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ALIEN DERIVATIVES OF THE WKB SOLUTIONS OF THE GAUSS HYPERGEOMETRIC DIFFERENTIAL EQUATION WITH A LARGE PARAMETER

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Abstract. We compute alien derivatives of the WKB solutions of the Gauss hypergeometric differential equation with a large parameter and discuss the singularity structures of the Borel transforms of the WKB solution expressed in terms of its alien derivatives.

Keywords: hypergeometric differential equation, WKB solution, Voros coefficient, alien derivative, Stokes curve, fixed singularity.

Mathematics Subject Classification: 33C05, 34M40, 34M60.

1. INTRODUCTION

For linear ordinary differential equations with large parameters, a kind of Stokes phenomena occur in the asymptotic behaviors of WKB solutions with a change of the parameters. Such Stokes phenomena are called parametric Stokes phenomena. In this paper we study the parametric Stokes phenomena for the Gauss hypergeometric differential equation from the viewpoint of the alien calculus.

The alien derivative is introduced by Écalle [10] in 1981. Delabaere-Dilinger-Pham [8] and Delabaere-Pham [9] studied the Stokes automorphisms and alien derivatives for WKB solutions of Schrödinger equations with polynomial potentials from the viewpoint of Ecalle's resurgent function theory. In these articles, Stokes automorphisms are described by using the intersection numbers of degenerate Stokes curves and integration paths for Voros coefficients. In this paper we apply these results to the Gauss hypergeometric differential equation with large parameters. We also refer to the works of Sauzin [15,16], where the relation between the Stokes automorphisms and the alien derivatives is clarified.

Parametric Stokes phenomena for second order linear ordinary differential equations with irregular singular points are studied in several papers: Takei [17] studied then for the Weber equation, Koike and Takei [14] for the Whittaker equation, and Aoki, Iwaki and Takahashi [1] for the Bessel equation. In these works, parametric Stokes phenomena are studied by analyzing the degeneration of Stokes curves and then calculating alien derivatives. In general, a Stokes curve emanates from a turning point and flows into a turning point or a singular point. A Stokes curve is said to be degenerate if it flows into a turning point. Then there are two types of degenerate Stokes curves. A degenerate Stokes curve which emanates from a turning point and flows into another turning point is called of Weber type, and one which flows into the same turning point is called of loop type. In [17] and [14], Stokes curves of Weber type are considered, and in [1] one of loop type is considered. Note that, in [8, 9, 15] and [16], the authors treat only the case of Weber type.

The Gauss hypergeometric differential equation has three parameters, and then there appears two types of degenerate Stokes curves. In our previous works [6] and [19], we considered degenerate Stokes curves of Weber type, and obtained the parametric Stokes phenomena by using the Borel sums of the Voros coefficients. In the present paper, first we consider the same cases as in our previous works, and obtain the parametric Stokes phenomena in another way – namely by using alien derivatives. Next we consider the case in which degenerate Stokes curves of loop type appear. Also in this case, we obtain the parametric Stokes phenomena by using alien derivatives.

In Section 2, we introduce the Gauss hypergeometric differential equation with large parameters. Then we review the definition of the Voros coefficients for this equation, and give their Borel transforms. The parametric Stokes phenomena are described in Section 3. First we divide the space of the parameters into several subregions such that the parametric Stokes phenomena occurs at the boundaries. Then we compute the alien derivatives to obtain the parametric Stokes phenomena. The main results are given in Theorems 3.2, 3.5, 3.7, 3.9 and 3.10.

2. VOROS COEFFICIENTS AND BOREL TRANSFORMS OF THEM

We consider the following Schrödinger-type equation:

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q(x)\Big)\psi=0 \tag{2.1}$$

with a large parameter $\eta>0$. Here we set $Q(x)=Q_0(x)+\eta^{-2}Q_1(x)$ with

$$Q_0(x)=\frac{(\alpha-\beta)^2x^2+2(2\alpha\beta-\alpha\gamma-\beta\gamma)x+\gamma^2}{4x^2(x-1)^2}$$

and

$$Q_1(x)=-\frac{x^2-x+1}{4x^2(x-1)^2},$$

where $\alpha,\beta,\ \gamma$ are complex parameters. Equation (2.1) is obtained from the Gauss hypergeometric differential equation:

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0.$$

We put

$$a = \frac{1}{2} + \alpha\eta,$$
$$b = \frac{1}{2} + \beta\eta,$$
$$c = 1 + \gamma\eta$$

and eliminate the first-order term by taking

$$\psi = x^{\frac{1}{2}(1+\gamma\eta)}(1-x)^{\frac{1}{2}(1+(\alpha+\beta-\gamma)\eta)}w$$

as unknown function. Then we have (2.1). In this paper, we call (2.1) the Gauss hypergeometric differential equation with large parameter η. By definition, WKB solutions of (2.1) are the following formal solutions:

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Bigg(\pm \int\limits_{x_0}^{x} S_{\mathrm{odd}}\, dx \Bigg),$$

where x_0 is a fixed point and S_{odd} denotes the odd-order part in η of the formal solution $S(x) = \sum_{l=-1}^{\infty} \eta^{-l} S_l$ of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q(x)$$

associated with (2.1). (See also [12, §2] for the notation and terminologies.) Eq. (2.1) has regular singular points $b_0 = 0$, $b_1 = 1$ and $b_2 = \infty$. A turning point a of (2.1) is, by definition, a simple zero of Q_0 (cf. [12, §2]).

We define a Stokes curve emanating from a by

$$\mathrm{Im} \int\limits_{a}^{x} \sqrt{Q_0}\, dx = 0.$$

A Stokes curve flows into a singular point or a turning point (cf. [12, §2]). If turning points are connected by a Stokes curve, the Stokes geometry of (2.1) is said to be degenerate. Let E_j $(j = 0, 1, 2)$ be the sets defined by the following:

$$E_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha\beta\gamma(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) = 0\},$$
$$E_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\alpha \cdot \mathrm{Re}\beta \cdot \mathrm{Re}(\gamma-\alpha) \cdot \mathrm{Re}(\gamma-\beta) = 0\},$$
$$E_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha-\beta) \cdot \mathrm{Re}(\alpha+\beta-\gamma) \cdot \mathrm{Re}\gamma = 0\}.$$

Stokes graph of (2.1) is, by definition, a two-color sphere graph that consists of all Stokes curves as edges, $\{a_0, a_1\}$ as vertices of the first color and $\{b_0, b_1, b_2\}$ as vertices of the second color (cf. [3] and [12, §3.2]).

Theorem 2.1 ([19, Theorem 3.1])**.** *We assume that (α,β,γ) is not contained in E_0.*
(i) *If two distinct turning points a_0 and a_1 are connected by a Stokes curve, then (α,β,γ) belongs to E_1. Conversely, if (α,β,γ) is contained in $E_1 - E_2$, the Stokes geometry of* (2.1) *has a Stokes curve which connects two distinct turning points a_0 and a_1.*
(ii) *If a Stokes curve forms a closed curve with a single turning point as the base point, then (α,β,γ) belongs to E_2. Conversely, if (α,β,γ) is contained in $E_2 - E_1$, the Stokes geometry of* (2.1) *has a Stokes curve which forms a closed path with a turning point as the base point.*

Let C_j $(j = 0,1,2)$ be a contour starting from the singular point b_j, going around a turning point in a counterclockwise direction and going back to b_j. We may assume that the other turning point and singular points are not included in C_j.

Definition 2.2 ([19])**.** Let V_j $(j = 0,1,2)$ be the formal power series in η^{-1} defined by the following integrals:

$$V_0 = V_0(\alpha,\beta,\gamma;\eta) := \frac{1}{2}\int_{C_0}(S_{\mathrm{odd}} - \eta S_{-1})dx,$$
$$V_1 = V_1(\alpha,\beta,\gamma;\eta) := \frac{1}{2}\int_{C_1}(S_{\mathrm{odd}} - \eta S_{-1})dx,$$
$$V_2 = V_2(\alpha,\beta,\gamma;\eta) := \frac{1}{2}\int_{C_2}(S_{\mathrm{odd}} - \eta S_{-1})dx.$$

Here the branch of $S_{-1} = \sqrt{Q_0(x)}$ on $C_j (j = 0,1,2)$ is taken as follows: We take a curve in $\mathbb{C} - \{0,1\}$ connecting the turning points as a branch cut and choose the branch of $S_{-1}(x)$ so that at the starting point b_j, we have

$$\sqrt{Q_0} \sim \frac{\gamma}{2x} \qquad \text{at } x = 0, \tag{2.2}$$
$$\sqrt{Q_0} \sim \frac{\alpha+\beta-\gamma}{2(x-1)} \qquad \text{at } x = 1, \tag{2.3}$$
$$\sqrt{Q_0} \sim \frac{\beta-\alpha}{2x} \qquad \text{at } x = \infty. \tag{2.4}$$

We call V_j the Voros coefficients of (2.1) with respect to b_j $(b_j = 0,1,2)$.

