Opuscula Math. 35, no. 6 (2015), 935–956
http://dx.doi.org/10.7494/OpMath.2015.35.6.935 Opuscula Mathematica

EXISTENCE, UNIQUENESS AND ESTIMATES OF CLASSICAL SOLUTIONS TO SOME EVOLUTIONARY SYSTEM

Lucjan Sapa

Communicated by Mirosław Lachowicz

Abstract. The theorem of the local existence, uniqueness and estimates of solutions in Hölder spaces for some nonlinear differential evolutionary system with initial conditions is formulated and proved. This system is composed of one partial hyperbolic second-order equation and an ordinary subsystem with a parameter. In the proof of the theorem we use the Banach fixed-point theorem, the Arzeli-Ascola lemma and the integral form of the differential problem.

Keywords: hyperbolic wave equation, telegraph equation, system of nonlinear equations, existence, uniqueness and estimates of solutions, Hölder space.

Mathematics Subject Classification: 35M31, 35A09, 35A01, 35A02, 35B45.

1. INTRODUCTION

Let functions $f : [0,T] \times \mathbb{R}^{2+n} \to \mathbb{R}$, $g = (g_1, \dots, g_n) : [0,T] \times \mathbb{R}^{2+n} \to \mathbb{R}^n$, $u_0, u_1 : \mathbb{R} \to \mathbb{R}$, $v_0 = (v_{01}, \dots, v_{0n}) : \mathbb{R} \to \mathbb{R}^n$ and a constant $c \geq 0$ be given. Consider a nonlinear second-order partial differential system of $(1+n)$ equations of the form

$$\begin{cases} u_{tt} - u_{xx} + cu_t = f(t,x,u,v), & (t,x) \in [0,T] \times \mathbb{R}, \\ v_t = g(t,x,u,v), & (t,x) \in [0,T] \times \mathbb{R} \end{cases} \tag{1.1}$$

with the initial conditions

$$\begin{cases} u\,(0,x) = u_0(x), & x \in \mathbb{R}, \\ v\,(0,x) = v_0(x), & x \in \mathbb{R}, \\ u_t\,(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $v = (v_1, \ldots, v_n)$. The equations are weakly coupled. If $c > 0$, then the first hyperbolic wave equation in (1.1) is called the telegraph equation. The others equations in (1.1) are of first-order with a space parameter x. Our results are true if we consider the first equation in (1.1) only with f not depending on v (see examples in Remark 4.4).

Some information especially about the maximum principles and the existence of time-periodic bounded weak solutions for the wave or telegraph equations is given in [12, 16].

Physical motivation of system (1.1) with $c = 1$ and f, g of a special form together with a construction of the solitary waves solutions and their stability are given in [13]. The existence, uniqueness and continuous dependence on the initial values of global classical solutions to a similar system but with the parabolic leading equation instead of our telegraph or wave equations were studied by J. Evans and N. Shenk [6, 7]. Moreover, J. Evans in [7–10] considered a stability in the suitable sense of stationary and traveling wave classical solutions to such systems. Those systems describe for example the dynamics of a nerve impulse in axons and they cover in particular the Hodgkin-Huxley system. Similar evolutionary systems appear also in [15]. In [11] there is described a connection between fast and slow waves in the FitzHugh-Nagumo system. Realistic view of wave mechanics was first proposed by de Broglie [1]. In his inspiring work Madelung related the linear Schrödinger equation to the hydrodynamic type system [14]. The various aspects of the hydrodynamic [2] and mechanistic [5] formulations of the nonlinear Schrödinger equation are still at the center of interest. These formulations lead to systems composed of the second or the third order partial differential evolution equations and the first order ones.

In this paper we consider local in time bounded together with their first derivatives solutions (u, v), $u \in C^{1+\alpha}_{loc}\left([0,\tau] \times \mathbb{R}, \mathbb{R}\right) \cap C^2\left([0,\tau] \times \mathbb{R}, \mathbb{R}\right)$, $v \in C^{1+\alpha}_{loc}\left([0,\tau] \times \mathbb{R}, \mathbb{R}^n\right)$ of the initial problem (1.1), (1.2), where $C^{1+\alpha}_{loc}\left([0,\tau] \times\ \mathbb{R}, \mathbb{R}\right)$ and $C^{1+\alpha}_{loc}\left([0,\tau] \times \mathbb{R}, \mathbb{R}^n\right)$ ($\alpha \in (0,1]$) are some Hölder spaces (see Section 2). If $\alpha = 1$, then $C^{1+\alpha}_{loc} = C^{1+\alpha}$ and they are global Lipschitz spaces. The main result of the paper is a theorem on the existence, uniqueness and estimates of such solutions. The estimates can be given a priori and they depend on the estimates of the initial conditions and the right-hand sides of equations. In the proof of the theorem we use the Banach fixed-point theorem, the Arzeli-Ascola lemma and the integral form of the differential problem. The similar construction of the proof was used by T. Człapiński [3, 4] for quasilinear hyperbolic partial differential functional equations of the first order.

The paper is organized in the following way. In Section 2 a notation is introduced and some definitions and assumptions are formulated. The integral system equivalent under some assumptions to the initial differential problem (1.1), (1.2) is given in Section 3. In Section 4 the theorem on the existence, uniqueness and estimates of solutions to (1.1), (1.2) is proved.

2. NOTATION, DEFINITIONS AND ASSUMPTIONS

Let $\|\cdot\|$ be the *maximum norm* in $\mathbb{R}^d$, i.e.

$$\|y\| = \max_{i=1,\ldots,d}|y_i|, \tag{2.1}$$

where $y \in \mathbb{R}^d$. In the space of bounded continuous functions $C_b\left(\Omega, \mathbb{R}^d\right)$ we define the *supremum norm*

$$\|z\|_0 = \sup\left\{\|z(\omega)\| : \ \omega \in \Omega\right\}, \tag{2.2}$$

where $z \in C_b\left(\Omega, \mathbb{R}^d\right)$, $\Omega = \mathbb{R}$ or $\Omega \subset \mathbb{R}^2$. Obviously $\left(C_b\left(\Omega, \mathbb{R}^d\right), \|\cdot\|_0\right)$ is the Banach space. The space of bounded continuous functions together with their first derivatives we denote by $C_b^1\left(\Omega, \mathbb{R}^d\right)$ and together with their first and second derivatives by $C_b^2\left(\Omega, \mathbb{R}^d\right)$. By the symbol

$$\|z\|_1 = \|z\|_0 + \|z_t\|_0 + \|z_x\|_0 \tag{2.3}$$

we denote the norm in $C_b^1\left(\Omega, \mathbb{R}^d\right)$.

For any $z \in C_b\left(\Omega, \mathbb{R}^d\right)$ and $\alpha \in (0, 1]$, let

$$[z]_{H,\alpha} = \sup\left\{\left\|z(t,x) - z(\bar{t},\bar{x})\right\| \left[|t-\bar{t}| + |x-\bar{x}|\right]^{-\alpha} : \ (t,x), (\bar{t},\bar{x}) \in \Omega\right\}. \tag{2.4}$$

Note that $[z]_{H,\alpha} < \infty$ is the smallest Hölder constant for the function z and the exponent α, it is usually called the *Hölder coefficient*. If $\alpha = 1$, then it is the *Lipschitz coefficient*. The *Hölder space* $C^{k+\alpha}\left(\Omega, \mathbb{R}^d\right)$, $k = 0, 1$, is the space of functions $z \in C_b^k\left(\Omega, \mathbb{R}^d\right)$ with the finite norm

$$\|z\|_{0+\alpha} = \|z\|_0 + [z]_{H,\alpha} \quad \text{if} \quad k = 0, \tag{2.5}$$

$$\|z\|_{1+\alpha} = \|z\|_1 + [z_t]_{H,\alpha} + [z_x]_{H,\alpha} \quad \text{if} \quad k = 1. \tag{2.6}$$

If $\Omega = \mathbb{R}$, then t and z_t in the above definitions do not appear.

Let $\Delta(x,\tau) \subset \mathbb{R} \times [0,\tau]$, $x \in \mathbb{R}$ be any isosceles triangle with the vertices $(x-\tau, 0)$, (x,τ), $(x+\tau, 0)$. We denote by $C^{1+\alpha}_{loc}([0,\tau] \times \mathbb{R}, \mathbb{R}^d)$ the *Hölder space* of functions $z \in C_b^1([0,\tau] \times \mathbb{R}, \mathbb{R}^d)$ such that $z \in C^{1+\alpha}(\Delta(x,\tau), \mathbb{R}^d)$ for any $\Delta(x,\tau)$ with the constants

$$\begin{aligned}
&[z_t]_{H,\alpha,\Delta(x,\tau)} \\
&= \sup\left\{\left\|z_t(t,x) - z_t(\bar{t},\bar{x})\right\| \left[|t-\bar{t}| + |x-\bar{x}|\right]^{-\alpha} : \ (t,x), (\bar{t},\bar{x}) \in \Delta(x,\tau)\right\}, \\
&[z_x]_{H,\alpha,\Delta(x,\tau)} \\
&= \sup\left\{\left\|z_x(t,x) - z_x(\bar{t},\bar{x})\right\| \left[|t-\bar{t}| + |x-\bar{x}|\right]^{-\alpha} : \ (t,x), (\bar{t},\bar{x}) \in \Delta(x,\tau)\right\}
\end{aligned} \tag{2.7}$$

independent of $\Delta(x,\tau)$. We say in this case that z_t, z_x are *uniformly Hölder continuous* in $[0,\tau] \times \mathbb{R}$. It is clear that if $\alpha = 1$, then $C^{1+\alpha}_{loc}([0,\tau] \times \mathbb{R}, \mathbb{R}^d) = C^{1+\alpha}([0,\tau] \times \mathbb{R}, \mathbb{R}^d)$.

