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AFFINE EXTENSIONS OF FUNCTIONS WITH A CLOSED GRAPH

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Abstract. Let A be a closed G_δ-subset of a normal space X. We prove that every function $f_0\colon A \to \mathbb{R}$ with a closed graph can be extended to a function $f\colon X \to \mathbb{R}$ with a closed graph, too. This is a consequence of a more general result which gives an affine and constructive method of obtaining such extensions.

Keywords: real-valued functions with a closed graph, points of discontinuity, affine extensions of functions.

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1. INTRODUCTION

Let $\mathcal{C}(A)$ denote the set of all continuous functions on a nonempty subset A of a Hausdorff space X. In this paper, every considered function is real. The set of all closed-graph functions on X is denoted by $\mathcal{U}(X)$. Obviously $\mathcal{C}(X) \subset \mathcal{U}(X)$. This paper deals with the following general problem in the theory of real functions, which is inspired by the Tietze extension theorem:

(P) *Let A be a nonempty subset of a topological space X and let $f_0 \in \mathbb{R}^A$ be a function with a certain property (W). Can f_0 be extended to a function $f \in \mathbb{R}^X$ with the same property (W)?*

It is well known that if X is a metric space, and A is a closed subset of X, the Tietze theorem can be significantly strengthened: In 1933 Borsuk [4] proved that there is a positive linear operator Ext from $\mathcal{C}(A)$ into $\mathcal{C}(X)$ such that $\mathrm{Ext}(f_0)_{\restriction A} = f_0$ for every $f_0 \in \mathcal{C}(A)$; furthermore, the restriction of Ext to the space $\mathcal{C}^b(A)$ of all bounded elements of $\mathcal{C}(A)$ is a positive isometry into $\mathcal{C}^b(X)$. Thus, the Borsuk's operator Ext was the first example of a linear extension operator: its existence proved it is possible to extend two functions $f, g \in \mathcal{C}(A)$ in such a way that the extension of $f + g$ to an element of $\mathcal{C}(X)$ is the sum of extensions of f and g, respectively (one should note that in 1951 Dugundji [7] generalized Borsuk's theorem for continuous mappings into a locally convex linear space, instead of $\mathbb{R}$, but in this paper we do not consider such kinds of extensions; we confine our studies only to real-valued functions).

The first results concerning the case of the Borsuk-Dugundji theorem for spaces of differentiable functions came from Merrien [11] and Bromberg [5], and for spaces of analytic mappings - from Aron and Berner [1]. In 2007, Fefferman [8] obtained a generalization of Merrien's and Bromberg's results. He proved that if $C^m(E)$ denotes the space of restrictions to $E \subset \mathbb{R}^n$ of m-differentiable functions $f\colon \mathbb{R}^n \to \mathbb{R}$, then there is a linear and continuous operator $T\colon \mathcal{C}^m(E) \to \mathcal{C}^m(\mathbb{R}^n)$ such that $T(f_{\restriction E}) = f$.

A natural question related to the above-mentioned results and problem (P) reads as follows: *Does there exist a larger class of functions, including the class of continuous functions, where Tietze-type theorems hold true?* This question has a few positive answers. A first result of this kind is due to Kuratowski [10]: in 1933 he obtained a Tietze-type result for functions of the first Baire class defined on G_δ-subsets of a metric space, and not until 2005 Kalenda and Spurný [9] extended Kuratowski's theorem for completely regular spaces. On the other hand, in 2010 we proved [12] that if X is a P-space (i.e., every G_δ-subset of X is open) then $\mathcal{C}(X) = \mathcal{U}(X)$, and thus (formally) for every closed subset A of X, every $f_0 \in \mathcal{U}(A)$ can be extended to $f \in \mathcal{U}(X)$. This observation has led us to the conjecture that a Tietze-type theorem should hold for the class of closed graph functions defined on some subsets of a Hausdorff space X. The conjecture is confirmed in our Theorem 3.2 below, where we show that there is a positively affine extension operator from $\mathcal{U}(A)$ into $\mathcal{U}(X)$, where A is a zero-subset of X.

