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q-ANALOGUE OF SUMMABILITY OF FORMAL SOLUTIONS OF SOME LINEAR q-DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. Let $q > 1$. The paper considers a linear q-difference-differential equation: it is a q-difference equation in the time variable t, and a partial differential equation in the space variable z. Under suitable conditions and by using q-Borel and q-Laplace transforms (introduced by J.-P. Ramis and C. Zhang), the authors show that if it has a formal power series solution $\hat{X}(t,z)$ one can construct an actual holomorphic solution which admits $\hat{X}(t,z)$ as a q-Gevrey asymptotic expansion of order 1.

Keywords: q-difference-differential equations, summability, formal power series solutions, q-Gevrey asymptotic expansions, q-Laplace transform.

Mathematics Subject Classification: 35C10, 35C20, 39A13.

1. INTRODUCTION

Let $m \geq 1$ be an integer, and let $(t,z) = (t, z_1, \ldots, z_d) \in \mathbb{C}_t \times \mathbb{C}_z^d$ be complex variables. For $r > 0$ we write $D_r = \{t \in \mathbb{C}\,;\, |t| \leq r\}$ and $D_r^* = \{t \in \mathbb{C}\,;\, 0 < |t| \leq r\}$. For $R > 0$ we write $D_R = \{z \in \mathbb{C}^d\,;\, |z| \leq R\}$ with $|z| = \max_{1 \leq i \leq d} |z_i|$. We denote by $\mathcal{O}_R$ the set of all holomorphic functions in a neighbourhood of D_R, and by $\mathcal{O}_R[[t]]$ the set of all formal power series in t with coefficients in $\mathcal{O}_R$.

For a holomorphic function $f(t,z)$ in a neighbourhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we define the order of the zeros of the function $f(t,z)$ at $t = 0$ (we denote this by $\mathrm{ord}_t(f)$) by

$$\mathrm{ord}_t(f) = \min\{k \in \mathbb{N}\,;\, (\partial_t^k f)(0,z) \not\equiv 0 \ \text{ near } z = 0\},$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let us consider the linear partial differential equation

$$\sum_{j+|\alpha| \leq m} a_{j,\alpha}(t,z)(t\partial_t)^j \partial_z^\alpha X = F(t,z) \tag{1.1}$$

with the unknown function $X = X(t,z)$, where $a_{j,\alpha}(t,z)$ $(j+|\alpha| \le m)$ and $F(t,z)$ are holomorphic functions in a neighbourhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. The Newton polygon $N(1.1)$ of (1.1) is defined by

$$N(1.1) = \text{the convex hull of} \bigcup_{j+|\alpha|\le m} C(j+|\alpha|, \mathrm{ord}_t(a_{j,\alpha}))$$

in $\mathbb{R}^2$, where $C(a,b) = \{(x,y) \in \mathbb{R}^2\,;\, x \le a, y \ge b\}$. See Miyake [8] and Ouchi [10] (though Ouchi used the word "the characteristic polygon" instead of "the Newton polygon"). Let us consider the following two cases:
Case 1. $N(1.1) = \{(x,y) \in \mathbb{R}^2\,;\, x \le m,\ y \ge 0\}$.
Case 2. There is an integer $0 \le m_0 < m$ such that

$$N(1.1) = \{(x,y) \in \mathbb{R}^2\,;\, x \le m,\ y \ge \max\{0, x-m_0\}\}.$$

In Case 1, about the convergence of formal solutions of (1.1), by Baouendi-Goulaouic [1] we have the following result.

Theorem 1.1. *Suppose the condition in Case 1,*

$$a_{m,0}(0,0) \neq 0, \quad \textit{and} \quad \mathrm{ord}_t(a_{j,\alpha}) \ge 1 \ \textit{ if } |\alpha| > 0.$$

Then, if (1.1) *has a formal solution* $\hat{X}(t,z) = \sum_{n\ge 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$, *it is convergent in a neighbourhood of the origin* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

In Case 2, even if (1.1) has a formal solution, it is not convergent in general, but we can give a meaning to this formal solution by using the notion of Borel summability. By [10], we have the following theorem.

