right\|_{L^2}.$$

In [10] it has been shown that

$$\left\|B^T I_{b-}^{1-\alpha}\Psi_u - B^T I_{b-}^{1-\alpha}\Psi_w\right\|_{L^2} \leq L \|u-w\|_{L^2}$$

with L given by (4.8). But (cf. [20, formula (2.72)])

$$\begin{aligned} \|u-w\|_{L^2} &= \left\|I_{a+}^{1-\alpha} D_{a+}^{1-\alpha} u - I_{a+}^{1-\alpha} D_{a+}^{1-\alpha} w\right\|_{L^2} \\ &\leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \left\|D_{a+}^{1-\alpha} u - D_{a+}^{1-\alpha} w\right\|_{L^2} = \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \|u-w\|_{I_{a+}^{1-\alpha}(L^2)} \end{aligned}$$

and the proof is complete. □

5. GRADIENT OF THE FUNCTIONAL TERM

Let $\alpha \in (\frac{1}{2}, 1)$ and consider the functional term

$$J_2(u) = \frac{1}{2} \|x_u(\cdot) - y(\cdot)\|_{L^2}^2, \; u \in I_{a+}^{1-\alpha}(L^2),$$

of the functional (1.2). J_2 can be written as the following superposition

$$u \in I_{a+}^{1-\alpha}(L^2) \longmapsto x_u \in AC_0^2 \longmapsto \frac{1}{2} \|x_u(\cdot) - y(\cdot)\|_{L^2}^2 \in \mathbb{R}.$$

It is easy to see that

$$J_2'(u)v = \int_a^b (x_u(t) - y(t)) h_v(t) dt$$

for u, $v \in I_{a+}^{1-\alpha}(L^2)$, where $h_v \in AC_0^2$ is such that

$$D_{a+}^{\alpha} h_v(t) = Ah_v(t) + Bv(t), \quad t \in [a,b] \text{ a.e.}$$

Denoting by Ψ_u^2 a unique in $AC_{b-}^{\alpha,2}$ solution of problem

$$\begin{cases} D_{b-}^{\alpha}\Psi(t) = A^T\Psi(t) + (x_u(t) - y(t)), \quad t \in [a,b] \text{ a.e.}, \\ I_{b-}^{1-\alpha}\Psi(b) = 0 \end{cases} \tag{5.1}$$

we obtain (cf. Theorem 2.1 and Corollary 2.6)

$$\begin{aligned} \int_a^b (x_u(t) - y(t))h_v(t)dt &= \int_a^b D_{b-}^{\alpha}\Psi_u^2(t)h_v(t)dt - \int_a^b A^T\Psi_u^2(t)h_v(t)dt \\ &= \int_a^b \Psi_u^2(t)D_{a+}^{\alpha}h_v(t)dt - \int_a^b A^T\Psi_u^2(t)h_v(t)dt \\ &= \int_a^b \Psi_u^2(t)(Ah_v(t) + Bv(t))dt - \int_a^b A^T\Psi_u^2(t))h_v(t)dt \\ &= \int_a^b \Psi_u^2(t)Bv(t)dt. \end{aligned} \tag{5.2}$$

In the same way as in the previous section

$$\int_a^b \Psi_u^2(t)Bv(t)dt = \left\langle I_{a+}^{1-\alpha}I_{b-}^{1-\alpha}(B^T\Psi_u^2), v\right\rangle_{I_{a+}^{1-\alpha}(L^2)}$$

and, consequently, for u, $v \in I_{a+}^{1-\alpha}(L^2)$,

$$J_2'(u)v = \int_a^b \Psi_u^2(t)Bv(t)dt = \left\langle I_{a+}^{1-\alpha}I_{b-}^{1-\alpha}(B^T\Psi_u^2), v\right\rangle_{I_{a+}^{1-\alpha}(L^2)}, \tag{5.3}$$

$$\nabla J_2(u) = I_{a+}^{1-\alpha}I_{b-}^{1-\alpha}(B^T\Psi_u^2), \tag{5.4}$$

where $\Psi_u^2 \in AC_{b-}^{\alpha,2}$ is a unique solution of problem (5.1), given by

$$\Psi_u^2(t) = \int_t^b \Phi_{\alpha,\alpha}^{A^T}(s-t)(x_u(s) - y(s))ds, \quad t \in [a,b] \text{ a.e.} \tag{5.5}$$

We also have the following result (to calculate D_3 we use integrability of function (4.7)).