Let $\psi_\pm$ and $\psi_\pm^{(j)}$ be the WKB solutions normalized at a turning point a ($a = a_0$ or $a = a_1$) and those normalized at the singular point b_j (cf. [7]):

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_a^x S_{\mathrm{odd}}dx\right),$$
$$\psi_\pm^{(j)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{b_j}^x (S_{\mathrm{odd}} - \eta S_{-1})dx \pm \eta\int_a^x S_{-1}dx\right),$$

respectively. For $j = 0, 1$ and 2, $V_j(\alpha, \beta, \gamma; \eta)$ describe the discrepancy between WKB solutions $\psi_\pm$ and $\psi_\pm^{(j)}$, that is, we have $\psi_\pm$ as

$$\psi_\pm = \exp(\mp V_j)\psi_\pm^{(j)}. \tag{2.5}$$

Since $S_{\rm odd}\, dx$ and $\eta S_{-1}\, dx$ have a simple pole at the singular points b_j and residues at b_j coincide (see [12] for the computation of residues of $S_{\rm odd}$), these integrals are well-defined for every homotopy class of the path of integration. Explicit forms of V_j are given in [5,6,19] (The choices of the branch of S_{-1} on C_j are slightly different in those references.):

Theorem 2.3 ([5, Theorem 2.1])**.** *The Voros coefficients V_j have the following forms*:

$$V_0(\alpha, \beta, \gamma; \eta) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left(\frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} + \frac{1}{(\gamma-\alpha)^{n-1}} + \frac{1}{(\gamma-\beta)^{n-1}} \right) + \frac{2}{\gamma^{n-1}} \right\},$$

$$V_1(\alpha, \beta, \gamma; \eta) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left(\frac{1}{\alpha^{n-1}} + \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma-\alpha)^{n-1}} - \frac{1}{(\gamma-\beta)^{n-1}} \right) + \frac{2}{(\alpha+\beta-\gamma)^{n-1}} \right\},$$

$$V_2(\alpha, \beta, \gamma; \eta) = \frac{1}{2}\sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left(\frac{1}{\alpha^{n-1}} - \frac{1}{\beta^{n-1}} - \frac{1}{(\gamma-\alpha)^{n-1}} + \frac{1}{(\gamma-\beta)^{n-1}} \right) - \frac{2}{(\beta-\alpha)^{n-1}} \right\}.$$

Here B_n are the Bernoulli numbers defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

We take the Borel transforms $V_{j,B}(\alpha, \beta, \gamma; y)$ $(j = 0, 1, 2)$ of V_j, then we obtain the following proposition (see [12, §2.1] for the definition of the Borel transform and [19] for the computation of $V_{j,B}$):

Proposition 2.4 ([19])**.** *The Borel transforms $V_{j,B}(\alpha, \beta, \gamma; y)$ of the Voros coefficients V_j have the following forms*:

$$V_{0,B}(\alpha, \beta, \gamma; y) = -\frac{1}{4}\{g_1(\alpha; y) + g_1(\beta; y) + g_1(\gamma-\alpha; y) + g_1(\gamma-\beta; y)\} + g_0(\gamma; y),$$

$$V_{1,B}(\alpha, \beta, \gamma; y) = \frac{1}{4}\{-g_1(\alpha; y) - g_1(\beta; y) + g_1(\gamma-\alpha; y) + g_1(\gamma-\beta; y)\} + g_0(\alpha+\beta-\gamma; y),$$

$$V_{2,B}(\alpha,\beta,\gamma;y) = \frac{1}{4}\{-g_1(\alpha;y)+g_1(\beta;y)+g_1(\gamma-\alpha;y)-g_1(\gamma-\beta;y)\}$$
$$- g_0(\beta-\alpha;y).$$

Here

$$g_0(t;y) = \frac{1}{y}\left(\frac{1}{\exp\frac{y}{t}-1}+\frac{1}{2}-\frac{t}{y}\right),$$
$$g_1(t;y) = \frac{1}{\exp\frac{y}{2t}-1}+\frac{1}{\exp\frac{y}{2t}+1}-\frac{2t}{y}.$$

For a fixed $t \neq 0$, $g_0(t;y)$ and $g_1(t;y)$ are holomorphic at $y=0$, and they have simple poles at $y = 2tm\pi i$ ($m \in \mathbb{Z}-\{0\}$) as functions of y. The residues of $g_0(t;y)$ and $g_1(t;y)$ ($j=0,1$) are given as follows:

$$\operatorname*{Res}_{y=2tm\pi i} g_0(t;y) = \frac{1}{2m\pi i},$$
$$\operatorname*{Res}_{y=2tm\pi i} g_1(t;y) = \frac{(-1)^m}{m\pi i}.$$

3. PARAMETRIC STOKES PHENOMENA AND THE ALIEN DERIVATIVE OF THE WKB SOLUTION

Let ω_h ($h=1,2,3,4$) be the sets of the parameter (α,β,γ) defined by

$$\omega_1 = \{(\alpha,\beta,\gamma)\in\mathbb{C}^3 \mid 0<\mathrm{Re}\alpha<\mathrm{Re}\gamma<\mathrm{Re}\beta\},$$
$$\omega_2 = \{(\alpha,\beta,\gamma)\in\mathbb{C}^3 \mid 0<\mathrm{Re}\alpha<\mathrm{Re}\beta<\mathrm{Re}\gamma<\mathrm{Re}\alpha+\mathrm{Re}\beta\},$$
$$\omega_3 = \{(\alpha,\beta,\gamma)\in\mathbb{C}^3 \mid 0<\mathrm{Re}\gamma<\mathrm{Re}\alpha<\mathrm{Re}\beta\},$$
$$\omega_4 = \{(\alpha,\beta,\gamma)\in\mathbb{C}^3 \mid 0<\mathrm{Re}\gamma<\mathrm{Re}\alpha+\mathrm{Re}\beta<\mathrm{Re}\beta\},$$

and let ι_j ($j=0,1,2$) be involutions in the space $\mathbb{C}^3$ of parameters (α,β,γ) defined by

$$\iota_0 : (\alpha,\beta,\gamma)\mapsto(-\alpha,-\beta,-\gamma),$$
$$\iota_1 : (\alpha,\beta,\gamma)\mapsto(\gamma-\beta,\gamma-\alpha,\gamma),$$
$$\iota_2 : (\alpha,\beta,\gamma)\mapsto(\beta,\alpha,\gamma).$$

Moreover, an open subset Π_h ($h=1,2,3,4$) in $\mathbb{C}^3$ is defined by

$$\Pi_h = \bigcup_{r\in G} r(\omega_h).$$

Here G is the group generated by ι_j ($j=0,1,2$). The union of Π_h covers most of $\mathbb{C}^3$:

$$\bigcup_{h=1}^{4}\Pi_h = \mathbb{C}^3 - \{(\alpha,\beta,\gamma)\,|\,\mathrm{Re}\,\alpha\mathrm{Re}\,\beta\mathrm{Re}\gamma\times$$
$$\mathrm{Re}(\gamma-\alpha)\mathrm{Re}(\gamma-\beta)\mathrm{Re}(\alpha-\beta)\mathrm{Re}(\alpha+\beta-\gamma)=0\}.$$

We note that the topological configuration of the Stokes graph is characterized by its order sequence (n_0, n_1, n_2). Here n_j is the number of Stokes curves that flow into b_j $(j = 0, 1, 2)$.