The following assumption on the functions u_0, v_0, u_1 will be needed.

Assumption $\mathcal{H}_1[u_0, v_0, u_1]$. Suppose that $u_0 \in C^{1+\alpha}(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$, $v_0 \in C^{1+\alpha}\left(\mathbb{R}, \mathbb{R}^n\right)$, $u_1 \in C^{0+\alpha}(\mathbb{R}, \mathbb{R}) \cap C^1(\mathbb{R}, \mathbb{R})$ and that

$$\|u_0\|_0 \le \Lambda_1, \quad \|u_1\|_0 \le \Lambda_1^{(1)}, \quad \|(u_0)_x\|_0 \le \Lambda_1^{(1)}, \tag{2.8}$$

$$[u_1]_{H,\alpha} \le \Lambda_1^{(2)}, \quad [(u_0)_x]_{H,\alpha} \le \Lambda_1^{(2)},$$

$$\|v_0\|_0 \le \Lambda_2, \quad \|g(0,\cdot,u_0(\cdot),v_0(\cdot))\|_0 \le \Lambda_2^{(1)}, \quad \|(v_0)_x\|_0 \le \Lambda_2^{(1)},$$

$$[g(0,\cdot,u_0(\cdot),v_0(\cdot))]_{H,\alpha} \le \Lambda_2^{(2)}, \quad [(v_0)_x]_{H,\alpha} \le \Lambda_2^{(2)},$$

where Λ_i, $\Lambda_i^{(j)}$, $i,j = 1,2$, are some non-negative constants.

Definition 2.1. Suppose that Assumption $\mathcal{H}_1[u_0,v_0,u_1]$ is satisfied and given non-negative constants Q_i, $Q_i^{(j)}$, $i,j = 1,2$, such that $Q_i \ge \Lambda_i$, $Q_i^{(j)} \ge \Lambda_i^{(j)}$ we will denote by $C^{1,\alpha}_{b,\tau}(Q)$, where $\tau \in (0,T]$, the set of functions $(u,v) \in C^{1+\alpha}_{loc}\left([0,\tau]\times\mathbb{R},\mathbb{R}^{1+n}\right)$, $v=(v_1,\dots,v_n)$, with the properties:

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \tag{2.9}$$

$$\|u\|_0 \le Q_1, \quad \|u_t\|_0 \le Q_1^{(1)}, \quad \|u_x\|_0 \le Q_1^{(1)}, \tag{2.10}$$

$$[u_t]_{H,\alpha,\Delta(x,\tau)} \le Q_1^{(2)}, \quad [u_x]_{H,\alpha,\Delta(x,\tau)} \le Q_1^{(2)},$$

$$\|v\|_0 \le Q_2, \quad \|v_t\|_0 \le Q_2^{(1)}, \quad \|v_x\|_0 \le Q_2^{(1)},$$

$$[v_t]_{H,\alpha,\Delta(x,\tau)} \le Q_2^{(2)}, \quad [v_x]_{H,\alpha,\Delta(x,\tau)} \le Q_2^{(2)}.$$

We will use the following assumptions on the functions f, g and their first derivatives.

Assumption $\mathcal{H}_2[f,g]$. The functions f, g and the derivatives f_x, f_p, $f_r = (f_{r_1},\dots,f_{r_n})$, g_x, g_p, $g_r = (g_{r_1},\dots,g_{r_n})$ are continuous. Moreover, there exist non-decreasing functions M_k, $M_k^{(1)}$, H_2, $H_2^{(1)} : \mathbb{R}^2_+ \to \mathbb{R}_+$, $k = 1,2$, such that for all $t,\bar{t} \in [0,T]$, $x,\overline{x} \in \mathbb{R}$, $(q_1,q_2) \in \mathbb{R}^2_+$, $|p|,|\overline{p}| \le q_1$, $\|r\|,\|\overline{r}\| \le q_2$, $i = 1,\dots,n$ we have

$$|f(t,x,p,r)| \le M_1(q_1,q_2), \tag{2.11}$$

$$|f_x(t,x,p,r)|, \ |f_p(t,x,p,r)|, \ |f_{r_i}(t,x,p,r)| \le M_1^{(1)}(q_1,q_2), \tag{2.12}$$

$$\|g(t,x,p,r)\| \le M_2(q_1,q_2), \tag{2.13}$$

$$\|g_x(t,x,p,r)\|, \ \|g_p(t,x,p,r)\|, \ \|g_{r_i}(t,x,p,r)\| \le M_2^{(1)}(q_1,q_2), \tag{2.14}$$

$$\|g(t,x,p,r) - g\left(\bar{t},x,p,r\right)\| \le H_2(q_1,q_2)|t-\bar{t}|^\alpha, \tag{2.15}$$

$$\begin{aligned} \|g_x(t,x,p,r) - g_x(t,\overline{x},\overline{p},\overline{r})\| &\le H_2^{(1)}(q_1,q_2)\left[|x-\overline{x}| + |p-\overline{p}| + \|r-\overline{r}\|\right]^\alpha, \\ \|g_p(t,x,p,r) - g_p(t,\overline{x},\overline{p},\overline{r})\| &\le H_2^{(1)}(q_1,q_2)\left[|x-\overline{x}| + |p-\overline{p}| + \|r-\overline{r}\|\right]^\alpha, \\ \|g_{r_i}(t,x,p,r) - g_{r_i}(t,\overline{x},\overline{p},\overline{r})\| &\le H_2^{(1)}(q_1,q_2)\left[|x-\overline{x}| + |p-\overline{p}| + \|r-\overline{r}\|\right]^\alpha. \end{aligned} \tag{2.16}$$

Put $M_k^* = M_k(Q_1, Q_2)$, $M_k^{(1)*} = M_k^{(1)}(Q_1, Q_2)$, $H_2^* = H_2(Q_1, Q_2)$, $H_2^{(1)*} = H_2^{(1)}(Q_1, Q_2)$, $k = 1, 2$. Some inequalities concerned Q_i, $Q_i^{(j)}$, $i, j = 1, 2$, have to be satisfied.

Assumption $\mathcal{H}_3[Q]$. Suppose that

$$\begin{aligned} Q_1 &> \Lambda_1, \\ Q_1^{(1)} &> \Lambda_1^{(1)}, \\ Q_1^{(2)} &> 2\Lambda_1^{(2)}, & \alpha \in (0,1), \\ Q_1^{(2)} &> M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{3}{2}c\Lambda_1^{(1)} + 2\Lambda_1^{(2)}, & \alpha = 1, \end{aligned} \tag{2.17}$$

$$\begin{aligned} Q_2 &> \Lambda_2, \\ Q_2^{(1)} &\geq M_2^*, \quad Q_2^{(1)} > \Lambda_2^{(1)}, \\ Q_2^{(2)} &> H_2^*, \quad Q_2^{(2)} > \Lambda_2^{(2)}, & \alpha \in (0,1), \\ Q_2^{(2)} &\geq M_2^{(1)*}\left[1 + 2\left(Q_1^{(1)} + nQ_2^{(1)}\right)\right] + H_2^*, & \alpha = 1, \\ Q_2^{(2)} &> M_2^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right) + \Lambda_2^{(2)}, & \alpha = 1. \end{aligned} \tag{2.18}$$

Remark 2.2. The conditions (2.11)–(2.14) in Assumption $\mathcal{H}_2[f, g]$ are equivalent to boundedness of f, g and $f_x, f_p, f_{r_i}, g_x, g_p, g_{r_i}$ on the sets $[0, T] \times \mathbb{R} \times [-q_1, q_1] \times [-q_2, q_2]^n$ for any $(q_1, q_2) \in \mathbb{R}_+^2$, i.e. globally with respect to t, x and locally with respect to p, r. If f, g are globally Lipschitz continuous with respect to x, p, r or f, g do not depend on t, x and are locally Lipschitz continuous with respect to p, r, then (2.12), (2.14) hold. But the global Lipschitz continuity of f, g with respect to t, x and local one with respect to p, r not always imply (2.12), (2.14), e.g. $f(t, x, p, r) = x \sin p$. On the other hand the function $f(t, x, p, r) = p^2 \sin x$ is locally Lipschitz continuous with respect to x, p only and (2.12) is true. The form of (2.11)–(2.14) is useful in applications because we do not have to calculate the norms. It is important for example if we consider differential equations with the polynomial right-hand sides. An analogous situation appears in Assumption $\mathcal{H}_1[u_0, v_0, u_1]$. An analysis of the local or global Hölder continuity of g, g_x, g_p, g_{r_i} in conditions (2.15), (2.16) is similar as the above one concerning the local or global Lipschitz continuity of f, g (see Remark 4.4 and examples (4.23), (4.24)).