Theorem 5.1. *Gradient ∇J_2 satisfies the Lipschitz condition:*

$$\|\nabla J_2(u) - \nabla J_2(w)\|_{I^{1-\alpha}_{a+}(L^2)} \leq D\,\|u-w\|_{I^{1-\alpha}_{a+}(L^2)}$$

for u, $w \in I^{1-\alpha}_{a+}(L^2)$ and some $D>0$.

Proof. We have

$$\|\nabla J_2(u) - \nabla J_2(w)\|^2_{I^{1-\alpha}_{a+}(L^2)} = \int_a^b \left|I^{1-\alpha}_{b-}(B^T(\Psi^2_u - \Psi^2_w))(t)\right|^2 dt$$
$$\leq D_1 \int_a^b \left|B^T(\Psi^2_u - \Psi^2_w)(t)\right|^2 dt \leq D_2 \int_a^b \left(\int_t^b \left|\Phi^{A^T}_{\alpha,\alpha}(s-t)(x_u(s)-x_w(s))\right| ds\right)^2 dt$$
$$\leq D_2 \int_a^b \left(\int_t^b \left|\Phi^{A^T}_{\alpha,\alpha}(s-t)\right|^2 ds\right) dt \int_a^b |(x_u(s)-x_w(s))|^2\, ds$$
$$= D_3 \int_a^b |(x_u(s)-x_w(s))|^2\, ds$$
$$\leq D_3 \int_a^b \left(\int_a^s \left|\Phi^{A}_{\alpha,\alpha}(s-t_1)\right|^2 dt_1 \int_a^s |B(u(t_1)-w(t_1))|^2\, dt_1\right) ds$$
$$\leq D_4 \int_a^b |B(u(t_1)-w(t_1))|^2\, dt_1 \leq D_5 \int_a^b |(u(t_1)-w(t_1))|^2\, dt_1 = D_5\,\|u-w\|^2_{L^2}$$
$$= D_5 \left\|I^{1-\alpha}_{a+}D^{1-\alpha}_{a+}u - I^{1-\alpha}_{a+}D^{1-\alpha}_{a+}w\right\|^2_{L^2} \leq D_6 \left\|D^{1-\alpha}_{a+}u - D^{1-\alpha}_{a+}w\right\|^2_{L^2}$$
$$= D^2\,\|u-w\|^2_{I^{1-\alpha}_{a+}(L^2)}$$

for u, $w \in I^{1-\alpha}_{a+}(L^2)$, where

$$D_1 = \left(\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2, \quad D_2 = D_1\,|B|^2, \quad D_3 = D_2 \int_a^b \left(\int_t^b \left|\Phi^{A^T}_{\alpha,\alpha}(s-t)\right|^2 ds\right) dt,$$

$$D_4 = D_3 \int_a^b\int_a^s \left|\Phi^{A}_{\alpha,\alpha}(s-t_1)\right|^2 dt_1 ds, \quad D_5 = D_4\,|B|^2,$$

$$D_6 = D_5 \left(\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2, \quad D = \sqrt{D_6}.$$