2. NOTATIONS AND DEFINITIONS

For every subset $A \subset X$, let $\operatorname{cl}(A)$, $\operatorname{int}(A)$ and $\operatorname{bd}(A)$ denote the *closure*, *interior* and *boundary* of A, respectively. The spaces $\mathbb{R}$ and $X \times \mathbb{R}$ are considered with their standard topologies. A function $f\colon X \to \mathbb{R}$ is *piecewise continuous* if there are nonempty closed sets $X_n \subset X$, $n \in \mathbb{N}$ such that $X = \bigcup_{n=0}^{\infty} X_n$ and the restriction $f_{\restriction X_n}$ is continuous for each $n \in \mathbb{N}$. For every function $f\colon X \to \mathbb{R}$, the symbol $G(f)$ denotes the graph of f, and the symbols $C(f)$ and $D(f)$ $(= X \setminus C(f))$ denote the sets of continuity and discontinuity points of f, respectively. We say that $f\colon X \to \mathbb{R}$ is a *function with a closed graph*, if $G(f)$ is a closed subset of $X \times \mathbb{R}$. The symbol $\mathcal{U}^+(X)$ stands for the set of all non-negative elements of $\mathcal{U}(X)$.

In 1985, Doboš [6] proved that the sum of two non-negative functions with a closed graph is a function with a closed graph. Since $0 \in \mathcal{U}^+(X)$, we have

$$\mathcal{U}^+(X) + \mathcal{U}^+(X) = \mathcal{U}^+(X). \tag{2.1}$$

Notice, however, that $\mathcal{U}^+(X) - \mathcal{U}^+(X) \neq \mathcal{U}(X)$, i.e. there is an example of a space X and functions $f, g \in \mathcal{U}^+(X)$ such that $f - g \notin \mathcal{U}(X)$ (see [6, p. 9]).

Definition 2.1. Let L_1, L_2 be two cones in linear spaces E_1, E_2, respectively (i.e. $L_i + L_i \subset L_i$, $aL_i \subset L_i$, $i = 1, 2$, for every $a \in \mathbb{R}^+$, and $L_i \cap (-L_i) = \{0\}$). We say that a mapping $T\colon L_1 \to L_2$ is *positively affine* if, for any elements $x, y \in L_1$ and $a, b \in \mathbb{R}^+$ such that $a + b = 1$, we have $T(ax + by) = aT(x) + bT(y)$.

3. MAIN THEOREM

Let X be a topological space, let A be a nonempty zero-set (i.e. $A = [g = 0] := g^{-1}(0)$ for some $g \in \mathcal{C}(X)$), and let $f_0\colon A \to \mathbb{R}$ be a function with a closed graph. The symbol $f_{(A,g)}$ denotes a real function defined on X of the form

$$f_{(A,g)}(x) = \begin{cases} f_0(x), & x \in A, \\ \frac{1}{g(x)}, & x \notin A. \end{cases} \tag{3.1}$$

To simplify notations, for A and g fixed, we write f instead of $f_{(A,g)}$. The symbol $\mathrm{Ext}_{(A,g)}$ denotes a mapping $\mathbb{R}^A \to \mathbb{R}^X$ defined by the formula

$$\mathrm{Ext}_{(A,g)}(f_0) = f.$$

Remark 3.1. From the above definitions it follows that if $A = g_1^{-1}(0) = g_2^{-1}(0)$ and $g_1 \neq g_2$, then $f_{(A,g_1)} \neq f_{(A,g_2)}$, and hence $\mathrm{Ext}_{(A,g_1)}(f) \neq \mathrm{Ext}_{(A,g_2)}(f)$ for every $f \in \mathbb{R}^A$.

The main result of this paper reads as follows.

Theorem 3.2. *Let X be a topological Hausdorff space, let A be a nonempty zero-subset of X, and let $f_0\colon A \to \mathbb{R}$ be a map with a closed graph. Then*

(a) *there is a function $f\colon X \to \mathbb{R}$ with a closed graph such that $f_{\restriction A} = f_0$, and*
(b) *the set $D(f)$, of points of discontinuity of f, is of the form*

$$D(f) = D(f_0) \cup \operatorname{bd} A. \tag{3.2}$$

More exactly, for every fixed function $g \in \mathcal{C}(X)$ such that $A = g^{-1}(0)$, the operator $\mathrm{Ext}_{(A,g)}$ *defined above maps $\mathcal{U}(A)$ into $\mathcal{U}(X)$ and is positively affine.*

One should note that from formula (2) it follows that the resulting function f is unbounded and discontinuous, in general, unless the set A is closed and open.