Remark 2.3. Note that in Assumption $\mathcal{H}_3[Q]$ a choice of the constants Q_i, $Q_i^{(j)}$, $i, j = 1, 2$, is possible for any given Λ_i, $\Lambda_i^{(j)}$, $i, j = 1, 2$. So we may give a priori the estimates of the solution of the differential problem (1.1), (1.2), its first derivatives and the Hölder constants of its first derivatives.

Define the constants

$$
\begin{aligned}
S_{1\tau} &= \frac{1}{2}\left[M_1^* + \frac{c^2}{4}Q_1\right]\tau^2 + \left[\frac{c}{2}\Lambda_1 + \Lambda_1^{(1)}\right]\tau + \Lambda_1,\\
S_{1\tau}^{(1.t)} &= \frac{c}{4}\left[M_1^* + \frac{c^2}{4}Q_1\right]\tau^2 + \left[M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{c}{2}\Lambda_1^{(1)}\right]\tau\\
&\quad + 2^{-1+\alpha}\Lambda_1^{(2)}\tau^\alpha + \Lambda_1^{(1)},\\
S_{1\tau}^{(1.x)} &= \left[M_1^* + \frac{c^2}{4}Q_1 + \frac{c}{2}\Lambda_1^{(1)}\right]\tau + 2^{-1+\alpha}\Lambda_1^{(2)}\tau^\alpha + \Lambda_1^{(1)},\\
S_{1\tau}^{(2.t)} &= \left[\frac{c^2}{8}M_1^* + \frac{c^4}{32}Q_1\right]\tau^{3-\alpha}\\
&\quad + \left[c\left(1+2^{-\alpha}\right)M_1^* + \left(1+2^{1-\alpha}\right)M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\right.\\
&\quad + \frac{c^3}{4}\left(1+2^{-\alpha}\right)Q_1 + \frac{c^2}{4}\left(1+2^{1-\alpha}\right)Q_1^{(1)}\\
&\quad \left.+\frac{c^3}{8}\Lambda_1 + \frac{c^2}{4}\left(1+2^{1-\alpha}\right)\Lambda_1^{(1)}\right]\tau^{2-\alpha}\\
&\quad + \frac{c}{2}\Lambda_1^{(2)}\tau + \left[M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{3}{2}c\Lambda_1^{(1)}\right]\tau^{1-\alpha} + 2\Lambda_1^{(2)},\\
S_{1\tau}^{(2.x)} &= \left[\frac{c}{2}M_1^* + \left(1+2^{1-\alpha}\right)M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right) + \frac{c^3}{8}Q_1\right.\\
&\quad \left.+\frac{c^2}{4}\left(1+2^{1-\alpha}\right)Q_1^{(1)}\right]\tau^{2-\alpha}\\
&\quad + \left[M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + c\left(1 + \frac{3^{1-\alpha}}{2}\right)\Lambda_1^{(1)}\right]\tau^{1-\alpha} + 2\Lambda_1^{(2)},
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
S_{2\tau} &= M_2^*\tau + \Lambda_2,\\
S_{2\tau}^{(1.t)} &= M_2^*,\\
S_{2\tau}^{(1.x)} &= M_2^{(1)*}\left[1 + Q_1^{(1)} + nQ_2^{(1)}\right]\tau + \Lambda_2^{(1)},\\
S_{2\tau}^{(2.t)} &= M_2^{(1)*}\left[2^{1-\alpha} + \left(1+2^{1-\alpha}\right)\left(Q_1^{(1)} + nQ_2^{(1)}\right)\right]\tau^{1-\alpha} + H_2^*,\\
S_{2\tau}^{(2.x)} &= \left[H_2^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\left(1 + Q_1^{(1)} + Q_2^{(1)}\right)^\alpha\right.\\
&\quad \left.+M_2^{(1)*}\left(Q_1^{(2)} + nQ_2^{(2)}\right)\right]\tau\\
&\quad + M_2^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\tau^{1-\alpha} + \Lambda_2^{(2)}.
\end{aligned}
\tag{2.20}
$$

Remark 2.4. Note that since

$$\lim_{\tau\to 0^+} S_{1\tau} = \Lambda_1, \quad \lim_{\tau\to 0^+} S_{1\tau}^{(1.t)} = \lim_{\tau\to 0^+} S_{1\tau}^{(1.x)} = \Lambda_1^{(1)},$$

$$\lim_{\tau\to 0^+} S_{1\tau}^{(2.t)} = \lim_{\tau\to 0^+} S_{1\tau}^{(2.x)} = 2\Lambda_1^{(2)} \quad \text{if} \quad \alpha \in (0,1),$$

$$\lim_{\tau\to 0^+} S_{1\tau}^{(2.t)} = \lim_{\tau\to 0^+} S_{1\tau}^{(2.x)} = M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{3}{2}c\Lambda_1^{(1)} + 2\Lambda_1^{(2)} \quad \text{if} \quad \alpha = 1,$$

$$\lim_{\tau\to 0^+} S_{2\tau} = \Lambda_2, \quad \lim_{\tau\to 0^+} S_{2\tau}^{(1.x)} = \Lambda_2^{(1)},$$

$$\lim_{\tau\to 0^+} S_{2\tau}^{(2.t)} = H_2^*, \quad \lim_{\tau\to 0^+} S_{2\tau}^{(2.x)} = \Lambda_2^{(2)} \quad \text{if} \quad \alpha \in (0,1),$$

$$\lim_{\tau\to 0^+} S_{2\tau}^{(2.x)} = M_2^{(1)^*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right) + \Lambda_2^{(2)} \quad \text{if} \quad \alpha = 1,$$

we may by Assumption $\mathcal{H}_3[Q]$ choose $\tau \in (0,T]$ sufficiently small in order that $S_{i\tau} \le Q_i$, $S_{i\tau}^{(j.t)} \le Q_i^{(j)}$, $S_{i\tau}^{(j.x)} \le Q_i^{(j)}$, $i,j = 1,2$.

3. INTEGRAL SYSTEM

In this section we give a lemma which will be crucial in our future considerations.

Consider a nonlinear integral system of $(1+n)$ equations of the form

$$\begin{cases} u(t,x) = \frac{1}{2}\int\limits_0^t \int\limits_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)}\left[f(s,y,u(s,y),v(s,y)) + \frac{c^2}{4}u(s,y)\right]dyds \\ \qquad + \frac{1}{2}e^{-\frac{c}{2}t}\int\limits_{x-t}^{x+t} u_1(y)dy + \frac{c}{4}e^{-\frac{c}{2}t}\int\limits_{x-t}^{x+t} u_0(y)dy \\ \qquad + \frac{1}{2}e^{-\frac{c}{2}t}\left[u_0(x+t) + u_0(x-t)\right], \qquad (t,x) \in [0,T]\times\mathbb{R}, \\ v(t,x) = v_0(x) + \int\limits_0^t g(s,x,u(s,x),v(s,x))\,ds, \qquad (t,x) \in [0,T]\times\mathbb{R}. \end{cases} \tag{3.1}$$

Lemma 3.1. *Under Assumptions $\mathcal{H}_1[u_0,v_0,u_1]$ and $\mathcal{H}_2[f,g]$, the differential initial problem* (1.1), (1.2) *and the integral system* (3.1) *are equivalent in the sense that any solution $(u,v) \in C^2([0,T]\times\mathbb{R},\mathbb{R}) \times C^1([0,T]\times\mathbb{R},\mathbb{R}^n)$ of* (1.1), (1.2) *is a solution of* (3.1) *and any solution $(u,v) \in C^1\left([0,T]\times\mathbb{R},\ \mathbb{R}^{1+n}\right)$ of* (3.1) *belongs to $C^2([0,T]\times\mathbb{R},\mathbb{R}) \times C^1([0,T]\times\mathbb{R},\mathbb{R}^n)$ and fulfills* (1.1), (1.2).