The proof is complete. □

6. GRADIENT OF THE CONTROL TERM

Let $\alpha \in (0,1)$ and consider the control term

$$J_3(u) = \frac{1}{2}\|u(\cdot)\|^2_{I^{1-\alpha}_{a+}(L^2)}, \quad u \in I^{1-\alpha}_{a+}(L^2),$$

of the functional (1.2). It is clear that

$$J_3'(u)v = \langle u, v\rangle_{I^{1-\alpha}_{a+}(L^2)} = \int_a^b D^{1-\alpha}_{a+}u(t)D^{1-\alpha}_{a+}v(t)dt \tag{6.1}$$

for u, $v \in I^{1-\alpha}_{a+}(L^2)$. Consequently,

$$\nabla J_3(u) = u. \tag{6.2}$$

Of course,

$$\|\nabla J_3(u) - \nabla J_3(w)\|_{I^{1-\alpha}_{a+}(L^2)} = \|u - w\|_{I^{1-\alpha}_{a+}(L^2)}$$

for u, $w \in I^{1-\alpha}_{a+}(L^2)$. So, ∇J_3 satisfies the Lipschitz condition with the constant 1.

7. EXISTENCE OF A SOLUTION TO (1.1)–(1.2)

Let us recall that a function $J : U \to \mathbb{R}$ where U is a convex subset of a Hilbert space H, is called strongly convex, if

$$J(\gamma u + (1-\gamma)v) \leq \gamma J(u) + (1-\gamma)J(v) - \gamma(1-\gamma)\kappa\|u - v\|^2$$

for some $\kappa > 0$ and all $\gamma \in [0,1]$, u, $v \in U$. One can show (cf. [23, Lemma 4.3.1] for the method of the proof) that J is strongly convex on U with a constant κ if and only if function $g(u) = J(u) - \kappa\|u\|^2$ is convex on U. So, for example, functional $H \ni u \longmapsto \|u\|^2 \in \mathbb{R}$ is strongly convex with constant $\kappa = 1$.

In [22, Theorem 1.3.8] the following theorem is proved.

Theorem 7.1. *If U is a convex closed subset of a Hilbert space H and a functional $J : U \to \mathbb{R}$ is strongly convex with a constant κ and lower semicontinuous on U, then $J_* := \inf_{u\in U} J(u) > -\infty$ and there exists a unique point u_* such that $J(u_*) = J_*$. Moreover, any minimizing sequence (u_k) (i.e. such that $J(u_k) \to J_*$) converges to u_* and*

$$\|u_k - u_*\| \leq \frac{1}{\kappa}(J(u_k) - J_*), \quad k = 1, 2, \ldots$$

Now, let $\alpha \in (0,1)$ and $M \subset \mathbb{R}^m$ be a fixed set. Consider problem (1.1)–(1.2) in the set $I^{1-\alpha}_{a+}(L^2_M)$, where

$$L^2_M = \{u \in L^2;\ u(t) \in M \text{ for } t \in [a,b] \text{ a.e.}\}.$$

Of course,

$$I_{a+}^{1-\alpha}(L_M^2) = \{u \in I_{a+}^{1-\alpha}(L^2);\ D_{a+}^{1-\alpha}u(t) \in M \quad \text{for } t \in [a,b] \text{ a.e.}\}.$$

We have the following theorem.

Theorem 7.2. *If $M \subset \mathbb{R}^m$ is a convex closed set, then there exists a unique point $u_* \in I_{a+}^{1-\alpha}(L_M^2)$ such that $J(u_*) = J_* := \inf_{u \in I_{a+}^{1-\alpha}(L_M^2)} J(u)$ and any minimizing sequence (u_k) converges to u_* and*

$$\|u_k - u_*\|_{I_{a+}^{1-\alpha}(L^2)} \leq 2(J(u_k) - J_*), \quad k = 1, 2, \ldots$$

Proof. The set $I_{a+}^{1-\alpha}(L_M^2)$ is convex and closed in $I_{a+}^{1-\alpha}(L^2)$. Convexity is obvious. To prove its closedness let us consider a sequence $(u_k) \subset I_{a+}^{1-\alpha}(L_M^2)$ converging in $I_{a+}^{1-\alpha}(L^2)$ to some u_0. Thus, the sequence $(D_{a+}^{1-\alpha}u_k)$ converges in L^2 to $D_{a+}^{1-\alpha}u_0$. So, there exists a subsequence of $(D_{a+}^{1-\alpha}u_k)$ that is converging pointwise a.e. on $[a,b]$ to $D_{a+}^{1-\alpha}u_0$. Since elements of this subsequence take their values a.e. on $[a,b]$ in the closed set M, therefore $D_{a+}^{1-\alpha}u_0$ has the same property. It means that $u_0 \in I_{a+}^{1-\alpha}(L_M^2)$.