Theorem 1.2. *Suppose the condition in Case 2,*

$$a_{m_0,0}(0,0) \neq 0, \quad \frac{a_{m,0}(t,0)}{t^{m-m_0}}\Big|_{t=0} \neq 0,$$

and

$$\mathrm{ord}_t(a_{j,\alpha}) \ge \max\{1, j+|\alpha|-m_0+1\} \ \textit{ if } |\alpha| > 0.$$

Then, if (1.1) *has a formal solution* $\hat{X}(t,z) = \sum_{n\ge 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$, *it is Borel summable in* t (*uniformly in* z *near* $z = 0$) *in a suitable direction.*

Let $q > 1$. For a function $f(t,z)$ we define the q-difference operator D_q by

$$(D_q f)(t,z) = \frac{f(qt,z) - f(t,z)}{qt - t}.$$

In this paper, we will try to *q-discretize* equation (1.1) with respect to the time variable t in the form

$$\sum_{j+|\alpha|\le m} a_{j,\alpha}(t,z)(tD_q)^j \partial_z^\alpha X = F(t,z), \tag{1.2}$$

and we will consider the following problem.

Problem 1.3. Let $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ be a formal solution of (1.2). Then:

(1) (q-analogue of Theorem 1.1) Under what condition can we get the convergence of the formal solution $\hat{X}(t,z)$?
(2) (q-analogue of Theorem 1.2) Under what condition can we get a true solution $W(t,z)$ of (1.2) which admits $\hat{X}(t,z)$ as a q-Gevrey asymptotic expansion of order 1 (in the sense of Definition 1.4 given below)?

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\epsilon > 0$, we set

$$\mathscr{Z}_\lambda = \{-\lambda q^m \in \mathbb{C}\,;\, m \in \mathbb{Z}\},$$
$$\mathscr{Z}_{\lambda,\epsilon} = \bigcup_{m\in\mathbb{Z}} \{t \in \mathbb{C} \setminus \{0\}\,;\, |1 + \lambda q^m/t| \leq \epsilon\}.$$

It is easy to see that if $\epsilon > 0$ is sufficiently small the set $\mathscr{Z}_{\lambda,\epsilon}$ is a disjoint union of closed disks. The following definition is due to Ramis-Zhang [11].

Definition 1.4. Let $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$ and let $W(t,z)$ be a holomorphic function on $(D_r^* \setminus \mathscr{Z}_\lambda) \times D_R$ for some $r > 0$. We say that $W(t,z)$ *admits* $\hat{X}(t,z)$ *as a q-Gevrey asymptotic expansion of order 1*, if there are $M > 0$ and $H > 0$ such that

$$\left| W(t,z) - \sum_{n=0}^{N-1} X_n(z)t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2} |t|^N \tag{1.3}$$

holds on $(D_r^* \setminus \mathscr{Z}_{\lambda,\epsilon}) \times D_R$ for any $N = 0, 1, 2, \ldots$ and any sufficiently small $\epsilon > 0$.

To solve Problem 1.3 we will use the framework of q-Laplace and q-Borel transforms via the Jacobi theta function, developed by Ramis-Zhang [11] and Zhang [15]. In the case of q-difference equations (corresponding to ordinary differential equations), q-analogues of summability of formal solutions have been studied quite well by Zhang [14], Marotte-Zhang [7] and Ramis-Sauloy-Zhang [12]. In the case of q-difference-differential equations, we have some references, Malek [5, 6], Lastra-Malek [3] and Lastra-Malek-Sanz [4], but their equations are different from ours.