Proof. The first equation in (1.1) is equivalent to the equation

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = f(t,x,u,v) + \frac{c^2}{4}u, \quad (t,x) \in [0,T]\times\mathbb{R}.$$

Hence, the anzats $u = we^{-\frac{c}{2}t}$ transforms problem (1.1), (1.2) to the equivalent differential system

$$\begin{cases} w_{tt} - w_{xx} = e^{\frac{c}{2}t} f\left(t,x,we^{-\frac{c}{2}t},v\right) + \frac{c^2}{4}w, & (t,x) \in [0,T]\times\mathbb{R}, \\ v_t = g\left(t,x,we^{-\frac{c}{2}t},v\right), & (t,x) \in [0,T]\times\mathbb{R} \end{cases} \tag{3.2}$$

with the equivalent initial conditions

$$\begin{cases} w\,(0,x) = u_0(x), & x \in \mathbb{R}, \\ v\,(0,x) = v_0(x), & x \in \mathbb{R}, \\ w_t\,(0,x) = \frac{c}{2}u_0(x) + u_1(x), & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

The continuity of f, g, u_0, v_0, u_1, the use of Riemann's method (see [16, p. 196]) for the first equation in (3.2) and integration of the second one imply that any solution $(w,v) \in C^2\left([0,T]\times\mathbb{R},\mathbb{R}\right) \times C^1\left([0,T]\times\mathbb{R},\mathbb{R}^n\right)$ of (3.2), (3.3) is a solution of a nonlinear integral system of $(1+n)$ equations of the form

$$\begin{cases} w(t,x) = \dfrac{1}{2}\displaystyle\int_0^t \int_{x-(t-s)}^{x+(t-s)} \left[e^{\frac{c}{2}s} f\left(s,y,w(s,y)e^{-\frac{c}{2}s}, v(s,y)\right) + \dfrac{c^2}{4}w(s,y)\right] dyds \\ \qquad + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} \left[\dfrac{c}{2}u_0(y) + u_1(y)\right] dy \\ \qquad + \dfrac{1}{2}\left[u_0(x+t) + u_0(x-t)\right], \qquad (t,x) \in [0,T]\times\mathbb{R}, \\ v(t,x) = v_0(x) + \displaystyle\int_0^t g\left(s,x,w(s,x)e^{-\frac{c}{2}s}, v(s,x)\right) ds, \qquad (t,x) \in [0,T]\times\mathbb{R}. \end{cases} \tag{3.4}$$

Multiplying the first equation in (3.4) by $e^{-\frac{c}{2}t}$ we have that any solution $(u,v) \in C^2\left([0,T]\times\mathbb{R},\mathbb{R}\right) \times C^1\left([0,T]\times\mathbb{R},\mathbb{R}^n\right)$ of the differential initial problem (1.1), (1.2) is a solution of the integral system (3.1).

On the other hand, let $(u,v) \in C^1\left([0,T]\times\mathbb{R},\mathbb{R}^{1+n}\right)$ be a solution of (3.1). Put for simplicity

$$A(t,x,s) = \left(s, x+(t-s), u(s,x+(t-s)), v(s,x+(t-s))\right),$$

$$B(t,x,s) = \left(s, x-(t-s), u(s,x-(t-s)), v(s,x-(t-s))\right).$$

From the regularity of f, g, u_0, v_0, u_1 and differentiation of the integrals in (3.1) we have

$$u_t(t,x) = -\frac{c}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f\left(s,y,u(s,y),v(s,y)\right) + \frac{c^2}{4}u(s,y)\right] dyds$$

$$+\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t,x,s)) + \frac{c^2}{4}u(s,x+(t-s))\right] ds$$

$$+\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t,x,s)) + \frac{c^2}{4}u(s,x-(t-s))\right] ds$$

$$-\frac{c}{4}e^{-\frac{c}{2}t}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}e^{-\frac{c}{2}t}\left[u_1(x+t)+u_1(x-t)\right]$$

$$-\frac{c^2}{8}e^{-\frac{c}{2}t}\int_{x-t}^{x+t} u_0(y)dy + \frac{1}{2}e^{-\frac{c}{2}t}\left[(u_0)_x(x+t)-(u_0)_x(x-t)\right],$$

$$u_{tt}(t,x) = \frac{c^2}{8}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f\left(s,y,u(s,y),v(s,y)\right) + \frac{c^2}{4}u(s,y)\right] dyds$$

$$-\frac{c}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t,x,s)) + \frac{c^2}{4}u(s,x+(t-s))\right] ds$$

$$-\frac{c}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t,x,s)) + \frac{c^2}{4}u(s,x-(t-s))\right] ds$$

$$+\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \Bigg[f_x(A(t,x,s)) + f_p(A(t,x,s))u_x(s,x+(t-s))$$

$$+\sum_{i=1}^n f_{r_i}(A(t,x,s))(v_i)_x(s,x+(t-s)) + \frac{c^2}{4}u_x(s,x+(t-s))\Bigg] ds$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)}\Bigg[f_x(B(t,x,s))+f_p(B(t,x,s))u_x(s,x-(t-s))\\
&+\sum_{i=1}^n f_{r_i}(B(t,x,s))(v_i)_x(s,x-(t-s))+\frac{c^2}{4}u_x(s,x-(t-s))\Bigg]ds\\
&+f\left(t,x,u(t,x),v(t,x)\right)+\frac{c^2}{4}u(t,x)\\
&+\frac{c^2}{8}e^{-\frac{c}{2}t}\int_{x-t}^{x+t}u_1(y)dy-\frac{c}{2}e^{-\frac{c}{2}t}\left[u_1(x+t)+u_1(x-t)\right]\\
&+\frac{1}{2}e^{-\frac{c}{2}t}\left[(u_1)_x(x+t)-(u_1)_x(x-t)\right]\\
&+\frac{c^3}{16}e^{-\frac{c}{2}t}\int_{x-t}^{x+t}u_0(y)dy-\frac{c^2}{8}e^{-\frac{c}{2}t}\left[u_0(x+t)+u_0(x-t)\right]\\
&-\frac{c}{4}e^{-\frac{c}{2}t}\left[(u_0)_x(x+t)-(u_0)_x(x-t)\right]\\
&+\frac{1}{2}e^{-\frac{c}{2}t}\left[(u_0)_{xx}(x+t)+(u_0)_{xx}(x-t)\right],
\end{aligned}
$$

$$
\begin{aligned}
u_{xx}(t,x)=&\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)}\Bigg[f_x(A(t,x,s))+f_p(A(t,x,s))u_x(s,x+(t-s))\\
&+\sum_{i=1}^n f_{r_i}(A(t,x,s))(v_i)_x(s,x+(t-s))+\frac{c^2}{4}u_x(s,x+(t-s))\Bigg]ds\\
&-\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)}\Bigg[f_x(B(t,x,s))+f_p(B(t,x,s))u_x(s,x-(t-s))\\
&+\sum_{i=1}^n f_{r_i}(B(t,x,s))(v_i)_x(s,x-(t-s))+\frac{c^2}{4}u_x(s,x-(t-s))\Bigg]ds\\
&+\frac{1}{2}e^{-\frac{c}{2}t}\left[(u_1)_x(x+t)-(u_1)_x(x-t)\right]\\
&+\frac{c}{4}e^{-\frac{c}{2}t}\left[(u_0)_x(x+t)-(u_0)_x(x-t)\right]\\
&+\frac{1}{2}e^{-\frac{c}{2}t}\left[(u_0)_{xx}(x+t)+(u_0)_{xx}(x-t)\right],
\end{aligned}
$$

$$v_t(t,x)=g\left(t,x,u(t,x),v(t,x)\right).$$

It is clear that $(u,v)\in C^2\left([0,T]\times\mathbb{R},\mathbb{R}\right)\times C^1\left([0,T]\times\mathbb{R},\mathbb{R}^n\right)$ and fulfills (1.1), (1.2). □

4. THE MAIN RESULT

Theorem 4.1. *If Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$, $\mathcal{H}_3[Q]$ are satisfied, then there is $\tau \in (0, T]$ such that problem* (1.1), (1.2) *has an unique solution $(u, v) \in C^{1,\alpha}_{b,\tau}(Q)$ and $u \in C^2([0, \tau] \times \mathbb{R}, \mathbb{R})$.*

Proof. Define the operator $\mathbf{H} = (F, G)$ on $C^{1,\alpha}_{b,\tau}(Q)$ by the formula

$$\begin{aligned} F[u,v](t,x) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s,y), v(s,y)) + \frac{c^2}{4} u(s,y) \right] dyds \\ + \frac{1}{2} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy + \frac{c}{4} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy \\ + \frac{1}{2} e^{-\frac{c}{2}t} \left[u_0(x+t) + u_0(x-t) \right], \end{aligned} \tag{4.1}$$

$$G[u,v](t,x) = v_0(x) + \int_0^t g(s, x, u(s,x), v(s,x)) \, ds, \tag{4.2}$$

where $(u, v) \in C^{1,\alpha}_{b,\tau}(Q)$, $(t, x) \in [0, \tau] \times \mathbb{R}$. It is obvious that $(u, v) \in C^{1,\alpha}_{b,\tau}(Q)$ is a solution of the integral system (3.1) and consequently, under additional assumptions, a solution of the initial differential problem (1.1), (1.2) (see Lemma 3.1) if and only if its a fix-point of $\mathbf{H}$.

Note that by the Arzeli-Ascola lemma, the set $C^{1,\alpha}_{b,\tau}(Q)$ is closed in the Banach space $\left(C_b \left([0, \tau] \times \mathbb{R}, \mathbb{R}^{1+n} \right), \| \cdot \|_0 \right)$. We will show that there is $\tau \in (0, T]$ such that $\mathbf{H}$ maps $C^{1,\alpha}_{b,\tau}(Q)$ into itself and it is a contraction in the complete metric space $(C^{1,\alpha}_{b,\tau}(Q), d)$ with the metric d generated by the norm $\| \cdot \|_0$. Then we will use the Banach fixed-point theorem.