Lemma 3.1 implies continuity of J on $I_{a+}^{1-\alpha}(L^2)$ (and, in consequence, on $I_{a+}^{1-\alpha}(L_M^2)$). So, since J is strongly convex on $I_{a+}^{1-\alpha}(L^2)$ (and, in consequence, on $I_{a+}^{1-\alpha}(L_M^2)$) with the constant $\kappa = \frac{1}{2}$, therefore from Theorem 7.1 the assertion follows. □

8. MAXIMUM PRINCIPLE

Let us start with the following classical result ([22, Theorem I.2.5]).

Lemma 8.1. *Let U be a convex subset of a Banach space X, J – a functional of class C^1 on U. If u_* is a global minimum point of J on U, then*

$$J'(u_*)u \geq J'(u_*)u_* \tag{8.1}$$

for any $u \in U$. If, additionally, J is convex on U, then condition (8.1) *is sufficient for u_* to be the global minimum point of J on U.*

Now, let us consider problem (1.1)–(1.2) in the set $I_{a+}^{1-\alpha}(L_M^2)$ with a convex set $M \subset \mathbb{R}^m$, in the case of $\alpha \in (\frac{1}{2}, 1)$. We have the following theorem.

Theorem 8.2. *Control u_* is a solution to problem* (1.1)–(1.2) *in the set $I_{a+}^{1-\alpha}(L_M^2)$ if and only if*

$$\begin{aligned}
&\min_{v \in M}(I_{b-}^{1-\alpha}((\Phi_{\alpha,\alpha}^{A^T}(b-\cdot)(x_{u_*}(b)-c) + \int_{\cdot}^{b} \Phi_{\alpha,\alpha}^{A^T}(s-\cdot)(x_{u_*}(s)-y(s))ds)B)(t) \\
&\qquad + D_{a+}^{1-\alpha}u_*(t))v \\
&= (I_{b-}^{1-\alpha}((\Phi_{\alpha,\alpha}^{A^T}(b-\cdot)(x_{u_*}(b)-c) + \int_{\cdot}^{b} \Phi_{\alpha,\alpha}^{A^T}(s-\cdot)(x_{u_*}(s)-y(s))ds)B)(t) \\
&\qquad + D_{a+}^{1-\alpha}u_*(t))(D_{a+}^{1-\alpha}u_*)(t)
\end{aligned}$$

for $t \in [a,b]$ a.e.

Proof. Let u_* be a solution of problem (1.1)–(1.2) in the set L^2_M. From Lemma 8.1, formulas (4.4), (5.3), (6.1) and convexity of functional (1.2) it follows that optimality of $u_* \in I^{1-\alpha}_{a+}(L^2_M)$ is equivalent to the functional condition of the form

$$\min_{u(\cdot)\in I^{1-\alpha}_{a+}(L^2_M)} \int_a^b (\Psi^1_{u_*}(t) + \Psi^2_{u_*}(t))BI^{1-\alpha}_{a+}(D^{1-\alpha}_{a+}u)(t)dt + \int_a^b D^{1-\alpha}_{a+}u_*(t)D^{1-\alpha}_{a+}u(t)dt$$
$$= \int_a^b (\Psi^1_{u_*}(t) + \Psi^2_{u_*}(t))BI^{1-\alpha}_{a+}(D^{1-\alpha}_{a+}u_*)(t)dt + \int_a^b D^{1-\alpha}_{a+}u_*(t)D^{1-\alpha}_{a+}u_*(t)dt,$$