2. MAIN RESULTS

Throughout this paper, we let $q > 1$ be a real number, $m \geq 1$ be an integer, and $\sigma > 0$ be a real number. As a generalization of (1.2), we will treat the following equation

$$\sum_{j+\sigma|\alpha|\leq m} a_{j,\alpha}(t,z)(tD_q)^j \partial_z^\alpha X = F(t,z) \tag{2.1}$$

with the unknown function $X = X(t,z)$, where $a_{j,\alpha}(t,z)$ $(j + \sigma|\alpha| \leq m)$ and $F(t,z)$ are holomorphic functions in a neighbourhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

In this case, we will use the t-Newton polygon (see the paper by Tahara-Yamazawa [13]): the t-Newton polygon $N_t(2.1)$ of equation (2.1) is defined by

$$N_t(2.1) = \text{the convex hull of } \bigcup_{j+\sigma|\alpha|\le m} C(j, \mathrm{ord}_t(a_{j,\alpha})).$$

in $\mathbb{R}^2$. Let us consider the following two cases:
Case 1. $N_t(2.1) = \{(x, y) \in \mathbb{R}^2\,;\, x \le m,\, y \ge 0\}$.
Case 2. There is an integer $0 \le m_0 < m$ such that

$$N_t(2.1) = \{(x, y) \in \mathbb{R}^2\,;\, x \le m,\, y \ge \max\{0, x - m_0\}\}.$$

In Case 1, we can give a q-analogue of Theorem 1.1 in the following form:

Theorem 2.1. *Suppose the condition in Case 1,*

$$a_{m,0}(0, 0) \ne 0, \quad \textit{and} \quad \mathrm{ord}_t(a_{j,\alpha}) \ge 1 \ \textit{if } |\alpha| > 0.$$

Then, if (2.1) *has a formal solution* $\hat{X}(t, z) = \sum_{n\ge0} X_n(z)t^n \in \mathcal{O}_R[[t]]$, *it is convergent in a neighbourhood of the origin* $(0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z$.

Example 2.2. Let us consider

$$(tD_q)^m X = A(z)t + B(z)t^p(tD_q)^j \partial_z^\alpha X,$$

where $A(z)$ and $B(z)$ are holomorphic functions in a neighbourhood of $z = 0$. In the case when $|\alpha| = 0$, if $j \le m$ and $p \ge 1$ we can apply Theorem 2.1 to this equation. In the case when $|\alpha| > 0$, if $j \le m - 1$ and $p \ge 1$ we can apply Theorem 2.1 to this equation. We note that for any $|\alpha| > 0$ by setting $\sigma = 1/|\alpha| > 0$ we have $j+\sigma|\alpha| \le m$.

In Case 2, by assumption we have the expression

$$a_{j,0}(t, z) = t^{j-m_0} b_{j,0}(t, z) \quad \text{for } m_0 < j \le m$$

for some holomorphic functions $b_{j,0}(t, z)$ ($m_0 < j \le m$) in a neighbourhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z$. We set

$$P(\xi, z) = \sum_{m_0<j\le m} \frac{b_{j,0}(0, z)}{(q-1)^j q^{j(j-1)/2}} \xi^{j-m_0} + \frac{a_{m_0,0}(0, z)}{(q-1)^{m_0} q^{m_0(m_0-1)/2}}.$$

If the conditions $a_{m_0,0}(0, 0) \ne 0$ and $b_{m,0}(0, 0) \ne 0$ are satisfied, we see that $P(\xi, 0)$ is a polynomial of degree $m - m_0$ and it has $m - m_0$ non-zero roots $\tau_1, \ldots, \tau_{m-m_0}$. Then, the set S of singular directions is defined by

$$S = \bigcup_{i=1}^{m-m_0} \{t\tau_i\,;\, t > 0\}.$$

As to a q-analogue of Theorem 1.2, we have the following result.

Theorem 2.3.

(1) *Suppose the condition in Case 2,*

$$a_{m_0,0}(0,0) \neq 0, \quad \text{and} \quad \mathrm{ord}_t(a_{j,\alpha}) \geq \max\{1, j-m_0+1\} \ \text{if } |\alpha|>0.$$

Then, if (2.1) *has a formal solution* $\hat{X}(t,z) = \sum_{n\geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$, *we can find* $A>0$, $h>0$ *and* $0<R_1<R$ *such that* $|X_n(z)| \leq Ah^n q^{n(n-1)/2}$ *on* D_{R_1} *for any* $n = 0,1,2,\ldots$.