Put for simplicity

$$A(t,x,s) = (s, x + (t-s), u(s, x + (t-s)), v(s, x + (t-s))),$$

$$B(t,x,s) = (s, x - (t-s), u(s, x - (t-s)), v(s, x - (t-s))),$$

$$C(x,s) = (s, x, u(s,x), v(s,x)).$$

Definitions (4.1), (4.2) and Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$ imply

$$\begin{aligned}
F_t[u,v](t,x) = &-\frac{c}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f\left(s,y,u(s,y),v(s,y)\right) + \frac{c^2}{4}u(s,y)\right] dyds \\
&+ \frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t,x,s)) + \frac{c^2}{4}u(s,x+(t-s))\right] ds \\
&+ \frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t,x,s)) + \frac{c^2}{4}u(s,x-(t-s))\right] ds \\
&- \frac{c}{4}e^{-\frac{c}{2}t}\int_{x-t}^{x+t} u_1(y)dy + \frac{1}{2}e^{-\frac{c}{2}t}\left[u_1(x+t)+u_1(x-t)\right] \\
&- \frac{c^2}{8}e^{-\frac{c}{2}t}\int_{x-t}^{x+t} u_0(y)dy + \frac{1}{2}e^{-\frac{c}{2}t}\left[(u_0)_x(x+t)-(u_0)_x(x-t)\right],
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
F_x[u,v](t,x) = &\frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t,x,s)) + \frac{c^2}{4}u(s,x+(t-s))\right] ds \\
&- \frac{1}{2}\int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t,x,s)) + \frac{c^2}{4}u(s,x-(t-s))\right] ds \\
&+ \frac{1}{2}e^{-\frac{c}{2}t}\left[u_1(x+t)-u_1(x-t)\right] \\
&+ \frac{c}{4}e^{-\frac{c}{2}t}\left[u_0(x+t)-u_0(x-t)\right] + \frac{1}{2}e^{-\frac{c}{2}t}\left[(u_0)_x(x+t)+(u_0)_x(x-t)\right],
\end{aligned} \tag{4.4}$$

$$G_t[u,v](t,x) = g\left(t,x,u(t,x),v(t,x)\right), \tag{4.5}$$

$$\begin{aligned}
G_x[u,v](t,x) = (v_0)_x(x) + \int_0^t &\left[g_x\left(C(x,s)\right) + g_p\left(C(x,s)\right)u_x(s,x)\right. \\
&\left. + \sum_{i=1}^n g_{r_i}\left(C(x,s)\right)(v_i)_x(s,x)\right] ds,
\end{aligned} \tag{4.6}$$

where $(u,v) \in C^{1,\alpha}_{b,\tau}(Q)$, $(t,x) \in [0,\tau] \times \mathbb{R}$.

We prove firstly that for a sufficiently small $\tau \in (0,T]$ the operator $\mathbf{H}$ maps $C^{1,\alpha}_{b,\tau}(Q)$ into itself. Let $(u,v) \in C^{1,\alpha}_{b,\tau}(Q)$ be fixed. Obviously, $\mathbf{H}[u,v] \in C^1_b\left([0,\tau]\times\mathbb{R}, \mathbb{R}^{1+n}\right)$ and for $x \in \mathbb{R}$

$$F[u,v](0,x) = u_0(x), \quad G[u,v](0,x) = v_0(x), \quad F_t[u,v](0,x) = u_1(x). \tag{4.7}$$

It is enough to show that there exists $\tau \in (0, T]$ such that for each $(t, x) \in [0, \tau] \times \mathbb{R}$

$$|F[u, v](t, x)| \le Q_1, \qquad |F_t[u, v](t, x)| \le Q_1^{(1)}, \qquad |F_x[u, v](t, x)| \le Q_1^{(1)}, \tag{4.8}$$

$$\|G[u, v](t, x)\| \le Q_2, \qquad \|G_t[u, v](t, x)\| \le Q_2^{(1)}, \qquad \|G_x[u, v](t, x)\| \le Q_2^{(1)}, \tag{4.9}$$

and for an arbitrarily fixed triangle $\Delta(X, \tau)$, $X \in \mathbb{R}$ and for each two points $(t, x), (\bar{t}, \overline{x}) \in \Delta(X, \tau)$

$$\begin{aligned} \left|F_t[u, v](t, x) - F_t[u, v](\bar{t}, \overline{x})\right| &\le Q_1^{(2)}[|t - \bar{t}| + |x - \overline{x}|]^\alpha, \\ \left|F_x[u, v](t, x) - F_x[u, v](\bar{t}, \overline{x})\right| &\le Q_1^{(2)}[|t - \bar{t}| + |x - \overline{x}|]^\alpha, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \left\|G_t[u, v](t, x) - G_t[u, v](\bar{t}, \overline{x})\right\| &\le Q_2^{(2)}[|t - \bar{t}| + |x - \overline{x}|]^\alpha, \\ \left\|G_x[u, v](t, x) - G_x[u, v](\bar{t}, \overline{x})\right\| &\le Q_2^{(2)}[|t - \bar{t}| + |x - \overline{x}|]^\alpha. \end{aligned} \tag{4.11}$$

From Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$, the properties of integrals and definitions (2.19), (2.20) we have

$$|F[u, v](t, x)| \le S_{1\tau},\ |F_t[u, v](t, x)| \le S_{1\tau}^{(1.t)},\ |F_x[u, v](t, x)| \le S_{1\tau}^{(1.x)}, \tag{4.12}$$

$$\|G[u, v](t, x)\| \le S_{2\tau},\ \|G_t[u, v](t, x)\| \le S_{2\tau}^{(1.t)},\ \|G_x[u, v](t, x)\| \le S_{2\tau}^{(1.x)} \tag{4.13}$$

for $(t, x) \in [0, \tau] \times \mathbb{R}$. Let $\Delta(X, \tau)$ be fixed and $(t, x), (\bar{t}, \overline{x}) \in \Delta(X, \tau)$. Using the mean value theorem for the functions:

$$\begin{aligned} f_1(t, x) &= \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f\left(s, y, u(s, y), v(s, y)\right) + \frac{c^2}{4} u(s, y) \right] dyds, \\ f_2(t, x) &= \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t - s)) \right] ds, \\ f_3(t, x) &= \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t - s)) \right] ds, \\ \widetilde{u}_1(t, x) &= e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y)dy, \qquad \widetilde{u}_0(t, x) = e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y)dy, \qquad u_0 \end{aligned}$$

and the relations: $|t-\bar{t}| = |t-\bar{t}|^{1-\alpha}|t-\bar{t}|^{\alpha}$, $|x-\overline{x}| = |x-\overline{x}|^{1-\alpha}|x-\overline{x}|^{\alpha}$, $|t-\bar{t}| \le \tau$, $|x-\overline{x}| \le 2\tau$ we obtain