where $\Psi^1_{u_*} \in AC^{\alpha,2}_{b-}$ is a unique solution of problem (4.2) and $\Psi^2_{u_*} \in AC^{\alpha,2}_{b-}$ is a unique solution to problem (5.1). Both solutions are given by (2.2). Using the classical fractional theorem on the integration by parts, in the integral form (cf. [20]), we can write the above equality in the following form

$$\min_{u(\cdot)\in I^{1-\alpha}_{a+}(L^2_M)} \int_a^b \left(I^{1-\alpha}_{b-}\left((\Psi^1_{u_*}(\cdot) + \Psi^2_{u_*}(\cdot))B\right)(t) + D^{1-\alpha}_{a+}u_*(t)\right)(D^{1-\alpha}_{a+}u)(t)dt$$
$$= \int_a^b \left(I^{1-\alpha}_{b-}\left((\Psi^1_{u_*}(\cdot) + \Psi^2_{u_*}(\cdot))B\right)(t) + D^{1-\alpha}_{a+}u_*(t)\right)(D^{1-\alpha}_{a+}u_*)(t)dt.$$

This condition is equivalent to

$$\min_{v(\cdot)\in L^2_M} \int_a^b \left(I^{1-\alpha}_{b-}\left((\Psi^1_{u_*}(\cdot) + \Psi^2_{u_*}(\cdot))B\right)(t) + D^{1-\alpha}_{a+}u_*(t)\right)v(t)dt$$
$$= \int_a^b \left(I^{1-\alpha}_{b-}\left((\Psi^1_{u_*}(\cdot) + \Psi^2_{u_*}(\cdot))B\right)(t) + D^{1-\alpha}_{a+}u_*(t)\right)(D^{1-\alpha}_{a+}u_*)(t)dt.$$

From [6, Lemma 6] it follows that the above condition is equivalent to the pointwise minimum condition given the theorem. □

9. GRADIENT METHOD

We have the following result concerning the convergence of the gradient method ([22, Theorem I.4.1]).

Lemma 9.1. *Let a functional $J : H \to \mathbb{R}$ be of class C^1, bounded below and with the gradient satisfying the Lipschitz condition. If $(u_k) \subset H$ is a sequence described by the formula*

$$u_{k+1} = u_k - \beta_k \nabla J(u_k), \quad k = 0, 1, \ldots, \tag{9.1}$$

with any fixed $u_0 \in H$, *where parameter* β_k *is a minimum point of the function*

$$f_k : [0, \infty) \ni \beta \longmapsto J(u_k - \beta \nabla J(u_k)) \in \mathbb{R}, \quad k = 0, 1, \ldots,$$

then the sequence $(J(u_k))$ *is nonincreasing and*

$$\lim_{k \to \infty} \|\nabla J(u_k)\| = 0. \tag{9.2}$$

If, additionally, J *is strongly convex with a constant* $\kappa > 0$, *then the sequence* (u_k) *converges to* u_* *– a unique minimum point of* J *and*

$$0 \le J(u_k) - J_* \le (J(u_0) - J_*)q^k, \tag{9.3}$$

$$\|u_k - u_*\|^2 \le \frac{1}{\kappa}(J(u_0) - J_*)q^k, \tag{9.4}$$

for $k = 0, 1, \ldots$, *where* $J_* = \inf_{u \in H} J(u)$, $q = 1 - \frac{2\kappa}{L_1} \in [0, 1)$ $(L_1$ *is a Lipschitz constant for* $\nabla J)$.

Remark 9.2. One can show that $2\kappa \le L_1$ (cf. [10]).