(2) *In addition, if the conditions*

$$\frac{a_{m,0}(t,0)}{t^{m-m_0}}\Big|_{t=0} \neq 0 \quad (\text{this is equivalent to } b_{m,0}(0,0)\neq 0),$$
$$\mathrm{ord}_t(a_{j,\alpha}) \geq j-m_0+2, \quad \text{if } |\alpha|>0 \text{ and } m_0 \leq j < m$$

are satisfied, for any $\lambda \in \mathbb{C}\setminus(\{0\}\cup S)$ *there are* $r>0$, $R_1>0$ *and a holomorphic solution* $W(t,z)$ *of* (2.1) *on* $(D_r^* \setminus \mathscr{Z}_\lambda)\times D_{R_1}$ *such that* $W(t,z)$ *admits* $\hat{X}(t,z)$ *as a q-Gevrey asymptotic expansion of order 1.*

Example 2.4. Let $0 \leq m_0 < m$ and let us consider

$$(tD_q)^{m_0}X = A(z)t + t^{m-m_0}(tD_q)^m X + B(z)t^p(tD_q)^j\partial_z^\alpha X,$$

where $A(z)$ and $B(z)$ are holomorphic functions in a neighbourhood of $z=0$. In the case $|\alpha|=0$, if $j\leq m$ and $p\geq \max\{1, j-m_0+1\}$ we can apply Theorem 2.3 to this equation. In the case $|\alpha|>0$, if $j\leq m-1$ and $p\geq\max\{1,j-m_0+2\}$ we can apply Theorem 2.3 to this equation. In both cases, S is given by

$$S = \{z = te^{\sqrt{-1}\theta}\in\mathbb{C}\,;\, t>0, \theta = 2\pi k/(m-m_0), 0\leq k\leq m-m_0-1\}.$$

The rest of this paper is organised as follows. In Section 3 we give a proof of Theorem 2.1, in Section 4 we show part (1) of Theorem 2.3, and in Sections 5 and 6 we prove part (2) of Theorem 2.3.

By the definition of D_q, we have

$$(tD_q f)(t,z) = \frac{f(qt,z)-f(t,z)}{q-1}.$$

If we define the operator σ_q by $\sigma_q(f)(t,z) = f(qt,z)$, we can rewrite equation (2.1) to the following linear equation

$$\sum_{j+\sigma|\alpha|\leq m} a_{j,\alpha}(t,z)(q-1)^{-j}(\sigma_q-1)^j\partial_z^\alpha X = F(t,z)$$

which is written in the form

$$\sum_{j+\sigma|\alpha|\leq m} a^*_{j,\alpha}(t,z)(\sigma_q)^j\partial_z^\alpha X = F(t,z) \tag{2.2}$$

with

$$a^*_{j,\alpha}(t,z) = \sum_{j \le k \le m-\sigma|\alpha|} a_{k,\alpha}(t,z)(q-1)^{-k}\binom{k}{j}(-1)^{k-j}, \quad j+\sigma|\alpha| \le m.$$

Therefore, in the proof of Theorems 2.1 and 2.3 in Sections 3–6 we will treat equation (2.2) instead of the original equation (2.1). In the discussion, we will use the norm $\|\varphi\|_s = \max_{|z|\le s}|\varphi(z)|$ and the following lemma.

Lemma 2.5. *If a holomorphic function $\varphi(z)$ on D_R satisfies*

$$\|\varphi\|_s \le \frac{A}{(R-s)^a} \quad \text{for any} \quad 0 < s < R,$$

for some $A > 0$ and $a \ge 0$, we have the estimates

$$\left\|\partial_{z_i}\varphi\right\|_s \le \frac{(a+1)eA}{(R-s)^{a+1}} \quad \text{for any} \quad 0 < s < R \text{ and } i = 1,\ldots,d.$$

For the proof, see [9] or Lemma 5.1.3 in [2].

3. PROOF OF THEOREM 2.1

Let us consider the equation

$$\sum_{j+\sigma|\alpha|\le m} a_{j,\alpha}(t,z)(\sigma_q)^j\partial_z^\alpha X = F(t,z), \tag{3.1}$$

where $a_{j,\alpha}(t,z)$ $(j+\sigma|\alpha| \le m)$ and $F(t,z)$ are holomorphic functions in a neighbourhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$. To prove Theorem 2.1 it is enough to show the following proposition.