Theorem 3.1 ([4, Theorem 3.2])**.** *Let* $\hat{n} = (n_0, n_1, n_2)$ *denote the order sequences of the Stokes graph with parameter* (α, β, γ)*.*
(1) *If* $(\alpha, \beta, \gamma) \in \Pi_1$, *then* $\hat{n} = (2, 2, 2)$.
(2) *If* $(\alpha, \beta, \gamma) \in \Pi_2$, *then* $\hat{n} = (4, 1, 1)$.
(3) *If* $(\alpha, \beta, \gamma) \in \Pi_3$, *then* $\hat{n} = (1, 4, 1)$.
(4) *If* $(\alpha, \beta, \gamma) \in \Pi_4$, *then* $\hat{n} = (1, 1, 4)$.

We introduce the following notations:

$$\begin{aligned} \iota_3 =& \iota_1\iota_2 \quad : (\alpha, \beta, \gamma) \mapsto (\gamma - \alpha, \gamma - \beta, \gamma), \\ \iota_4 =& \iota_0\iota_2 \quad : (\alpha, \beta, \gamma) \mapsto (-\beta, -\alpha, -\gamma), \\ \iota_5 =& \iota_0\iota_1 \quad : (\alpha, \beta, \gamma) \mapsto (\beta - \gamma, \alpha - \gamma, -\gamma), \\ \iota_6 =& \iota_0\iota_1\iota_2 : (\alpha, \beta, \gamma) \mapsto (\alpha - \gamma, \beta - \gamma, -\gamma). \end{aligned}$$

We denote $\iota_m(\omega_h)$ by ω_{hm} $(m = 0, 1, \ldots, 6;\ h = 1, \ldots, 4)$.

3.1. ANALYSIS OF STOKES CURVES OF WEBER TYPE

We derive alien derivatives and parametric Stokes phenomena on the WKB solutions for the hypergeometric differential equation with a large parameter for the Weber type. Several formulas describing the parametric Stokes phenomena are proved in [6] by using Borel sums of the WKB solutions. The same formulas can be derived from the alien derivatives, and hence, in this section, we obtain another proof of the parametric Stokes phenomena. The notion of alien derivatives was introduced by J. Ecall [10]. We refer the reader to [15, §2], and [16, p. 78, §28], for the definition of the alien derivative. We assume that arguments of α, β, $\alpha - \gamma$, $\beta - \gamma$, γ, $\alpha + \beta - \gamma$ and $\alpha - \beta$ are mutually distinct.

3.1.1. Analysis on the boundary between ω_1 and ω_2

We discuss the case where (α, β, γ) is contained the boundary between ω_1 and ω_2, that is, $\mathrm{Re}(\gamma - \beta) = 0$. We assume that $\mathrm{Im}(\beta - \gamma)$ is negative. In this case, we consider the case where $(\alpha, \beta, \gamma) = (0.5, 1 - \hat{\epsilon} i, 1) \in E_1$. It follows from (i) of Theorem 2.1 that we can take $(\alpha, \beta, \gamma) = (0.5, 1 - \hat{\epsilon} i, 1) \in E_1$ without loss of generality.

We consider the WKB solutions

$$\psi_{\pm,k} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{a_k}^{x} S_{\mathrm{odd}}\, dx \right)$$

in a neighborhood of a_k $(k = 0, 1)$. Here we take the straight line connecting a_k to x as the path of integration. We expand the WKB solutions $\psi_{\pm,k}$ as

$$\psi_{\pm,k} = \exp\left(\eta y_{\pm,k}(x)\right) \sum_{n=0}^{\infty} \psi_{\pm,n}(x) \eta^{-n-\frac{1}{2}},$$

where

$$y_{\pm,k}(x) = \pm \int_{a_k}^{x} \sqrt{Q}\, dx.$$

Let us show examples of Stokes curves of those two cases in Figures 3.1 and 3.3 and a degenerate case, namely, one case of the Weber type in Figure 3.2 (cf. [6] and [19]). Here $\epsilon > 0$ and $\hat{\epsilon} > 0$ are sufficiently small. Here open circles and closed circles are the singular points and turning points, respectively. We place the cut as shown by the wavy lines in Figure 3.h ($h = 1, 3$). In this case, we use the branch of $S_{-1} = \sqrt{Q_0}$ at 0, i.e. we take the branch of S_{-1} as (2.2). Hence, ψ_- is dominant (cf. [6] for the computation). Let $\mathcal{R}^{\mathrm{I}}$, $\mathcal{R}^{\mathrm{II}}$, $\mathcal{R}^{\mathrm{I}}_{\omega_h}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_h}$ ($h = 1, 2$) denote regions surrounded by the Stokes curves as shown in Figures 3.1, 3.2 and 3.3.

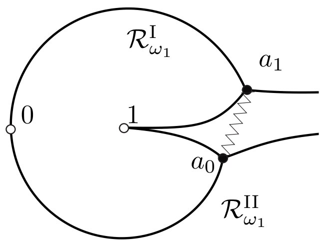

Fig. 3.1. $(\alpha, \beta, \gamma) = (0.5, 1 + \epsilon - \hat{\epsilon}i, 1)$ in ω_1

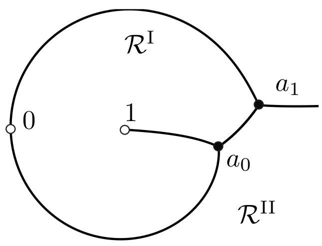

Fig. 3.2. $(0.5, 1 - \hat{\epsilon}i, 1)$

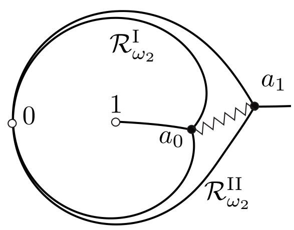

Fig. 3.3. $(0.5, 1 - \epsilon - \hat{\epsilon}i, 1)$ in ω_2

The WKB solutions $\psi_{\pm,k} (k = 0, 1)$ (resp. $\psi^{(0)}_{\pm}$) are Borel summable in $\mathcal{R}^{\mathrm{I}}$ (resp. $\mathcal{R}^{\mathrm{II}}$). (See [12] and [13] for the notation and terminologies.) Let us denote the Borel transform of $\psi^{(0)}_{+}$ by $\psi^{(0),\mathrm{I}}_{+,B}$ (resp. by $\psi^{(0),\mathrm{II}}_{+,B}$). It follows from a result given in [13] that the Borel transform $\psi^{(0),\mathrm{I}}_{+,B}$ (resp. $\psi^{(0),\mathrm{II}}_{+,B}$) is free from singularities on the half line

$$\left\{ y \in \mathbb{C}; y = -\int_{a_k}^{x} \sqrt{Q_0} dx + \rho; \rho > 0 \right\}. \tag{3.1}$$

Moreover, the Borel transform $V_{0,B}(\alpha, \beta, \gamma; y)$ of V_0 is holomorphic at $y = 0$ and it has simple poles at $y = 2m(\gamma - \beta)\pi i$ for every non-zero integer $m \in \mathbb{Z} - \{0\}$. Therefore, the Borel transform of $\psi_{+,k}$ has singularities at

$$y = -y_{\pm,k}(x) + 2m(\beta - \gamma)\pi i$$

when $\mathrm{Re}(\gamma - \beta) = 0$ ($m \in \mathbb{Z}$). The following description of the alien derivative of the WKB solutions in the formal model is due to [1] (for the definition of the alien derivative of the WKB solutions in the convolution model) and [17]:

$$\triangle \psi_{+,k} = \mathcal{B}^{-1} \log \left(\mathcal{L}_-^{-1} \mathcal{L}_+ \right) \mathcal{B} \psi_{+,k}. \tag{3.2}$$

Here $\mathcal{B}$ and $\mathcal{L}_+$ (resp. $\mathcal{L}_-$) is the Borel transformation and the Laplace transformation along a path which avoids the singular points from above (resp. from below), respectively. We decompose (3.2) as follows:

$$\triangle\psi_{+,k} = \sum_{m=1}^{\infty} \triangle_{y=-y_{+,k}+2m(\beta-\gamma)\pi i}\psi_{+,k}$$

with

$$\triangle_{y=-y_{+,k}+2m(\beta-\gamma)\pi i}\psi_{+,k} = \mathcal{B}^{-1}\left[(t_+^{(m)} - t_-^{(m)}) \sum_{\epsilon_n=\pm} \frac{p_+!p_-!}{m!} t_{\epsilon_{m-1}}^{(m-1)} \cdots t_{\epsilon_1}^{(1)}\right] \mathcal{B}\psi_{+,k},$$

where $t_+^{(n)}$ (resp. $t_-^{(n)}$) denotes the operator of analytic continuation which does not pass the n-th singular point $y = -y_{+,k} + 2n(\beta-\gamma)\pi i$ from above (resp. below) and p_+ (resp. p_-) is the number of times and $\epsilon_n = +$ (resp. $\epsilon_n = -$) for $1 \le n \le m-1$.