$$
\begin{aligned}
&\left|F_t[u,v](t,x) - F_t[u,v](\bar{t},\overline{x})\right| \\
&\le \frac{c}{4}\Bigg| \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f\left(s,y,u(s,y),v(s,y)\right) + \frac{c^2}{4}u(s,y)\right] \\
&\qquad - \int_0^{\bar{t}} \int_{\overline{x}-(\bar{t}-s)}^{\overline{x}+(\bar{t}-s)} \int e^{-\frac{c}{2}(\bar{t}-s)} \left[f\left(s,y,u(s,y),v(s,y)\right) + \frac{c^2}{4}u(s,y)\right] \Bigg| \\
&+\frac{1}{2}\Bigg| \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t,x,s)) + \frac{c^2}{4}u(s,x+(t-s))\right] ds \\
&\qquad - \int_0^{\bar{t}} e^{-\frac{c}{2}(\bar{t}-s)} \left[f\left(A\left(\bar{t},\overline{x},s\right)\right) + \frac{c^2}{4}u\left(s,\overline{x}+\left(\bar{t}-s\right)\right)\right] ds \Bigg| \\
&+\frac{1}{2}\Bigg| \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t,x,s)) + \frac{c^2}{4}u(s,x-(t-s))\right] ds \\
&\qquad - \int_0^{\bar{t}} e^{-\frac{c}{2}(\bar{t}-s)} \left[f\left(B\left(\bar{t},\overline{x},s\right)\right) + \frac{c^2}{4}u\left(s,\overline{x}-\left(\bar{t}-s\right)\right)\right] ds \Bigg| \\
&+ \frac{c}{4}\left| e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y)dy - e^{-\frac{c}{2}\bar{t}} \int_{\overline{x}-\bar{t}}^{\overline{x}+\bar{t}} u_1(y)dy \right| \\
&+ \frac{1}{2}\left| e^{-\frac{c}{2}t}u_1(x+t) - e^{-\frac{c}{2}\bar{t}}u_1(x+t) + e^{-\frac{c}{2}\bar{t}}u_1(x+t) - e^{-\frac{c}{2}\bar{t}}u_1(\overline{x}+\bar{t})\right| \\
&+ \frac{1}{2}\left| e^{-\frac{c}{2}t}u_1(x-t) - e^{-\frac{c}{2}\bar{t}}u_1(x-t) + e^{-\frac{c}{2}\bar{t}}u_1(x-t) - e^{-\frac{c}{2}\bar{t}}u_1(\overline{x}-\bar{t})\right| \\
&+ \frac{c^2}{8}\left| e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y)dy - e^{-\frac{c}{2}\bar{t}} \int_{\overline{x}-\bar{t}}^{\overline{x}+\bar{t}} u_0(y)dy \right| \\
&+\frac{1}{2}\Big| e^{-\frac{c}{2}t}(u_0)_x(x+t) - e^{-\frac{c}{2}\bar{t}}(u_0)_x(x+t) \\
&\qquad + e^{-\frac{c}{2}\bar{t}}(u_0)_x(x+t) - e^{-\frac{c}{2}\bar{t}}(u_0)_x(\overline{x}+\bar{t})\Big| \\
&+\frac{1}{2}\Big| e^{-\frac{c}{2}t}(u_0)_x(x-t) - e^{-\frac{c}{2}\bar{t}}(u_0)_x(x-t) \\
&\qquad + e^{-\frac{c}{2}\bar{t}}(u_0)_x(x-t) - e^{-\frac{c}{2}\bar{t}}(u_0)_x(\overline{x}-\bar{t})\Big|
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
\leq \frac{c}{4}\Bigg| & -\frac{c}{2}\int_0^{t_1}\int_{x_1-(t_1-s)}^{x_1+(t_1-s)} e^{-\frac{c}{2}(t_1-s)}\left[f\left(s,y,u(s,y),v(s,y)\right)+\frac{c^2}{4}u(s,y)\right]ds \\
& +\int_0^{t_1} e^{-\frac{c}{2}(t_1-s)}\left[f\left(A(t_1,x_1,s)\right)+\frac{c^2}{4}u\left(s,x_1+(t_1-s)\right)\right]ds \\
& +\int_0^{t_1} e^{-\frac{c}{2}(t_1-s)}\left[f\left(B(t_1,x_1,s)\right)+\frac{c^2}{4}u\left(s,x_1-(t_1-s)\right)\right]ds\Bigg||t-\bar{t}| \\
+\frac{c}{4}\Bigg| & \int_0^{t_1} e^{-\frac{c}{2}(t_1-s)}\left[f\left(A(t_1,x_1,s)\right)+\frac{c^2}{4}u\left(s,x_1+(t_1-s)\right)\right]ds \\
& -\int_0^{t_1} e^{-\frac{c}{2}(t_1-s)}\left[f\left(B(t_1,x_1,s)\right)+\frac{c^2}{4}u\left(s,x_1-(t_1-s)\right)\right]ds\Bigg||x-\overline{x}| \\
+\frac{1}{2}\Bigg| & \int_0^{t_2}\Bigg[-\frac{c}{2}e^{-\frac{c}{2}(t_2-s)}\left[f(A(t_2,x_2,s))+\frac{c^2}{4}u(s,x_2+(t_2-s))\right] \\
& \qquad +e^{-\frac{c}{2}(t_2-s)}\Bigg[f_x(A(t_2,x_2,s))+f_p(A(t_2,x_2,s))u_x(s,x_2+(t_2-s)) \\
& \qquad\qquad +\sum_{i=1}^n f_{r_i}(A(t_2,x_2,s))(v_i)_x(s,x_2+(t_2-s)) \\
& \qquad\qquad +\frac{c^2}{4}u_x(s,x_2+(t_2-s))\Bigg]\Bigg]ds \\
& +f\left(t_2,x_2,u(t_2,x_2),v(t_2,x_2)\right)+\frac{c^2}{4}u(t_2,x_2)\Bigg||t-\bar{t}| \\
+\frac{1}{2}\Bigg| & \int_0^{t_2} e^{-\frac{c}{2}(t_2-s)}\Bigg[f_x(A(t_2,x_2,s))+f_p(A(t_2,x_2,s))u_x(s,x_2+(t_2-s)) \\
& \qquad +\sum_{i=1}^n f_{r_i}(A(t_2,x_2,s))(v_i)_x(s,x_2+(t_2-s)) \\
& \qquad +\frac{c^2}{4}u_x(s,x_2+(t_2-s))\Bigg]ds\Bigg||x-\overline{x}|
\end{aligned}
$$

$$+\frac{1}{2}\left|\int_{0}^{t_3}\left[-\frac{c}{2}e^{-\frac{c}{2}(t_3-s)}\left[f\left(B\left(t_3,x_3,s\right)\right)+\frac{c^2}{4}u\left(s,x_3-\left(t_3-s\right)\right)\right]\right.\right.$$

$$-e^{-\frac{c}{2}(t_3-s)}\left[f_x\left(B\left(t_3,x_3,s\right)\right)+f_p\left(B\left(t_3,x_3,s\right)\right)u_x\left(s,x_3-\left(t_3-s\right)\right)\right.$$

$$\left.\left.+\sum_{i=1}^{n}f_{r_i}\left(B\left(t_3,x_3,s\right)\right)\left(v_i\right)_x\left(s,x_3-\left(t_3-s\right)\right)+\frac{c^2}{4}u_x\left(s,x_3-\left(t_3-s\right)\right)\right]\right]ds$$

$$\left.+f\left(t_3,x_3,u\left(t_3,x_3\right),v\left(t_3,x_3\right)\right)+\frac{c^2}{4}u\left(t_3,x_3\right)\right|\left|t-\bar{t}\right|$$

$$+\frac{1}{2}\left|\int_{0}^{t_3}e^{-\frac{c}{2}(t_3-s)}\left[f_x\left(B\left(t_3,x_3,s\right)\right)+f_p\left(B\left(t_3,x_3,s\right)\right)u_x\left(s,x_3-\left(t_3-s\right)\right)\right.\right.$$

$$\left.\left.+\sum_{i=1}^{n}f_{r_i}\left(B\left(t_3,x_3,s\right)\right)\left(v_i\right)_x\left(s,x_3-\left(t_3-s\right)\right)+\frac{c^2}{4}u_x\left(s,x_3-\left(t_3-s\right)\right)\right]ds\right|$$

$$\times\left|x-\overline{x}\right|$$

$$+\frac{c}{4}\left[\left|-\frac{c}{2}e^{-\frac{c}{2}t_4}\int_{x_4-t_4}^{x_4+t_4}u_1(y)dy+e^{-\frac{c}{2}t_4}\left[u_1\left(x_4+t_4\right)+u_1\left(x_4-t_4\right)\right]\right|\left|t-\bar{t}\right|\right.$$

$$\left.+e^{-\frac{c}{2}t_4}\left|u_1\left(x_4+t_4\right)-u_1\left(x_4-t_4\right)\right|\left|x-\overline{x}\right|\right]$$

$$+\frac{1}{2}\left|-\frac{c}{2}e^{-\frac{c}{2}t_5}(t-\bar{t})\right|\left|u_1(x+t)\right|+\frac{1}{2}e^{-\frac{c}{2}\bar{t}}\left|u_1(x+t)-u_1(\overline{x}+\bar{t})\right|$$

$$+\frac{1}{2}\left|-\frac{c}{2}e^{-\frac{c}{2}t_5}(t-\bar{t})\right|\left|u_1(x-t)\right|+\frac{1}{2}e^{-\frac{c}{2}\bar{t}}\left|u_1(x-t)-u_1(\overline{x}-\bar{t})\right|$$

$$+\frac{c^2}{8}\left[\left|-\frac{c}{2}e^{-\frac{c}{2}t_6}\int_{x_6-t_6}^{x_6+t_6}u_0(y)dy+e^{-\frac{c}{2}t_6}\left[u_0\left(x_6+t_6\right)+u_0\left(x_6-t_6\right)\right]\right|\left|t-\bar{t}\right|\right.$$

$$\left.+e^{-\frac{c}{2}t_6}\left|u_0\left(x_6+t_6\right)-u_0\left(x_6-t_6\right)\right|\left|x-\overline{x}\right|\right]$$

$$+\frac{1}{2}\left|-\frac{c}{2}e^{-\frac{c}{2}t_5}(t-\bar{t})\right|\left|(u_0)_x(x+t)\right|+\frac{1}{2}e^{-\frac{c}{2}\bar{t}}\left|(u_0)_x(x+t)-(u_0)_x(\overline{x}+\bar{t})\right|$$

$$+\frac{1}{2}\left|-\frac{c}{2}e^{-\frac{c}{2}t_5}(t-\bar{t})\right|\left|(u_0)_x(x-t)\right|+\frac{1}{2}e^{-\frac{c}{2}\bar{t}}\left|(u_0)_x(x-t)-(u_0)_x(\overline{x}-\bar{t})\right|$$

$$
\begin{aligned}
&\leq \Bigg\{ \frac{c}{4}\left[\frac{c}{2}\left(M_1^* + \frac{c^2}{4}Q_1\right)\tau^2\tau^{1-\alpha} + 2\left(M_1^* + \frac{c^2}{4}Q_1\right)\tau\tau^{1-\alpha}\right.\\
&\quad \left. +2\left(M_1^* + \frac{c^2}{4}Q_1\right)\tau(2\tau)^{1-\alpha}\right] + 2\cdot\frac{1}{2}\left[\frac{c}{2}\left(M_1^* + \frac{c^2}{4}Q_1\right)\tau\tau^{1-\alpha}\right.\\
&\quad + M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\tau\tau^{1-\alpha} + \frac{c^2}{4}Q_1^{(1)}\tau\tau^{1-\alpha} + \left(M_1^* + \frac{c^2}{4}Q_1\right)\tau^{1-\alpha}\\
&\quad \left. +M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\tau(2\tau)^{1-\alpha} + \frac{c^2}{4}Q_1^{(1)}\tau(2\tau)^{1-\alpha}\right]\\
&\quad + \frac{c}{4}\left[\left(\frac{c}{2}\Lambda_1^{(1)}2\tau + 2\Lambda_1^{(1)}\right)\tau^{1-\alpha} + \Lambda_1^{(2)}(2\tau)^{\alpha}(2\tau)^{1-\alpha}\right] + \frac{c}{2}\Lambda_1^{(1)}\tau^{1-\alpha} + \Lambda_1^{(2)}\\
&\quad +\frac{c^2}{8}\left[\left(\frac{c}{2}\Lambda_1 2\tau + 2\Lambda_1\right)\tau^{1-\alpha} + \Lambda_1^{(1)}2\tau(2\tau)^{1-\alpha}\right] + \frac{c}{2}\Lambda_1^{(1)}\tau^{1-\alpha} + \Lambda_1^{(2)}\Bigg\}\\
&\quad \times [|t-\bar{t}| + |x-\overline{x}|]^{\alpha}\\
&= S_{1\tau}^{(2.t)}[|t-\bar{t}| + |x-\overline{x}|]^{\alpha}.
\end{aligned}
$$