Proof. We shall prove first that the mapping $f = f_{(A,g)}$ defined by formula (3.1) has a closed graph. Let (x_δ) be a Moore-Smith (MS) sequence such that $x_\delta \to x$ and $f(x_\delta) \to t$.

If $x \notin A$, the continuity of g implies that $t = \frac{1}{g(x)} = f(x)$.

For $x \in A$, we consider the following two cases:

(i) $x \in \operatorname{int} A \neq \emptyset$,
(ii) $x \in A \setminus \operatorname{int} A$.

In case (i), the nonempty set $\operatorname{int} A$ is open, thus there is α_0 such that $x_\alpha \in \operatorname{int} A$ for every $\alpha > \alpha_0$. Therefore $f(x_\alpha) = f_0(x_\alpha) \to t$ and $t = f_0(x) = f(x)$ because f_0 has a closed graph.

In case (ii), we have $f(x) = f_0(x)$ and $g(x) = 0$. We claim there is β such that, for every $\alpha > \beta$, we have $x_\alpha \in A$. Indeed, otherwise, for every index β there would be an index $\alpha_\beta > \beta$ such that $x_{\alpha_\beta} = y_\beta \in X \setminus A$. Then

$$f(y_\beta) = \frac{1}{g(y_\beta)} \to t \neq 0$$

(the case $t = 0$ is impossible, because then we would have $|g(y_\beta)| \to \infty$ with $y_\beta \to x$, which contradicts the continuity of g at x). Hence

$$g(y_\beta) \to \frac{1}{t} \in (0, \infty). \tag{3.3}$$

On the other hand, the continuity of g *implies that* $g(y_\beta) \to g(x) = 0$, which contradicts (3.3). Thus, there is an element β such that, for any index $\alpha > \beta$, we have $f(x_\alpha) = f_0(x_\alpha) \to t$. Now the closedness of the graph of f_0 implies that $t = f_0(x) = f(x)$. We thus have showed that f has a closed graph, as claimed.

Now we shall prove equality (3.2); equivalently,

$$D(f) = (X \setminus C(f_0)) \cup \Big(A \cap (X \setminus \operatorname{int} A)\Big). \tag{3.4}$$

Let us fix $x \in D(f)$. Suppose, by way of contradiction, that $x \notin D(f_0) \cup \operatorname{bd} A$. Then, by (3.4), we have $x \in C(f_0) \cap \big[(X \setminus A) \cup \operatorname{int} A\big]$, whence $x \in C(f_0)$ and $x \in (X \setminus A) \cup \operatorname{int} A$. If $x \in X \setminus A$, we have $f(x) = \frac{1}{g(x)}$, whence $x \in C(g) \subset C(f)$, and if $x \in \operatorname{int} A \neq \emptyset$, we have $f(x) = f_0(x)$, and hence $x \in C(f\restriction_{\operatorname{int} A}) \subset C(f)$. In both the cases we thus have $x \in C(f)$, contrary to our hypothesis. We thus have shown that

$$D(f) \subset D(f_0) \cup \operatorname{bd} A. \tag{3.5}$$

For the proof of the reversed inclusion to (3.5), let us fix $x \in D(f_0) \cup \operatorname{bd} A$. Assume first that $x \in D(f_0)$. Since each point of the discontinuity of f_0 is a point of the discontinuity of f, we obtain $x \in D(f)$. Moreover, if $x \in \operatorname{bd} A = A \cap (X \setminus \operatorname{int} A)$, there is an MS-sequence $(x_\delta) \subset X \setminus A$ convergent to x. By the continuity of g, we obtain $\frac{1}{f(x_\delta)} = g(x_\delta) \to 0$. Therefore $|f(x_\alpha)| \to \infty$, whence $x \in D(f)$. We thus have shown that if $x \in D(f_0) \cup \operatorname{bd} A$ then $x \in D(f)$, i.e.,

$$D(f_0) \cup \operatorname{bd} A \subset D(f). \tag{3.6}$$

Combining inclusions (3.5) and (3.6), we obtain (3.2). Obviously, $\operatorname{Ext}_{(A,g)}$ is positively affine. The proof is complete. □

The following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let A be a closed and G_δ (closed, respectively) subset of a normal (perfectly normal, respectively) space X. Then there is a positively affine extension operator* $\operatorname{Ext}\colon \mathcal{U}(A) \to \mathcal{U}(X)$.