Proposition 3.1. *Suppose the conditions*

$$a_{m,0}(0,0) \ne 0, \quad \text{and} \quad \mathrm{ord}_t(a_{j,\alpha}) \ge 1 \text{ if } |\alpha| > 0. \tag{3.2}$$

Then, if (3.1) has a formal solution $\hat{X}(t,z) = \sum_{n\ge0} X_n(z)t^n \in \mathcal{O}_R[[t]]$, it is convergent in a neighbourhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z^d$.

Proof. By the assumption, we can expand $a_{j,\alpha}(t,z)$ $(j+\sigma|\alpha| \le m)$ and $F(t,z)$ into the forms:

$$\begin{aligned} a_{j,0}(t,z) &= \sum_{k\ge0} c_{j,0,k}(z)t^k \quad (0 \le j \le m), \\ a_{j,\alpha}(t,z) &= \sum_{k\ge1} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0), \\ F(t,z) &= \sum_{k\ge0} F_k(z)t^k. \end{aligned}$$

We may suppose that $R > 0$ is sufficiently small. Therefore, we may suppose $0 < R < 1$, that $c_{j,\alpha,k}(z)$ and $F_k(z)$ are all holomorphic functions on D_R, and that there are $B > 0$ and $h > 0$ satisfying $|c_{j,\alpha,k}(z)| \le Bh^k$ ($j + \sigma|\alpha| \le m$ and $k \ge 1$) and $|F_k(z)| \le Bh^k$ ($k \ge 0$) on D_R. Since $a_{m,0}(0,0) \neq 0$ is supposed, we may also assume that $a_{m,0}(0,z) \neq 0$ on D_R. We set

$$C(\lambda, z) = \sum_{j \le m} a_{j,0}(0,z)\lambda^j.$$

It is clear that there are constant $c_0 > 0$ and a positive integer μ such that

$$|C(q^n, z)| \ge c_0 (q^n)^m \quad \text{on } D_R \text{ for any } n \ge \mu. \tag{3.3}$$

Since $a_{j,0}(0,z) = c_{j,0,0}(z)$ ($0 \le j \le m$) holds, our equation (3.1) is written in the form

$$C(\sigma_q, z)X = F(t,z) - \sum_{j+\sigma|\alpha| \le m} \sum_{k \ge 1} c_{j,\alpha,k}(z) t^k (\sigma_q)^j \partial_z^\alpha X. \tag{3.4}$$

Let

$$\hat{X}(t,z) = \sum_{n \ge 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$$

be a formal solution of (3.1). By substituting this into (3.4) and by comparing the coefficients of t^n in both sides of the equation, we have the following recurrent formulas:

$$C(q^0, z)X_0 = F_0(z)$$

and for $n \ge 1$

$$C(q^n, z)X_n = F_n(z) - \sum_{j+\sigma|\alpha| \le m} \sum_{k=1}^{n} c_{j,\alpha,k}(z)(q^j)^{n-k} \partial_z^\alpha X_{n-k}. \tag{3.5}$$

We set $L = m/\sigma$; if $j + \sigma|\alpha| \le m$ we have $|\alpha| \le L$. To prove Proposition 3.1 it is enough to show the following lemma.

Lemma 3.2. *There are $A > 0$ and $H > 0$ such that the estimate*

$$\|\partial_z^\alpha X_n\|_s \le \frac{AH^n}{(R-s)^{Ln}} \quad \text{for any } 0 < s < R \text{ and } |\alpha| \le L \tag{3.6}$$

holds for any $n = 0, 1, 2, \ldots$.

Proof of Lemma 3.2. Let μ be as in (3.3). Since $\partial_z^\alpha X_n(z)$ ($n = 0, 1, \ldots, \mu$ and $|\alpha| \le L$) are holomorphic functions on D_R, by taking $A > 0$ and $H > 0$ sufficiently large we have the condition (3.6) for $n = 0, 1, \ldots, \mu$.