The points $(t_i, x_i) \in \Delta(X,\tau)$, $i = 1,\dots,5$, are intermediate ones. Making similar calculations as above we get

$$
\begin{aligned}
&\left|F_x[u,v](t,x) - F_x[u,v](\bar{t},\overline{x})\right|\\
&\leq \Bigg\{\left[\left(\frac{c}{2}M_1^* + M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)\right)\tau + M_1^*\right]\tau^{1-\alpha}\\
&\quad + M_1^{(1)*}\left(1 + Q_1^{(1)} + nQ_2^{(1)}\right)(2\tau)^{1-\alpha}\tau + \frac{c^2}{4}\left[\left(\frac{c}{2}Q_1 + Q_1^{(1)}\right)\tau + Q_1\right]\tau^{1-\alpha}\\
&\quad + \frac{c^2}{4}Q_1^{(1)}(2\tau)^{1-\alpha}\tau + \Lambda_1^{(2)} + \frac{c}{2}\Lambda_1^{(1)}\tau^{1-\alpha} + \frac{c}{2}\left[\Lambda_1^{(1)}(3\tau)^{1-\alpha} + \frac{c}{2}\Lambda_1\tau^{1-\alpha}\right]\\
&\quad +\Lambda_1^{(2)} + \frac{c}{2}\Lambda_1^{(1)}\tau^{1-\alpha}\Bigg\}[|t-\bar{t}| + |x-\overline{x}|]^{\alpha}\\
&= S_{1\tau}^{(2.x)}[|t-\bar{t}| + |x-\overline{x}|]^{\alpha},
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
\left\|G_t[u,v](t,x) - G_t[u,v](\bar{t},\overline{x})\right\| &\leq \Big\{H_2^* + M_2^{(1)*}\Big[(2\tau)^{1-\alpha} + Q_1^{(1)}\left(\tau^{1-\alpha} + (2\tau)^{1-\alpha}\right)\\
&\quad +nQ_2^{(1)}\left(\tau^{1-\alpha} + (2\tau)^{1-\alpha}\right)\Big]\Big\}[|t-\bar{t}| + |x-\overline{x}|]^{\alpha}\\
&= S_{2\tau}^{(2.t)}[|t-\bar{t}| + |x-\overline{x}|]^{\alpha}
\end{aligned}
\tag{4.16}
$$

for a fixed $\Delta(X,\tau)$ and $(t,x), (\bar{t},\overline{x}) \in \Delta(X,\tau)$. Because of the non-sufficient regularity of g we can not use the mean value theorem for $g_1(t,x) = G_x[u,v](t,x)$. But using the additivity of an integral we have

$$
\begin{aligned}
&\left\|G_x[u,v](t,x)-G_x[u,v](\bar{t},\overline{x})\right\| \\
&\leq \|(v_0)_x(x)-(v_0)_x(\overline{x})\| + \left\| \int_0^t g_x\left(C(x,s)\right)ds - \int_0^{\bar{t}} g_x\left(C(\overline{x},s)\right)ds \right\| \\
&\quad + \left\| \int_0^t g_p(C(x,s))u_x(s,x)ds - \int_0^{\bar{t}} g_p(C(\overline{x},s))u_x(s,\overline{x})ds \right\| \\
&\quad + \sum_{i=1}^n \left\| \int_0^t g_{r_i}(C(x,s))(v_i)_x(s,x)ds - \int_0^{\bar{t}} g_{r_i}(C(\overline{x},s))(v_i)_x(s,\overline{x})ds \right\| \\
&\leq \Lambda_2^{(2)}|x-\overline{x}|^\alpha + \left\| \int_0^t g_x\left(C(x,s)\right)ds - \int_0^t g_x\left(C(\overline{x},s)\right)ds - \int_t^{\bar{t}} g_x\left(C(\overline{x},s)\right)ds \right\| \\
&\quad + \left\| \int_0^t g_p(C(x,s))u_x(s,x)ds - \int_0^t g_p(C(\overline{x},s))u_x(s,\overline{x})ds - \int_t^{\bar{t}} g_p(C(\overline{x},s))u_x(s,\overline{x})ds \right\| \\
&\quad + \sum_{i=1}^n \Bigg\| \int_0^t g_{r_i}(C(x,s))(v_i)_x(s,x)ds - \int_0^t g_{r_i}(C(\overline{x},s))(v_i)_x(s,\overline{x})ds \\
&\qquad\qquad - \int_t^{\bar{t}} g_{r_i}(C(\overline{x},s))(v_i)_x(s,\overline{x})ds \Bigg\| \\
&\leq \Lambda_2^{(2)}|x-\overline{x}|^\alpha + \int_0^t \left\|g_x\left(C(x,s)\right)-g_x\left(C(\overline{x},s)\right)\right\|ds + \left| \int_t^{\bar{t}} M_2^{(1)*}ds \right| \\
&\quad + \int_0^t \left\|g_p(C(x,s))u_x(s,x)-g_p(C(x,s))u_x(s,\overline{x})\right\|ds \\
&\quad + \int_0^t \left\|g_p(C(x,s))u_x(s,\overline{x})-g_p(C(\overline{x},s))u_x(s,\overline{x})\right\|ds + \left| \int_t^{\bar{t}} M_2^{(1)*}Q_1^{(1)}ds \right| \\
&\quad + \sum_{i=1}^n \int_0^t \left\|g_{r_i}(C(x,s))(v_i)_x(s,x)-g_{r_i}(C(x,s))(v_i)_x(s,\overline{x})\right\|ds \\
&\quad + \sum_{i=1}^n \int_0^t \left\|g_{r_i}(C(x,s))(v_i)_x(s,\overline{x})-g_{r_i}(C(\overline{x},s))(v_i)_x(s,\overline{x})\right\|ds + \left| \int_t^{\bar{t}} M_2^{(1)*}Q_2^{(1)}ds \right|
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\leq \Lambda_2^{(2)}|x-\overline{x}|^\alpha + \int_0^t H_2^{(1)*}\,[|x-\overline{x}| + |u(s,x)-u(s,\overline{x})| \\
&\quad + \|v(s,x)-v(s,\overline{x})\|]^\alpha\, ds + M_2^{(1)*}|t-\overline{t}| \\
&\quad + \int_0^t M_2^{(1)*}Q_1^{(2)}|x-\overline{x}|^\alpha ds + \int_0^t H_2^{(1)*}\,[|x-\overline{x}| + |u(s,x)-u(s,\overline{x})| \\
&\quad + \|v(s,x)-v(s,\overline{x})\|]^\alpha\, Q_1^{(1)}ds + M_2^{(1)*}Q_1^{(1)}|t-\overline{t}| \\
&\quad + \sum_{i=1}^n \Bigg[\int_0^t M_2^{(1)*}Q_2^{(2)}|x-\overline{x}|^\alpha ds \\
&\qquad + \int_0^t H_2^{(1)*}\Big[|x-\overline{x}| + |u(s,x)-u(s,\overline{x})| \\
&\qquad\qquad + \|v(s,x)-v(s,\overline{x})\|\Big]^\alpha Q_2^{(1)}ds + M_2^{(1)*}Q_2^{(1)}|t-\overline{t}|\Bigg] \\
&\leq \Lambda_2^{(2)}|x-\overline{x}|^\alpha + H_2^{(1)*}\left[|x-\overline{x}| + Q_1^{(1)}|x-\overline{x}| + Q_2^{(1)}|x-\overline{x}|\right]^\alpha \tau + M_2^{(1)*}|t-\overline{t}| \\
&\quad + M_2^{(1)*}Q_1^{(2)}|x-\overline{x}|^\alpha\tau + H_2^{(1)*}\left[|x-\overline{x}| + Q_1^{(1)}|x-\overline{x}| + Q_2^{(1)}|x-\overline{x}|\right]^\alpha Q_1^{(1)}\tau \\
&\quad + M_2^{(1)*}Q_1^{(1)*}|t-\overline{t}| \\
&\quad + n\Big[M_2^{(1)*}Q_2^{(2)}|x-\overline{x}|^\alpha\tau + H_2^{(1)*}\left[|x-\overline{x}| + Q_1^{(1)}|x-\overline{x}| + Q_2^{(1)}|x-\overline{x}|\right]^\alpha Q_2^{(1)}\tau \\
&\qquad + M_2^{(1)*}Q_2^{(1)*}|t-\overline{t}|\Big] \\
&\leq \Big\{\Lambda_2^{(2)} + H_2^{(1)*}\left(1+Q_1^{(1)}+Q_2^{(1)}\right)^\alpha\tau + M_2^{(1)*}\tau^{1-\alpha} \\
&\quad + M_2^{(1)*}Q_1^{(2)}\tau + H_2^{(1)*}\left(1+Q_1^{(1)}+Q_2^{(1)}\right)^\alpha Q_1^{(1)}\tau + M_2^{(1)*}Q_1^{(1)}\tau^{1-\alpha} \\
&\quad + n\left[M_2^{(1)*}Q_2^{(2)}\tau + H_2^{(1)*}\left(1+Q_1^{(1)}+Q_2^{(1)}\right)^\alpha Q_2^{(1)}\tau + M_2^{(1)*}Q_2^{(1)}\tau^{1-\alpha}\right]\Big\} \\
&\quad \times [|t-\overline{t}| + |x-\overline{x}|]^\alpha \\
&= S_{2\tau}^{(2.x)}[|t-\overline{t}| + |x-\overline{x}|]^\alpha
\end{aligned}
$$