Notice that the Tietze theorem asserts that if A is a closed subset of a normal space X, then the restriction from $\mathcal{C}(X)$ to $\mathcal{C}(A)$ is surjective. From Theorem 3.2 we obtain a similar result.

Corollary 3.4. *Let X be a topological Hausdorff space, and let A be a zero-set. Then the restriction operator $r_A\colon \mathcal{U}(X) \to \mathcal{U}(A)$ (given by $r_A(f) = f_{\restriction A}$) is a surjection.*

In two examples below we show that the requirement in Corollary 3.3, "A to be a closed subset of X" cannot be replaced by the weaker condition: "A to be an F_σ-set". We do not know, however, if the hypothesis of Theorem 1 about A is essential, i.e., we cannot indicate a closed and non zero-subset A of a Hausdorff space X such that some $f_0 \in \mathcal{U}(A)$ cannot be extended to an element of $\mathcal{U}(X)$.

In Example 3.5 we address an "extremely bad" case: there is a nonempty F_σ-subset A of a metric space X and $f \in \mathcal{U}(A)$ such that, for every subset B of A such that $\operatorname{int}(\operatorname{cl}(B)) \neq \emptyset$, the restriction $f_{\restriction B}$ cannot be extended to an element of $\mathcal{U}\left(\operatorname{cl}(B)\right)$.

Example 3.5. Let $X = [0,1]$ be the unit interval with the standard topology. Set $A = (0,1) \cap \mathbb{Q} \subset X$, and let B be any fixed subset of A such that $\operatorname{int}(\operatorname{cl} B) \neq \emptyset$. Let $f\colon A \to \mathbb{R}$ be a function defined as $f(\frac{m}{n}) = n$ with m, n positive integers and $\frac{m}{n}$ irreducible. Then f is a function with a closed graph which is discontinuous at every point of A (due to the fact, that the number of irreducible fractions in A with a given denominator is finite). Since $\operatorname{int}(\operatorname{cl} B) \neq \emptyset$, there are real numbers $0 < a < b < 1$ such that $[a,b] \subset \operatorname{cl} B$. Suppose that $f_B := f_{\restriction B}$ can be extended to $\overline{f_B} \in \mathcal{U}\left(\operatorname{cl} B\right)$. Then (see [3, Lemma 2.2]) $\overline{f_B}$ is piecewise continuous, and thus there is a sequence (B_n) of closed subsets of $[a,b]$ such that $[a,b] = \bigcup_{n=1}^{\infty} B_n$ and the restriction $\overline{f_B}_{\restriction B_n}$ is continuous for each $n \in \mathbb{N}$. Then, by the Baire property, there is a number $n_0 \in \mathbb{N}$ such that $\operatorname{int}\left(B_{n_0}\right) \neq \emptyset$. Hence there is a nonempty interval (c,d) contained in B_{n_0}. Thus, by the continuity of the restrictions $\overline{f_B}_{\restriction B_n}$, every rational number $\xi \in (c,d)$ would be the point of continuity of $\overline{f_B}$, and thus the point of continuity of $f_B = f_{\restriction B}$, but this contradicts the discontinuity of f.

In the next example we show that the hypothesis in Corollary 3.3: "A is closed" cannot be replaced by "A is open F_σ". But now, in contrast to Example 3.5, there are subsets $B \subset A$ such that $\operatorname{int}(B) \neq \emptyset$ and $f_{\restriction B}$ has an extension to an element of $\mathcal{U}\left(\operatorname{cl}(B)\right)$.

Example 3.6. Let $X = \mathbb{R}$ and $A = (0,\infty)$. Thus A is an open and F_σ subset of X. Let $f_0\colon (0,\infty) \to \mathbb{R}$ be a map given by the formula $f_0(x) = \sin\frac{1}{x}$. The function f_0 is of course continuous at every point $x \in A$, whence $f_0 \in \mathcal{U}(A)$. However, the function f_0 cannot be extended to any function $f\colon [0,\infty) \to \mathbb{R}$ with a closed graph because $\operatorname{cl} G(f_0) \supset \{0\} \times [-1,1]$.

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