Let us consider the alien derivatives $\triangle\psi_{+,0}$ and $\triangle\psi_{+,1}$. Since the Borel transform $V_{0,B}$ of V_0 is a single-valued analytic function with the simple pole at $y = 2m(\beta-\gamma)\pi i$, we have

$$\begin{aligned}
&\triangle_{y=2m(\beta-\gamma)\pi i}\,(-V_0) \\
&= \mathcal{B}^{-1}\left[\sum_{\epsilon_k=\pm} \frac{p_+!p_-!}{m!} \operatorname{sing}_{y=-y_{+,k}+2m(\beta-\gamma)\pi i} t_{\epsilon_{m-1}}^{(m-1)} \cdots t_{\epsilon_1}^{(1)}\right](-V_{0,B}) \\
&= \mathcal{B}^{-1}\left[2\pi i \operatorname*{Res}_{y=2m(\beta-\gamma)\pi i}(-V_{0,B})\right] \\
&= \frac{(-1)^{m+1}}{2m}
\end{aligned}$$

(cf. [17]). Hence, the chain rule in the alien calculus leads to

$$\triangle_{y=2m(\beta-\gamma)\pi i}(\exp(-V_0)) = \frac{(-1)^{m+1}}{2m} \exp(-V_0). \tag{3.3}$$

Since the Borel transform $\psi_{+,B}^{(0),\mathrm{I}}$ (resp. $\psi_{+,B}^{(0),\mathrm{II}}$) is free from singularities on (3.1), we obtain

$$\triangle\left(\exp(-y_{+,k}(x)\eta)\psi_+^{(0)}\right) = 0, \tag{3.4}$$

where $k = 0$ (resp. 1). Combining (2.5), (3.3) and (3.4), we have

$$\begin{aligned}
&\triangle_{y=2m(\beta-\gamma)\pi i}\,(\exp(-y_{+,0}(x)\eta)\psi_{+,0}) \\
&= \triangle_{y=2m(\beta-\gamma)\pi i}\left(\exp(-y_{+,0}(x)\eta)\exp(-V_0(\alpha,\beta,\gamma;\eta))\psi_+^{(0)}\right) \\
&= \frac{(-1)^{m+1}}{2m}\left(\exp(-y_{+,0}(x)\eta)\exp(-V_0(\alpha,\beta,\gamma;\eta))\psi_+^{(0)}\right) \\
&= \frac{(-1)^{m+1}}{2m}\,(\exp(-y_{+,0}(x)\eta)\psi_{+,0})
\end{aligned}$$

and

$$\triangle_{y=2m(\beta-\gamma)\pi i}\left(\exp(-y_{+,1}(x)\eta)\psi_{+,1}\right)=\frac{(-1)^{m+1}}{2m}\left(\exp(-y_{+,1}(x)\eta)\psi_{+,1}\right).$$

Similarly, we can compute the alien derivatives $\triangle_{y=-y_{-,k}(x)+2m(\beta-\gamma)\pi i}\psi_{-,k}$ of $\psi_{-,k}(x,\eta)$. Hence, we obtain the following theorem.

Theorem 3.2. *Let $\psi_{\pm,k}(x,\eta)$ denote the WKB solutions of* (2.1) *normalized at the turning point a_k $(k=0,1)$. Let us consider the case where (α,β,γ) is contained within the boundary between ω_1 and ω_2. The Borel transforms of $\psi_{\pm,k}(x,\eta)$ have fixed singular points at*

$$y=-y_{\pm,k}(x)+2m(\beta-\gamma)\pi i,\quad m\in\mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,k}(x)+2m(\beta-\gamma)\pi i}\psi_{\pm,k}$ of $\psi_{\pm,k}(x,\eta)$ $(k=0,1)$ satisfies the following relations:

$$(\triangle_{y=-y_{\pm,0}(x)+2m(\beta-\gamma)\pi i}\psi_{\pm,0})_B(x,y)=\pm\frac{(-1)^{m+1}}{2m}\psi_{\pm,0,B}(x,y-2m(\beta-\gamma)\pi i)\quad(3.5)$$

for x in $\mathcal{R}^{\mathrm{I}}$ and

$$(\triangle_{y=-y_{\pm,1}(x)+2m(\beta-\gamma)\pi i}\psi_{+,1})_B(x,y)=\pm\frac{(-1)^{m+1}}{2m}\psi_{+,1,B}(x,y-2m(\beta-\gamma)\pi i)\quad(3.6)$$

for x in $\mathcal{R}^{\mathrm{II}}$.

Next we describe the actions of Stokes automorphisms on the WKB solutions. The WKB solutions $\psi_{\pm,0}$ (resp. $\psi_{\pm,1}$) are Borel summable in $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{I}}_{\omega_2}$ (resp. $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_2}$) (cf. [12] and [13] for the notation and terminologies). Let us denote the Borel sums of $\psi_{+,0}$ (resp. $\psi_{+,1}$) by $\psi^{\mathrm{I}}_{\omega_1}$ and $\psi^{\mathrm{I}}_{\omega_2}$ (resp. by $\psi^{\mathrm{II}}_{\omega_1}$ and $\psi^{\mathrm{II}}_{\omega_2}$). In this case, the Stokes automorphism on the WKB solutions is defined by

$$\mathfrak{S}\psi_{\pm,k}=\exp\Big[\sum_{m=1}^{\infty}\triangle_{y=-y_{\pm,k}(x)+2m(\beta-\gamma)\pi i}\Big]\psi_{\pm,k}.$$

Here $\mathfrak{S}$ denote the Stokes automorphism associated with the change to ω_1 from ω_2 (cf. [8, 9, 14]). Using Theorem 3.2 and the discussion given in [14, 17], we have the following theorem:

Theorem 3.3. *When (α,β,γ) moves ω_1 to ω_2 and $\mathrm{Im}(\beta-\gamma)$ is negative, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,k}$ on the WKB solutions $\psi_{\pm,k}$ $(k=0,1)$:*

$$\mathfrak{S}\psi_{\pm,0}=(1+\exp(2\pi i(\gamma-\beta)\eta))^{-\frac{1}{2}}\psi_{\pm,0}\quad(3.7)$$

for x in $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{I}}_{\omega_2}$, and

$$\mathfrak{S}\psi_{\pm,1}=(1+\exp(2\pi i(\gamma-\beta)\eta))^{-\frac{1}{2}}\psi_{\pm,1}\quad(3.8)$$

for x in $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_2}$.

Proof. We obtain

$$\begin{aligned}
\triangle\psi_{+,k} &= \sum_{m=1}^{\infty} \triangle_{y=-y_{+,k}(x)+2m(\beta-\gamma)\pi i}\psi_{+,k} \\
&= \sum_{m=1}^{\infty} \mathcal{B}^{-1}[\triangle_{y=-y_{+,k}(x)+2m(\beta-\gamma)\pi i}\psi_{+,k,B}] \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m}\mathcal{B}^{-1}[\psi_{+,k,B}(x, y-2m(\beta-\gamma)\pi i)] \\
&= \frac{1}{2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \exp\left(-2m(\beta-\gamma)\pi i\right)\psi_{+,k} \\
&= \frac{1}{2}\log(1+\exp\left(-2(\beta-\gamma)\pi i\right))\psi_{+,k},
\end{aligned}$$

where $\mathcal{B}$ denotes the Borel transform. Then we have

$$\mathfrak{S}\psi_{+,k} = (1+\exp\left(-2(\beta-\gamma)\pi i\right))^{\frac{1}{2}}\psi_{+,k}.$$

Similarly, we can compute $\mathfrak{S}\psi_{-,k}$. Hence, we have Theorem 3.3. □

By taking the Borel sums of (3.7) and (3.8), we get the parametric Stokes phenomena of the WKB solutions:

Theorem 3.4. (cf. [6, Theorem 4.4], [19, Thorem 6.3]) *Between the Borel sums $\psi^{\mathrm{I}}_{\omega_1}$ and $\psi^{\mathrm{I}}_{\omega_2}$ of the WKB solution $\psi_{+,1}$ the following relation holds*:

$$\psi^{\mathrm{I}}_{\omega_1} = (1+\exp(2\pi i(\gamma-\beta)\eta))^{\frac{1}{2}}\psi^{\mathrm{I}}_{\omega_2}.$$

Between the Borel sums $\psi^{\mathrm{II}}_{\omega_1}$ and $\psi^{\mathrm{II}}_{\omega_2}$ of the WKB solution $\psi_{+,1}$ the following relation holds:

$$\psi^{\mathrm{II}}_{\omega_1} = (1+\exp(2\pi i(\gamma-\beta)\eta))^{\frac{1}{2}}\psi^{\mathrm{II}}_{\omega_2}.$$

Hence, we can give the another proof of the parametric Stokes phenomena from the viewpoint of alien calculus.