Now, let us consider problem (1.1)–(1.2) in the set $U = I_{a+}^{1-\alpha}(L^2)$ with $\alpha \in (\frac{1}{2}, 1)$. Using the above lemma we obtain

Theorem 9.3. *A sequence* (u_k) *given by* (9.1) *with* β_k *described in Lemma 9.1 converges to* u_* *– a unique minimum point of* J *on* $I_{a+}^{1-\alpha}(L^2)$, *the sequence* $(J(u_k))$ *is nonincreasing and conditions* (9.2), (9.3), (9.4) *are satified with*

$$L_1 = \frac{(b-a)^{1-\alpha} L}{\Gamma(2-\alpha)} + D + 1,$$

where L *is given by* (4.8) *and* D *is described in the proof of Theorem 5.1.*

Remark 9.4. Let us observe that

$$\begin{aligned}
f_k(\beta) = J(u_k - \beta \nabla J(u_k)) &= \frac{1}{2} \left| x_{u_k - \beta \nabla J(u_k)}(b) - c \right|^2_{\mathbb{R}^n} + \frac{1}{2} \left\| x_{u_k - \beta \nabla J(u_k)}(\cdot) - y(\cdot) \right\|^2_{L^2} \\
&+ \frac{1}{2} \left\| u_k(\cdot) - \beta \nabla J(u_k)(\cdot) \right\|^2_{I_{a+}^{1-\alpha}(L^2)} = \frac{1}{2} \left| x_{u_k}(b) - c - \beta x_{\nabla J(u_k)}(b) \right|^2_{\mathbb{R}^n} \\
&+ \frac{1}{2} \left\| x_{u_k}(\cdot) - y(\cdot) - \beta x_{\nabla J(u_k)}(\cdot) \right\|^2_{L^2} + \frac{1}{2} \left\| u_k(\cdot) - \beta \nabla J(u_k)(\cdot) \right\|^2_{I_{a+}^{1-\alpha}(L^2)} \\
= J_1(u_k) &- \beta (x_{u_k}(b) - c) x_{\nabla J(u_k)}(b) + \frac{1}{2}\beta^2 \left| x_{\nabla J(u_k)}(b) \right|^2_{\mathbb{R}^n} \\
&+ J_2(u_k) - \beta \left\langle x_{u_k}(\cdot) - y(\cdot), x_{\nabla J(u_k)}(\cdot) \right\rangle_{L^2} + \frac{1}{2}\beta^2 \left\| x_{\nabla J(u_k)}(\cdot) \right\|^2_{L^2} \\
&+ J_3(u_k) - \beta \left\langle u_k(\cdot), \nabla J(u_k)(\cdot) \right\rangle_{I_{a+}^{1-\alpha}(L^2)} + \frac{1}{2}\beta^2 \left\| \nabla J(u_k)(\cdot) \right\|^2_{I_{a+}^{1-\alpha}(L^2)}.
\end{aligned}$$

So, if

$$\left| x_{\nabla J(u_k)}(b) \right|_{\mathbb{R}^n} + \left\| x_{\nabla J(u_k)}(\cdot) \right\|_{L^2} + \left\| \nabla J(u_k)(\cdot) \right\|_{I_{a+}^{1-\alpha}(L^2)} = 0$$

for some k, then one can put $\beta_k = 0$ and, consequently, $\nabla J(u_k) = 0$. This means that u_k is a unique minimum point of J. If

$$\left|x_{\nabla J(u_k)}(b)\right|_{\mathbb{R}^n} + \left\|x_{\nabla J(u_k)}(\cdot)\right\|_{L^2} + \|\nabla J(u_k)(\cdot)\|_{I_{a+}^{1-\alpha}(L^2)} \neq 0,$$

then

$$\beta_k = \frac{(x_{u_k}(b) - c)x_{\nabla J(u_k)}(b) + \left\langle x_{u_k}(\cdot) - y(\cdot), x_{\nabla J(u_k)}(\cdot)\right\rangle_{L^2} + \langle u_k(\cdot), \nabla J(u_k)(\cdot)\rangle_{I_{a+}^{1-\alpha}(L^2)}}{\left|x_{\nabla J(u_k)}(b)\right|_{\mathbb{R}^n}^2 + \left\|x_{\nabla J(u_k)}(\cdot)\right\|_{L^2}^2 + \|\nabla J(u_k)(\cdot)\|_{I_{a+}^{1-\alpha}(L^2)}^2}$$