Let $n > \mu$, and suppose that (3.6) with n replaced by p is already proved for all $p < n$. Then, by (3.3), (3.5) and the induction hypothesis, we have

$$\begin{aligned}\|X_n\|_s &\le \frac{1}{c_0(q^n)^m}\Bigl[Bh^n + \sum_{j+\sigma|\alpha|\le m}\sum_{1\le k\le n} Bh^k(q^j)^{n-k} \times \frac{AH^{n-k}}{(R-s)^{L(n-k)}}\Bigr] \\ &\le \frac{AH^n}{(R-s)^{L(n-1)}c_0(q^n)^m}\Bigl[\frac{B}{A}\Bigl(\frac{h}{H}\Bigr)^n + \sum_{j+\sigma|\alpha|\le m}\sum_{1\le k\le n} B\Bigl(\frac{h}{H}\Bigr)^k(q^j)^{n-k}\Bigr],\end{aligned}$$

and so, by Lemma 2.5, we have

$$\begin{aligned}\|\partial_z^\alpha X_n\|_s \le \frac{AH^n e^{|\alpha|}(L(n-1)+1)\dots(L(n-1)+|\alpha|)}{(R-s)^{L(n-1)+|\alpha|}c_0(q^n)^m}\times \\ \times\Bigl[\frac{B}{A}\Bigl(\frac{h}{H}\Bigr)^n + \sum_{j+\sigma|\alpha|\le m}\sum_{1\le k\le n} B\Bigl(\frac{h}{H}\Bigr)^k(q^j)^{n-k}\Bigr] \qquad (3.7)\end{aligned}$$

for any $0 < s < R$. Here, we note that $n/q^{\sigma n} \to 0$ (as $n \to \infty$), and so $n/q^{\sigma n} \le c_1$ $(n = 1, 2, \dots)$ hold for some $c_1 > 1$. Since

$$(L(n-1)+1)\dots(L(n-1)+|\alpha|) \le (Ln)^{|\alpha|} \le L^{|\alpha|}(c_1q^{\sigma n})^{|\alpha|}$$

holds, by applying this to (3.7) and by using $(q^n)^{j+\sigma|\alpha|} \le (q^n)^m$ we have

$$\|\partial_z^\alpha X_n\|_s \le \frac{AH^n}{(R-s)^{Ln}} \times \frac{(eLc_1)^L}{c_0}\Bigl[\frac{B}{A}\Bigl(\frac{h}{H}\Bigr)^n + \sum_{j+\sigma|\alpha|\le m}\sum_{1\le k\le n} B\Bigl(\frac{h}{H}\Bigr)^k\Bigr].$$

Thus, if $A \ge B$ and H is sufficiently large with $H > h$, we have

$$\begin{aligned}&\frac{(eLc_1)^L}{c_0}\Bigl[\frac{B}{A}\Bigl(\frac{h}{H}\Bigr)^n + \sum_{j+\sigma|\alpha|\le m}\sum_{1\le k\le n} B\Bigl(\frac{h}{H}\Bigr)^k\Bigr] \\ &\le \frac{(eLc_1)^L}{c_0}\Bigl[\Bigl(\frac{h}{H}\Bigr)^n + \sum_{j+\sigma|\alpha|\le m} B \times \frac{h/H}{(1-h/H)}\Bigr] \le 1.\end{aligned}$$

This proves that if we take $A > 0$ and $H > 0$ sufficiently large we have the estimate (3.6). This proves Lemma 3.2. □

Thus, we have proved Proposition 3.1. □

Example 3.3. Let $A > 0$, $B > 0$, $m \in \mathbb{N}$, $j \in \mathbb{N}$, $p \in \mathbb{N}^*$ $(= \{1, 2, \dots\})$, $\alpha \in \mathbb{N}^*$, and let us consider

$$(\sigma_q)^m X = \frac{A}{1-z}\,t + B\,t^p(\sigma_q)^j\partial_z^\alpha X.$$

This equation has a unique formal power series solution and it is given by

$$\hat{X}(t,z) = \sum_{n\ge 0} AB^n \frac{q^j(q^{p+1})^j\dots(q^{(n-1)p+1})^j}{q^m(q^{p+1})^m\dots(q^{np+1})^m}\,\frac{(n\alpha)!}{(1-z)^{n\alpha+1}}\,t^{np+1}.$$

It is easy to see that $\hat{X}(t,z)$ is convergent if and only if $j \le m-1$ holds: in this case, by setting $\sigma = 1/\alpha$ we have $j + \sigma\alpha \le m$.