3.1.2. Analysis on the boundary between ω_1 and ω_3

We consider the case where (α, β, γ) is contained within the boundary ω_1 and ω_3, i.e., $\mathrm{Re}(\gamma-\alpha) = 0$. Similarly, we can describe parametric Stokes phenomena in terms of alien derivatives $\triangle\psi_{\pm,0}$ and $\triangle\psi_{\pm,1}$. We assume that $\mathrm{Im}(\alpha-\gamma)$ is positive. In this case, we consider the case where $(\alpha, \beta, \gamma) = (1+\hat{\epsilon}i, 2, 1) \in E_1$. It follows from (i) of Theorem 2.1 that we can take $(\alpha, \beta, \gamma) = (1+\hat{\epsilon}i, 2, 1)$ without loss of generality. Let us show an example of Stokes curves of those two cases in Figures 3.4 and 3.6 and a degenerate case, that is, this case is one of the Weber type in Figure 3.5. We place the cut as shown by the wavy lines in Figure 3.h ($h = 4, 6$). In this case, we use the branch of $S_{-1} = \sqrt{Q_0}$ at 1, i.e. we take the branch of S_{-1} as (2.3). Then ψ_- is dominant (cf. [6] for the computation). Let us denote regions surrounded by the

Stokes curves as shown in Figures 3.4, 3.5 and 3.6 by $\mathcal{R}^{\mathrm{I}}$, $\mathcal{R}^{\mathrm{III}}$, $\mathcal{R}^{\mathrm{I}}_{\omega_h}$, $\mathcal{R}^{\mathrm{III}}_{\omega_h}$ $(h = 1, 3)$. The WKB solutions $\psi_{\pm,0}$ (resp. $\psi_{\pm,1}$) and $\psi^{(1)}_{\pm}$ are Borel summable in $\mathcal{R}^{\mathrm{I}}$ (resp. $\mathcal{R}^{\mathrm{III}}$) (cf. [13]).

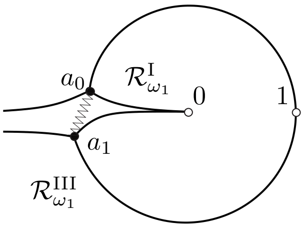

Fig. 3.4. $(\alpha, \beta, \gamma) = (1 - \epsilon + \hat{\epsilon}i, 2, 1)$ in ω_1

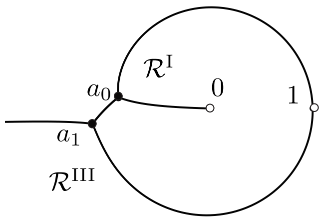

Fig. 3.5. $(1 + \hat{\epsilon}i, 2, 1)$

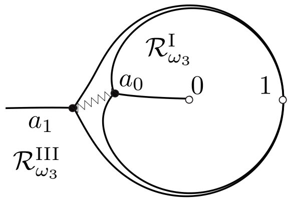

Fig. 3.6. $(1+\epsilon+\hat{\epsilon}i, 2, 1)$ in ω_3

We denote the Borel transform of $\psi^{(1)}_{+}$ by $\psi^{(1),\mathrm{I}}_{+,B}$ and $\psi^{(1),\mathrm{III}}_{+,B}$. The Borel transform $\psi^{(1),\mathrm{I}}_{+,B}$ (resp. $\psi^{(1),\mathrm{III}}_{+,B}$) is free from singularities on (3.1). Moreover, the Borel transform $V_{1,B}(\alpha, \beta, \gamma; y)$ is holomorphic at $y = 0$ and it has simple poles at $y = 2m(\gamma - \alpha)\pi i$ for every non-zero integer $m \in \mathbb{Z} - \{0\}$. Therefore, the Borel transform $\psi^{(1),\mathrm{I}}_{+,B}$ (resp. $\psi^{(1),\mathrm{III}}_{+,B}$) has singularities at

$$y = -y_{+,k}(x) + 2m(\gamma - \alpha)\pi i$$

when $\mathrm{Re}(\gamma - \alpha) = 0$ $(m \in \mathbb{Z})$. The following description of the alien derivative is due to [17]: We consider the alien derivatives $\triangle\psi_{+,0}$ and $\triangle\psi_{+,1}$. Since the Borel transform $V_{1,B}$ of V_1 is a single-valued analytic function with the simple pole at $y = 2m(\gamma-\alpha)\pi i$, we obtain

$$\triangle_{y=2m(\gamma-\alpha)\pi i}(-V_1) = \frac{(-1)^{m+1}}{2m}.$$

Hence, the chain rule in alien calculus leads to

$$\triangle_{y=2m(\gamma-\alpha)\pi i}(\exp(-V_1)) = \frac{(-1)^{m+1}}{2m}\exp(-V_1). \tag{3.9}$$

Since the Borel transform $\psi^{(1),\mathrm{I}}_{+,B}$ (resp. $\psi^{(1),\mathrm{III}}_{\omega_3,B}$) is free from singularities on (3.1), we obtain

$$\triangle\left(\exp(-y_{+,k}(x)\eta)\psi^{(1)}_{+}\right) = 0 \tag{3.10}$$

$(k = 0$ (resp. $1))$. Combining (2.5), (3.9) and (3.10), we have

$$\triangle_{y=2m(\gamma-\alpha)\pi i}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right) = \frac{(-1)^{m+1}}{2m}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right)$$

and

$$\triangle_{y=2m(\gamma-\alpha)\pi i}\left(\exp(-y_{+,1}(x)\eta)\psi_{+,1}\right) = \frac{(-1)^{m+1}}{2m}\left(\exp(-y_{+,1}(x)\eta)\psi_{+,1}\right).$$

Similarly, we can compute the alien derivatives $\triangle_{y=-y_{-,k}(x)+2m(\gamma-\alpha)\pi i}\psi_{-,k}$ of $\psi_{-,k}(x,\eta)$. Then we have the following theorem:

Theorem 3.5. *Let $\psi_{\pm,k}(x,\eta)$ denote the WKB solutions of* (2.1) *normalized at the turning point a_k $(k=0,1)$. Let us consider the case where (α,β,γ) is contained within the boundary between ω_1 and ω_3. The Borel transforms of $\psi_{\pm,k}(x,\eta)$ have fixed singular points at*

$$y=-y_{\pm,k}(x)+2m(\gamma-\alpha)\pi i,\quad m\in\mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,k}(x)+2m(\gamma-\alpha)\pi i}\psi_{\pm,k}$ of $\psi_{\pm,k}(x,\eta)$ $(k=0,1)$ satisfy the following relations:

$$(\triangle_{y=-y_{\pm,0}(x)+2m(\gamma-\alpha)\pi i}\psi_{\pm,0})_B(x,y)=\pm\frac{(-1)^{m+1}}{2m}\psi_{\pm,0,B}(x,y-2m(\gamma-\alpha)\pi i)\quad(3.11)$$

for x in $\mathcal{R}^{\mathrm{I}}$ and

$$(\triangle_{y=-y_{\pm,1}(x)+2m(\gamma-\alpha)\pi i}\psi_{+,1})_B(x,y)=\pm\frac{(-1)^{m+1}}{2m}\psi_{\pm,1,B}(x,y-2m(\gamma-\alpha)\pi i)\quad(3.12)$$

for x in $\mathcal{R}^{\mathrm{III}}$.