is the unique minimum point of f_k. It is worth observing (cf. (4.3), (4.5), (5.4), (6.2)) that

$$(x_{u_k}(b) - c)x_{\nabla J(u_k)}(b) = \int_a^b \left|B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t)\right|^2 dt$$
$$+ \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t))(B^T I_{b-}^{1-\alpha}\Psi_{u_k}^2(t))dt + \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t))(D_{a+}^{1-\alpha}u_k(t))dt,$$

where $\Psi_{u_k}^1$ is given by (4.6) and $\Psi_{u_k}^2$ – by (5.5) (with u replaced by u_k). Similarly (cf. (5.2), (4.5), (5.4), (6.2)),

$$\left\langle x_{u_k}(\cdot) - y(\cdot), x_{\nabla J(u_k)}(\cdot)\right\rangle_{L^2} = \int_a^b \left|I_{b-}^{1-\alpha}(B^T\Psi_{u_k}^2)(t)\right|^2 dt$$
$$+ \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^2(t))(B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t))dt + \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^2(t))(D_{a+}^{1-\alpha}u_k(t))dt$$

and

$$\langle u_k(\cdot), \nabla J(u_k)(\cdot)\rangle_{I_{a+}^{1-\alpha}(L^2)} = \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t))(D_{a+}^{1-\alpha}u_k(t))dt$$
$$+ \int_a^b (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^2(t))(D_{a+}^{1-\alpha}u_k(t))dt + \int_a^b (D_{a+}^{1-\alpha}u_k(t))(D_{a+}^{1-\alpha}u_k(t))dt.$$

So,

$$\beta_k = \frac{\int_a^b \left|(B^T I_{b-}^{1-\alpha}\Psi_{u_k}^1(t)) + (B^T I_{b-}^{1-\alpha}\Psi_{u_k}^2(t)) + (D_{a+}^{1-\alpha}u_k(t))\right|^2 dt}{\left|x_{\nabla J(u_k)}(b)\right|_{\mathbb{R}^n}^2 + \left\|x_{\nabla J(u_k)}(\cdot)\right\|_{L^2}^2 + \|\nabla J(u_k)(\cdot)\|_{I_{a+}^{1-\alpha}(L^2)}^2}$$

$$= \frac{\|\nabla J(u_k)(\cdot)\|_{I_{a+}^{1-\alpha}(L^2)}^2}{\left|x_{\nabla J(u_k)}(b)\right|_{\mathbb{R}^n}^2 + \left\|x_{\nabla J(u_k)}(\cdot)\right\|_{L^2}^2 + \|\nabla J(u_k)(\cdot)\|_{I_{a+}^{1-\alpha}(L^2)}^2}.$$

10. PROJECTION OF THE GRADIENT METHOD

By $P_U(u)$ we denote the projection of a point $u \in H$ on a convex closed subset U of a Hilbert space H. We shall use the following result on the convergence of the projection of the gradient method ([22, Theorem I.4.4]).

Lemma 10.1. *Let U be a convex closed subset of a Hilbert space H and $J : U \to \mathbb{R}$ – a functional of class C^1, bounded below and with the gradient ∇J satisfying the Lipschitz condition with a constant L. If $(u_k) \subset H$ is a sequence described by the formula*

$$u_{k+1} = P_U(u_k - \beta_k \nabla J(u_k)), \quad k = 0, 1, \ldots, \tag{10.1}$$

with any fixed $u_0 \in H$, where β_k, $k = 0, 1, \ldots$, is such that

$$\varepsilon_0 \le \beta_k \le \frac{2}{L + 2\varepsilon} \tag{10.2}$$

(here ε_0, ε are fixed positive parameters such that $\varepsilon_0 \le \frac{2}{L+2\varepsilon}$), then the sequence $(J(u_k))$ is nonincreasing and