for a fixed $\Delta(X,\tau)$ and $(t,x),(\overline{t},\overline{x}) \in \Delta(X,\tau)$. The inequalities (4.12)–(4.17) and Remark 2.4 imply the existence of $\tau \in (0,T]$ such that (4.8)–(4.11) are true.

Now we prove that for a sufficiently small $\tau \in (0,T]$ the operator $\mathbf{H}$ is a contraction. Indeed, if $(u,v),(\overline{u},\overline{v}) \in C^{1,\alpha}_{b,\tau}(Q)$ and $(t,x) \in [0,\tau]\times\mathbb{R}$, then using $\mathcal{H}_2[f,g]$, the mean value theorem for f and the theorem on the estimate of an increment of g we have

$$\begin{aligned}
|F[u,v](t,x) - F[\overline{u},\overline{v}](t,x)| &\le \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left| f\left(s,y,u(s,y),v(s,y)\right) \right. \\
&\quad \left. - f\left(s,y,\overline{u}(s,y),\overline{v}(s,y)\right)\right| dyds \\
&\quad + \frac{c^2}{8}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \big|u(s,y)-\overline{u}(s,y)\big| dyds \\
&\le \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \Bigg[\left|f_p\left(P_1\right)\right| \left|u(s,y)-\overline{u}(s,y)\right| \\
&\quad + \sum_{i=1}^n \left|f_{r_i}\left(P_1\right)\right| \left|v_i(s,y)-\overline{v}_i(s,y)\right| \Bigg] dyds \\
&\quad + \frac{c^2}{8}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \left|u(s,y)-\overline{u}(s,y)\right| dyds \\
&\le \frac{1}{2}\left[(1+n)M_1^{(1)*} + \frac{c^2}{4}\right] \tau^2 \left\|(u-\overline{u}, v-\overline{v})\right\|_0 ,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\left\|G[u,v](t,x) - G[\overline{u},\overline{v}](t,x)\right\| \\
&\le \int_0^t \left\| g\left(s,x,u(s,x),v(s,x)\right) - g\left(s,x,\overline{u}(s,x),\overline{v}(s,x)\right)\right\| ds \\
&\le \int_0^t \left[\left\|g_p\left(P_2\right)\right\| \left|u(s,x)-\overline{u}(s,x)\right| + \sum_{i=1}^n \left\|g_{r_i}\left(P_2\right)\right\| \left|v_i(s,x)-\overline{v}_i(s,x)\right| \right] ds \\
&\le (1+n)M_2^{(1)*}\tau \left\|(u-\overline{u}, v-\overline{v})\right\|_0 ,
\end{aligned} \tag{4.19}$$

where $P_1, P_2 \in [0,\tau]\times\mathbb{R}\times[-Q_1,Q_1]\times[-Q_2,Q_2]^n$ are intermediate points. Inequalities (4.18), (4.19) imply

$$\left\|\mathbf{H}[u,v] - \mathbf{H}(\overline{u},\overline{v})\right\|_0 \le D_\tau \left\|(u-\overline{u}, v-\overline{v})\right\|_0 , \tag{4.20}$$

where

$$D_\tau = \max\left\{ \frac{1}{2}\left[(1+n)M_1^{(1)*} + \frac{c^2}{4}\right]\tau^2, (1+n)M_2^{(1)*}\tau \right\}. \tag{4.21}$$

It is obvious that there is $\tau \in (0,T]$ such that

$$D_\tau < 1. \tag{4.22}$$

The use of the Banach fixed-point theorem and Lemma 3.1 finishes the proof. □

Remark 4.2. It follows from the proof of Theorem 4.1 that the set $[0,\tau] \times \mathbb{R}$ of the existence and uniqueness of solutions to (1.1), (1.2) can be found by solving the inequalities $S_{i\tau} \le Q_i$, $S_{i\tau}^{(j.t)} \le Q_i^{(j)}$, $S_{i\tau}^{(j.x)} \le Q_i^{(j)}$, $i,j = 1,2$, (see Remark 2.4) and inequality (4.22).

Remark 4.3. If additionally, in Theorem 4.1, $u_0 \in C_b^2(\mathbb{R},\mathbb{R})$ and $u_1 \in C_b^1(\mathbb{R},\mathbb{R})$, then $u \in C_b^2\left([0,\tau] \times \mathbb{R},\mathbb{R}\right)$.

Remark 4.4. If the functions f, g generating the system (1.1) do not depend on t, x and f is of C^1 class and g is of C^2 class on $\mathbb{R}^{1+n}$, then all the assumptions of Theorem 4.1 are satisfied. It is true especially for the equations with any polynomial right-hand sides with respect to p, r. Add that Theorem 4.1 works if we consider the first equation in (1.1) only (the nonlinear wave equation for $c = 0$ and the nonlinear telegraph equation for $c > 0$). The examples of such equations are the forced dissipative sine-Gordon equation with a variable coefficient

$$u_{tt} - u_{xx} + cu_t + a(t,x)\sin u = h(t,x) \tag{4.23}$$

and the superlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t + du^m = h(t,x), \tag{4.24}$$

where $d \in \mathbb{R}$, $m \in \mathbb{N}$ and a, a_x, h, h_x are continuous and bounded on $[0,T] \times \mathbb{R}$ (see [12]).

Acknowledgements
The research of the author was partially supported by the Polish Ministry of Science and Higher Education.

REFERENCES

[1] L. de Broglie, *La mécanique ondulatoire et la structure atomique de la matière et du rayonnement*, J. Phys. Rad. **8** (1927) 5, 225–241.

[2] R. Carles, R. Danchin, J.C. Saut, *Madelung, Gross-Pitaevskii and Korteveg*, Nonlinearity **25** (2012), 2843–2873.

[3] T. Członpiński, *On the Cauchy problem for quasilinear hyperbolic differential-functional systems of in the Schauder canonic form*, Discuss. Math. **10** (1990), 47–68.

[4] T. Członpiński, *On the mixed problem for quasilinear partial differential-functional equations of the first order*, Z. Anal. Anwend. **16** (1997), 463–478.

[5] M. Danielewski, *The Planck-Kleinert crystal*, Z. Naturforsch. **62a** (2007), 564–568.

[6] J. Evans, N. Shenk, *Solutions to axon equations*, Biophys. J. **10** (1970), 1090–1101.

[7] J.W. Evans, *Nerve axon equations: I linear approximations*, Indiana Univ. Math. J. **21** (1972) 9, 877–885.

[8] J.W. Evans, *Nerve axon equations: II stability at rest*, Indiana Univ. Math. J. **22** (1972) 1, 75–90.

[9] J.W. Evans, *Nerve axon equations: III stability of the nerve impulse*, Indiana Univ. Math. J. **22** (1972) 6, 577–593.

[10] J.W. Evans, *Nerve axon equations: IV the stable and the unstable impulse*, Indiana Univ. Math. J. **24** (1975) 12, 1169–1190.

[11] M. Krupa, B. Sanstede, P. Szmolyan, *Fast and slow waves in the FitzHugh-Nagumo equation*, J. Differential Equations **133** (1997), 49–97.

[12] Y. Li, *Maximum principles and the method of upper and lower solutions for time periodic problems of the telegraph equations*, J. Math. Anal. Appl. **327** (2007), 997–1009.

[13] W. Likus, V.A. Vladimirov, *Solitary waves in the model of active media, taking into account relaxing effects*, to appear in Rep. Math. Phys. (2015).

[14] E. Madelung, *Quantentheorie in hydrodynamischer form*, Z. Phys. A-Hadron. Nucl. **40** (1927), 322–326.

[15] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum, New York, 1992.

[16] M.H. Protter, H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.

Lucjan Sapa
sapa@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Received: May 12, 2014.
Revised: February 12, 2015.
Accepted: February 16, 2015.