Next we study the actions of Stokes automorphisms on the WKB solutions. The WKB solutions $\psi_{\pm,0}$ (resp. $\psi_{\pm,1}$) are Borel summable in $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{I}}_{\omega_3}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{III}}_{\omega_3}$) (cf. [12] and [13] for the notation and terminologies). In this case, the Stokes automorphism on the WKB solutions is defined by

$$\mathfrak{S}\psi_{\pm,k}=\exp\Big[\sum_{m=1}^{\infty}\triangle_{y=-y_{\pm,k}(x)+2m(\gamma-\alpha)\pi i}\Big]\psi_{\pm,k},$$

where $\mathfrak{S}$ denote the Stokes automorphism associated with the change to ω_1 from ω_3 (cf. [8, 9, 14]). We use Theorem 3.5 and the discussion given in [14, 17]. Then we have the following theorem:

Theorem 3.6. *When (α,β,γ) moves ω_1 to ω_3 and $\mathrm{Im}(\alpha-\gamma)$ is positive, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,k}$ on the WKB solutions $\psi_{\pm,k}(k=0,1)$:*

$$\mathfrak{S}\psi_{\pm,0}=(1+\exp(2\pi i(\alpha-\gamma)\eta))^{\frac{1}{2}}\psi_{\pm,0}\quad(3.13)$$

for x in $\mathcal{R}^{\mathrm{I}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{I}}_{\omega_3}$, and

$$\mathfrak{S}\psi_{\pm,1}=(1+\exp(2\pi i(\alpha-\gamma)\eta))^{\frac{1}{2}}\psi_{\pm,1}\quad(3.14)$$

for x in $\mathcal{R}^{\mathrm{III}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{III}}_{\omega_3}$.

Similarly, by taking the Borel sum of (3.13) and (3.14), we can get the parametric Stokes phenomena of the WKB solutions. Then we can give another proof of the parametric Stokes phenomena from view point of alien calculus.

3.1.3. Analysis on the boundary between ω_1 and ω_4

We consider the case where (α, β, γ) belong to the boundary between $\omega_1$and ω_4, i.e., $\mathrm{Re}\alpha = 0$. We assume that $\mathrm{Im}\alpha$ is negative. In this case, we discuss the case where $(\alpha, \beta, \gamma) = (-\hat{\epsilon}i, 2, 1) \in E_1$. It follows from (i) of Theorem 2.1 that we can take $(\alpha, \beta, \gamma) = (-\hat{\epsilon}i, 2, 1)$ without loss of generality. Let us show an example of Stokes curves of those two cases in Figures 3.7 and 3.9 and a degenerate case, namely, one of the Weber type in Figure 3.8. We place the cut as shown by the wavy lines in Figure 3.h ($h = 7, 9$). We use the branch of $S_{-1} = \sqrt{Q_0}$ at ∞, i.e. we take the branch of S_{-1} as (2.4). Then ψ_+ is dominant (cf. [6] for the computation). We denote regions surrounded by the Stokes curves as shown in Figures 3.7, 3.8 and 3.9 by $\mathcal{R}^{\mathrm{II}}$, $\mathcal{R}^{\mathrm{III}}$, $\mathcal{R}^{\mathrm{II}}_{\omega_h}$, $\mathcal{R}^{\mathrm{III}}_{\omega_h}$ ($h = 1, 4$). The WKB solutions $\psi_{\pm,0}$ (resp. $\psi_{\pm,1}$) and $\psi^{(2)}_{\pm}$ are Borel summable in $\mathcal{R}^{\mathrm{II}}$ (resp. $\mathcal{R}^{\mathrm{III}}$ (cf. [13]).

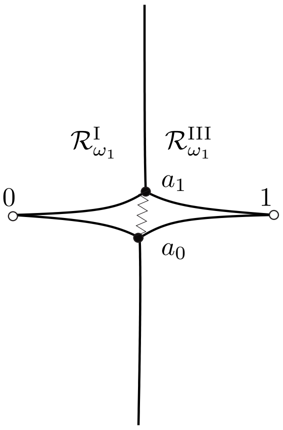

Fig. 3.7. $(\alpha, \beta, \gamma) = (\epsilon - \hat{\epsilon}i, 2, 1)$ in ω_1

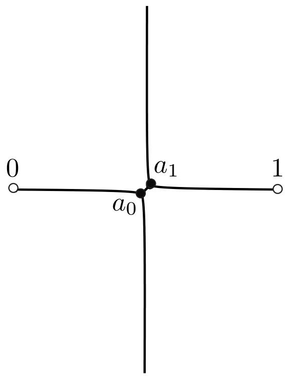

Fig. 3.8. $(-\hat{\epsilon}i, 2, 1)$

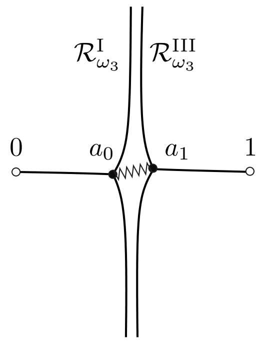

Fig. 3.9. $(-\epsilon - \hat{\epsilon}i, 2, 1)$ in ω_4

We denote the Borel transform of $\psi^{(2)}_{+}$ by $\psi^{(2),\mathrm{II}}_{+,B}$ and $\psi^{(2),\mathrm{III}.}_{+,B}$ The Borel transform $\psi^{(2),\mathrm{II}}_{+,B}$ (resp. $\psi^{(2),\mathrm{III}}_{+,B}$) is free from singularities on (3.1). Moreover, the Borel transform $V_{2,B}(\alpha, \beta, \gamma; y)$ is holomorphic at $y = 0$ and it has simple poles at $y = 2m\alpha\pi i$ for every non-zero integer $m \in \mathbb{Z} - \{0\}$. Therefore, the Borel transform $\psi^{(2),\mathrm{II}}_{+,B}$ (resp. $\psi^{(2),\mathrm{III}}_{+,B}$) of $\psi_{+,k}$ has singularities at

$$y = -y_{+,k}(x) + 2m\alpha\pi i$$

when $\mathrm{Re}\alpha = 0$ ($m \in \mathbb{Z}$). The following description of the alien derivative is due to [17]: We consider the alien derivatives $\triangle\psi_{+,0}$ and $\triangle\psi_{+,1}$ in the case where (α, β, γ) moves from ω_1 to ω_4. Since the Borel transform $V_{2,B}$ of V_2 is a single-valued analytic function with simple pole at $y = 2m\alpha\pi i$, we obtain

$$\triangle_{y=2m\alpha\pi i}(-V_2) = \frac{(-1)^m}{2m}$$

(cf. [17]). Hence, the chain rule in alien calculus leads to

$$\triangle_{y=2m\alpha\pi i}(\exp(-V_2)) = \frac{(-1)^m}{2m}\exp(-V_2). \tag{3.15}$$

Since the Borel transforms $\psi^{(2),\mathrm{II}}_{\omega_1,B}$ and $\psi^{(2),\mathrm{III}}_{\omega_1,B}$ (resp. $\psi^{(2),\mathrm{II}}_{\omega_4,B}$ and $\psi^{(2),\mathrm{III}}_{\omega_4,B}$) are free from singularities on (3.1), we have

$$\triangle\left(\exp(-y_{+,k}(x)\eta)\psi^{(2)}_{+}\right) = 0 \tag{3.16}$$

($k = 0$ (resp. 1)). Combining (2.5), (3.15) and (3.16), we have

$$\triangle_{y=2m\alpha\pi i}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right) = \frac{(-1)^m}{2m}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right)$$

and

$$\triangle_{y=2m\alpha\pi i}\left(\exp(-y_{+,1}(x)\eta)\psi_{\pm,1}\right) = \frac{(-1)^m}{2m}\left(\exp(-y_{+,1}(x)\eta)\psi_{+,1}\right).$$

In a similar way, we can compute the alien derivative $\triangle_{y=-y_{-,k}(x)+2m\alpha\pi i}\psi_{-,k}$ of $\psi_{-,k}(x,\eta)$. Then we have the following theorem:

Theorem 3.7. *Let $\psi_{\pm,k}(x,\eta)$ denote the WKB solutions of* (2.1) *normalized at the turning point a_k ($k = 0, 1$). Let us consider the case where (α,β,γ) is contained within the boundary between ω_1 and ω_4. The Borel transforms of $\psi_{\pm,k}(x,\eta)$ have the fixed singular points at*

$$y = -y_{\pm,k}(x) + 2m\alpha\pi i, \quad m \in \mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,k}(x)+2m\alpha\pi i}\psi_{\pm,k}$ of $\psi_{\pm,k}(x,\eta)$ ($k = 0, 1$) satisfy the following relations:

$$(\triangle_{y=-y_{\pm,0}(x)+2m\alpha\pi i}\psi_{\pm,0})_B(x,y) = \pm\frac{(-1)^m}{2m}\psi_{\pm,0,B}(x, y-2m\alpha\pi i) \tag{3.17}$$

for x in $\mathcal{R}^{\mathrm{II}}$ and

$$(\triangle_{y=-y_{\pm,1}(x)+2m\alpha\pi i}\psi_{\pm,1})_B(x,y) = \pm\frac{(-1)^m}{2m}\psi_{\pm,1,B}(x, y-2m\alpha\pi i) \tag{3.18}$$

for x in $\mathcal{R}^{\mathrm{III}}$.

Next we discuss the actions of Stokes automorphisms on the WKB solutions. The WKB solutions $\psi_{\pm,0}$ (resp. $\psi_{\pm,1}$) are Borel summable in $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_4}$ (resp. $\mathcal{R}^{\mathrm{III}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{III}}_{\omega_4}$) (cf. [12] and [13] for the notation and terminologies). In this case, the Stokes automorphism on the WKB solutions is defined by

$$\mathfrak{S}\psi_{\pm,k} = \exp\Big[\sum_{m=1}^{\infty}\triangle_{y=-y_{\pm,k}(x)+2m(\gamma-\alpha)\pi i}\Big]\psi_{+,k},$$

where $\mathfrak{S}$ denote the Stokes automorphism associated with the change to ω_1 from ω_4 (cf. [8, 9, 14]). Using Theorem 3.5 and the discussion given in [14, 17], we have the following theorem.

Theorem 3.8. *When (α, β, γ) moves ω_1 to ω_4 and* $\mathrm{Im}\alpha$ *is negative, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,k}$ on the WKB solutions $\psi_{\pm,k}$ $(k = 0, 1)$:*

$$\mathfrak{S}\psi_{\pm,0} = (1 + \exp(-2\pi i\alpha\eta))^{\frac{1}{2}}\psi_{\pm,0} \tag{3.19}$$

for x in $\mathcal{R}^{\mathrm{II}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_4}$, and

$$\mathfrak{S}\psi_{\pm,1} = (1 + \exp(-2\pi i\alpha\eta))^{\frac{1}{2}}\psi_{\pm,1} \tag{3.20}$$

for x in $\mathcal{R}^{\mathrm{III}}_{\omega_1}$ and $\mathcal{R}^{\mathrm{III}}_{\omega_4}$.

Similarly, by taking the Borel sum of (3.19) and (3.20), we can get the parametric Stokes phenomena of the WKB solutions. Then we can give another proof of the parametric Stokes phenomena from the view point of alien calculus.

Next we discuss alien derivatives when (α, β, γ) belongs to E_1 and Stokes automorphisms between ω_{1m} and $\omega_{hm}(h = 2, 3, 4; m = 0, \ldots, 6)$. Since the potential Q is invariant under involution ι_m, the Stokes geometry for $(\alpha, \beta, \gamma) \in \iota_{hm}$ is the same as that for $(\alpha, \beta, \gamma) \in \iota_h$. Applying ι_m to the relations (3.5), (3.6), (3.11), (3.12), (3.17) and (3.18), we have the formulas of the alien derivatives. Similarly, we give the Stokes automorphisms between ω_{1m} and ω_{hm}.

3.2. ANALYSIS OF STOKES CURVES OF LOOP TYPE

We give the computations of the alien derivatives for a loop-type. In [1] and [11], they will give a concrete form of the alien derivative of the WKB solutions of the general linear second-order differential equation with a large parameter. In this paper, we consider the alien derivatives of the WKB solutions for the Gauss hypergeometrc differential equations of the loop-type. We consider the case where $(\alpha, \beta, \gamma)$is contained within the boundary between ω_4 and ω_{41}, i.e., $\mathrm{Re}(\alpha + \beta - \gamma) = 0$. We assume that $\mathrm{Im}(\alpha + \beta - \gamma)$ is positive. In this case, we consider the case where $(\alpha, \beta, \gamma) = (-0.5, 1.5 + \hat{\epsilon}i, 1) \in E_2$. If $\mathrm{Re}(\alpha + \beta - \gamma) = 0$, the Stokes geometry has a Stokes which is a loop around 1. Hence, we can take $(\alpha, \beta, \gamma) = (-0.5, 1.5 + \hat{\epsilon}i, 1)$ without loss of generality. Let us show an example of Stokes curves of a derivative case, namely, the loop-type in Figure 3.10.

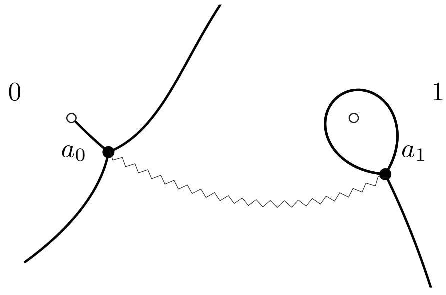

Fig. 3.10. $(\alpha, \beta, \gamma) = (-0.5, 1.5 + \hat{\epsilon}i, 1)$

In this case, a Stokes curve forms a closed curve with 1 in Figure 3.10. We place the cut as shown by the wavy lines in Figure 3.10. We use the branch of $S_{-1} = \sqrt{Q_0}$ at 1, i.e. we take the branch of S_{-1} as (2.3). Hence ψ_- is dominant (cf. [6] for the computation). We denote a region surrounded by the Stokes curve which forms a closed curve with 0 by $\mathcal{R}^{\mathrm{IV}}$. The WKB solutions $\psi_{\pm,1}$ and $\psi_{\pm}^{(1)}$ are Borel summable in $\mathcal{R}^{\mathrm{IV}}$ (cf. [1, 11, 13]). We denote the Borel transforms of $\psi_{+,1}$ and $\psi_{+}^{(1)}$ by $\psi_{+,B}^{\mathrm{IV}}$ and $\psi_{+,B}^{(1),\mathrm{IV}}$, respectively.

The Borel transform $\psi_{+,B}^{(1),\mathrm{IV}}$ is free from singularities on (3.1) (cf. [13]) Moreover, the Borel transform $V_{1,B}(\alpha,\beta,\gamma;y)$ is holomorphic at $y=0$ and it has simple poles at $y = 2m(\alpha+\beta-\gamma)\pi i$ for every non-zero integer $m \in \mathbb{Z}-\{0\}$. Therefore, the Borel transforms of $\psi_{+,k}$ has singularities at

$$y = -y_{+,k}(x) + 2m(\alpha+\beta-\gamma)\pi i$$

when $\mathrm{Re}(\alpha+\beta-\gamma) = 0$ $(m \in \mathbb{Z})$. The following description of the alien derivative of the loop-type is due to [1]. Since the Borel transform $V_{1,B}$ of V_1 is a single-valued analytic function with the simple pole at $y = 2m(\alpha+\beta-\gamma)\pi i$, we have

$$\triangle_{y=2m(\alpha+\beta-\gamma)\pi i}(-V_1) = \frac{1}{m}$$

(cf. [17]). Hence, the chain rule in the alien calculus leads to

$$\triangle_{y=2m(\alpha+\beta-\gamma)\pi i}(\exp(-V_1)) = \frac{1}{m}\exp(-V_1). \tag{3.21}$$

For the Borel transform $\psi_{+,B}^{(1),\mathrm{IV}}$ is free from singularities on (3.1), we have

$$\triangle\left(\exp(-y_{+,1}(x)\eta)\psi_{+}^{(1)}\right) = 0. \tag{3.22}$$

Combining (2.5), (3.21) and (3.22), we have

$$\triangle_{y=2m(\alpha+\beta-\gamma)\pi i}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right) = \frac{1}{m}\left(\exp(-y_{+,0}(x)\eta)\psi_{+,0}\right). \tag{3.23}$$

Similarly, we can compute the alien derivative $\triangle_{y=-y_{-,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\psi_{-,k}$ of $\psi_{-,k}(x,\eta)$. Then we have the following theorem:

Theorem 3.9. *Let $\psi_{\pm,1}(x,\eta)$ denote the WKB solutions of* (2.1) *normalized at the turning point a_1. Let us consider the case where (α,β,γ) is contained within the boundary between ω_4 and ω_{41}. The Borel transforms of $\psi_{\pm,1}(x,\eta)$ have fixed singular points at*

$$y = -y_{\pm,1}(x) + 2m(\alpha+\beta-\gamma)\pi i, \quad m \in \mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\psi_{\pm,1}$ of $\psi_{\pm,1}(x,\eta)$ satisfy the following relations:

$$(\triangle_{y=-y_{\pm,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\psi_{\pm,1})_B(x,y) = \pm\frac{1}{m}\psi_{\pm,1,B}(x, y-2m(\alpha+\beta-\gamma)\pi i)$$

for x in $\mathcal{R}^{\mathrm{IV}}$.