$$\lim_{k\to\infty} \|u_k - u_{k+1}\| = 0. \tag{10.3}$$

If, additionally, J is strongly convex on U, then the sequence (u_k) converges to a unique minimum point u_* of J on U and there exists a constant $c \ge 0$ such that

$$\|u_k - u_*\|^2 \le \frac{c}{k} \tag{10.4}$$

for $k = 1, 2, \ldots$

Remark 10.2. The constant c can be calulated (cf. [22, Theorem I.4.4] and [23, Theorem V.2.2] for the method of calculation).

Now, let us consider problem (1.1)–(1.2) with $\alpha \in (\frac{1}{2}, 1)$ in the set $I_{a+}^{1-\alpha}(L_M^2) \subset I_{a+}^{1-\alpha}(L^2)$, where M is a convex closed subset of $\mathbb{R}^m$. In the proof of Theorem 7.2 it has been shown that $I_{a+}^{1-\alpha}(L_M^2)$ is a convex closed set in $I_{a+}^{1-\alpha}(L^2)$. So, from the above lemma and Theorem 7.2 we obtain the following result.

Theorem 10.3. *If ε_0, $\varepsilon > 0$ are such that $\varepsilon_0 \leq \frac{2}{L+2\varepsilon}$, where $L > 0$ is a Lipschitz constant for ∇J, then the sequence (u_k) given by (10.1)–(10.2) converges to u_* – a unique minimum point of J on $I_{a+}^{1-\alpha}(L_M^2)$ and condition (10.4) is satisfied. Moreover, the sequence $(J(u_k))$ is nonincreasing.*

Remark 10.4. Let $M = [n_1, N_1] \times \ldots \times [n_m, N_m] \subset \mathbb{R}^m$. In [22, Part I.4] one can find the form of the projection $P_{L_M^2} : L^2 \to L_M^2$ of the space L^2 on the set L_M^2. This allows us to describe the projection $P_{I_{a+}^{1-\alpha}(L_M^2)} : I_{a+}^{1-\alpha}(L^2) \to I_{a+}^{1-\alpha}(L_M^2)$ of the space $I_{a+}^{1-\alpha}(L^2)$ on the set $I_{a+}^{1-\alpha}(L_M^2)$. Indeed, for any $u \in I_{a+}^{1-\alpha}(L^2)$, we have

$$\begin{aligned}
\min_{v \in I_{a+}^{1-\alpha}(L_M^2)} \|u - v\|_{I_{a+}^{1-\alpha}(L^2)} &= \min_{v \in I_{a+}^{1-\alpha}(L_M^2)} \left\|D_{a+}^{1-\alpha}u - D_{a+}^{1-\alpha}v\right\|_{L^2} \\
&= \min_{f \in L_M^2} \left\|D_{a+}^{1-\alpha}u - f\right\|_{L^2} = \left\|D_{a+}^{1-\alpha}u - P_{L_M^2}(D_{a+}^{1-\alpha}u)\right\|_{L^2} \\
&= \left\|D_{a+}^{1-\alpha}u - D_{a+}^{1-\alpha}I_{a+}^{1-\alpha}(P_{L_M^2}(D_{a+}^{1-\alpha}u))\right\|_{L^2} \\
&= \left\|u - I_{a+}^{1-\alpha}(P_{L_M^2}(D_{a+}^{1-\alpha}u))\right\|_{I_{a+}^{1-\alpha}(L^2)}.
\end{aligned}$$

So,

$$P_{I_{a+}^{1-\alpha}(L_M^2)}(u) = I_{a+}^{1-\alpha}(P_{L_M^2}(D_{a+}^{1-\alpha}u))$$

for $u \in I_{a+}^{1-\alpha}(L^2)$.

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