Using the alien derivatives on the WKB solutions, we compute the action of a Stokes automorphisms on the WKB solutions $\psi_\pm$ of the loop-type. We discuss the case where (α,β,γ) moves ω_4 to ω_{41}. We show examples which are enlarged near the singular point 1 of Stokes curves of those two cases in Figures 3.11 and 3.12. Let us denote regions surrounded by the Stokes curves as shown in Figures 3.11 and 3.12 by $\mathcal{R}^{\mathrm{IV}}_{\omega_4}$ and $\mathcal{R}^{\mathrm{IV}}_{\omega_{41}}$, respectively.

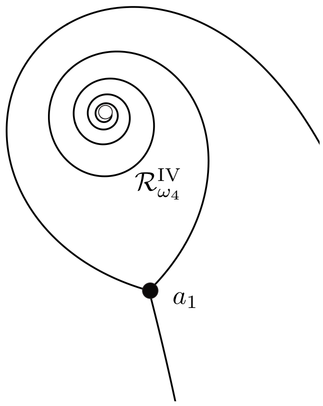

Fig. 3.11. $(\alpha,\beta,\gamma)=(-0.4+\epsilon,1.5+\hat{\epsilon}i,1)\in\omega_4$

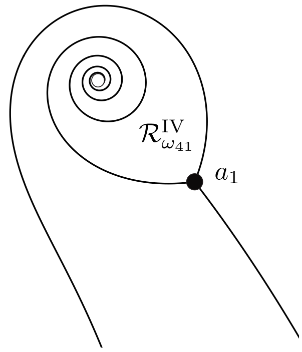

Fig. 3.12. $(-0.6-\epsilon,1.5+\hat{\epsilon}i,1)\in\omega_{41}$

By the definition of the alien derivatives $\triangle\psi_{\pm,1}$, we have

$$\mathfrak{S}\psi_{\pm,1}=\exp\Big[\sum_{m=1}^{\infty}\triangle_{y=-y_{\pm,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\Big]\psi_{\pm,k}.$$

Here $\mathfrak{S}$ denote Stokes automorphism associated with the change to ω_4 from ω_{41} (cf. [8,9,14]).

Theorem 3.10. *When (α,β,γ) moves ω_4 to ω_{41} and $\mathrm{Im}(\alpha+\beta-\gamma)$ is positive, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,1}$ of the WKB solutions $\psi_{\pm,1}$*:

$$\mathfrak{S}\psi_{\pm,1}=(1+\exp(-2(\alpha+\beta-\gamma)\pi i))\psi_{\pm,1}. \tag{3.24}$$

Proof. By the definition of the alien derivatives and the Stokes automorphism, we have

$$\begin{aligned}\triangle\psi_{+,1}&=\sum_{m=1}^{\infty}\triangle_{y=-y_{+,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\psi_{+,1}\\&=\sum_{m=1}^{\infty}\mathcal{B}^{-1}[\triangle_{y=-y_{+,1}(x)+2m(\alpha+\beta-\gamma)\pi i}\psi_{+,1,B}]\\&=\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{B}^{-1}[\psi_{+,1,B}(x,y-2m(\alpha+\beta-\gamma)\pi i)]\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m} \exp\left(-2m(\alpha+\beta-\gamma)\pi i\right)\psi_{+,1}$$
$$= \log(1-\exp\left(-2(\alpha+\beta-\gamma)\pi i\right))\psi_{+,1}.$$

Similarly, we can compute $\triangle\psi_{-,1}$. Hence, we have (3.24). □

Finally, we discuss alien derivatives and Stokes automorphisms when $(\alpha, \beta\gamma)$ belongs to E_2 and moves from ω_h to $\omega_{hm}(h = 1, 2, 3, 4; m = 0, 1, \ldots, 6)$, respectively. If $(\alpha, \beta\gamma)$ is contained $\mathrm{Re}\gamma = 0$ (resp. $\mathrm{Re}(\alpha-\beta) = 0$), a Stokes geometry has a loop around 0 (resp. ∞). We can compute them in a similar manner as the computation of Theorem 3.9 and 3.10 and we have the following theorems:

Theorem 3.11. *Let $\psi_{\pm,0}(x,\eta)$ denote the WKB solutions of* (2.1) *normalized at the turning point $a_k(k = 0, 1)$. Let us consider the case where (α, β, γ) is contained within the boundary between ω_m and ω_{hm} $(h = 1, 2, 3, 4; m = 0, 1, \ldots, 6)$ and $\mathrm{Re}\gamma = 0$. The Borel transforms of $\psi_{\pm,0}(x,\eta)$ have the fixed singular points at*

$$y = -y_{\pm,k}(x) + 2m\gamma\pi i, \quad m \in \mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,k}(x)+2m\gamma\pi i}\psi_{\pm,k}$ of $\psi_{\pm,k}(x,\eta)$ satisfy the following relations:

$$(\triangle_{y=-y_{\pm,k}(x)+2m\gamma\pi i}\psi_{\pm,k})_B(x,y) = \pm\frac{1}{m}\psi_{\pm,k,B}(x, y-2m\gamma\pi i)$$

for x inside of the loop. Let us consider the case where (α, β, γ) is contained within the boundary between ω_m and ω_{hm} $(h = 1, 2, 3, 4; m = 0, 1, \ldots, 6)$ and $\mathrm{Re}(\beta-\alpha) = 0$. The Borel transforms of $\psi_{\pm,k}(x,\eta)$ have fixed singular points at

$$y = -y_{\pm,k}(x) + 2m(\beta-\alpha)\pi i, \quad m \in \mathbb{Z}.$$

Furthermore, the alien derivatives $\triangle_{y=-y_{\pm,k}(x)+2m(\beta-\alpha)\pi i}\psi_{\pm,k}$ of $\psi_{\pm,k}(x,\eta)$ satisfy the following relations:

$$(\triangle_{y=-y_{\pm,k}(x)+2m(\beta-\alpha)\pi i}\psi_{\pm,k})_B(x,y) = \mp\frac{1}{m}\psi_{\pm,k,B}(x, y-2m\gamma\pi i)$$

for x inside of the loop.

Theorem 3.12. *When (α, β, γ) moves ω_h to $\omega_{hm}(h = 1, 2, 3, 4; m = 0, 1, \ldots, 6)$ and $\mathrm{Re}\gamma = 0$, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,k}$ of the WKB solutions $\psi_{\pm,k}$*:

$$\mathfrak{S}\psi_{\pm,k} = (1+\exp\left(-2\gamma\pi i\right))\psi_{\pm,k}.$$

When (α, β, γ) moves ω_h to $\omega_{hm}(h = 1, 2, 3, 4; m = 0, 1, \ldots, 6)$ and $\mathrm{Re}(\beta-\alpha) = 0$, the Stokes automorphisms $\mathfrak{S}\psi_{\pm,k}$ of the WKB solutions $\psi_{\pm,k}$:

$$\mathfrak{S}\psi_{\pm,k} = -(1+\exp\left(-2(\beta-\alpha)\pi i\right))\psi_{\pm,k}.$$

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