4. PROOF OF (1) OF THEOREM 2.3

Let us consider the same equation (3.1) under the assumption that there is an integer m_0 with $0 \le m_0 < m$ such that

$$\begin{cases} \mathrm{ord}_t(a_{j,\alpha}) \ge \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\ \mathrm{ord}_t(a_{j,\alpha}) \ge \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0 \end{cases} \tag{4.1}$$

and that $a_{m_0,0}(0,z) \ne 0$ on D_R for some $R > 0$. We set

$$C(\lambda, z) = \sum_{j=0}^{m_0} a_{j,0}(0,z)\lambda^j$$

which is a polynomial of degree m_0 in λ with holomorphic coefficients. Since the condition $a_{m_0,0}(0,z) \ne 0$ is assumed, we have a constant $c_0 > 0$ and a positive integer μ such that

$$|C(q^n, z)| \ge c_0(q^n)^{m_0} \quad \text{on } D_R \text{ for any } n \ge \mu. \tag{4.2}$$

For simplicity, we set $\Lambda = \{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^d\,;\, j + \sigma|\alpha| \le m\}$ and set $L = m/\sigma$. We have $(j,0) \in \Lambda$ for any $j = 0, 1, \ldots, m$, and if $(j,\alpha) \in \Lambda$ we have $|\alpha| \le L$. By condition (4.1), we see that:

if $j \le m_0$ and $|\alpha| = 0$, we have $a_{j,0}(t,z) = a_{j,0}(0,z) + tb_{j,0}(t,z)$,
if $m_0 < j \le m$ and $|\alpha| = 0$, we have $a_{j,0}(t,z) = t^{j-m_0}b_{j,0}(t,z)$,
if $|\alpha| > 0$, we have $a_{j,\alpha}(t,z) = t^{\max\{1, j-m_0+1\}}b_{j,\alpha}(t,z)$

for some holomorphic functions $b_{j,\alpha}(t,z)$ in a neighbourhood of $(0,0) \in \mathbb{C} \times \mathbb{C}^d$. By setting

$$\begin{cases} p_{j,0} = 1, & \text{if } j \le m_0 \text{ and } |\alpha| = 0, \\ p_{j,0} = j - m_0, & \text{if } m_0 < j \le m \text{ and } |\alpha| = 0, \\ p_{j,\alpha} = \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0 \end{cases} \tag{4.3}$$

we see that our equation (3.1) is written in the form

$$C(\sigma_q, z)X + \sum_{(j,\alpha)\in\Lambda} t^{p_{j,\alpha}} b_{j,\alpha}(t,z)(\sigma_q)^j \partial_z^\alpha X = F(t,z). \tag{4.4}$$

Since $|\alpha|/L \le 1$ holds for any $(j,\alpha) \in \Lambda$, by the definition of $p_{j,\alpha}$ $((j,\alpha) \in \Lambda)$ we have

$$1 \ge \frac{j + |\alpha|/L - m_0}{p_{j,\alpha}}, \quad (j,\alpha) \in \Lambda. \tag{4.5}$$

To prove (1) of Theorem 2.3 it is enough to show the following result.

Proposition 4.1. *Suppose the conditions* (4.2), (4.3) *and* (4.5) *hold. Then, if*

$$\hat{X}(t,z) = \sum_{n \ge 0} X_n(z) t^n \in \mathcal{O}_R[[t]]$$

is a formal solution of (4.4), *there are* $A > 0$, $H > 0$ *and* $R_1 > 0$ *such that*

$$|X_n(z)| \le AH^n q^{n(n-1)/2} \quad \text{on } D_{R_1}, \quad n = 0, 1, 2, \ldots. \tag{4.6}$$

Proof. By assumption, we can expand $b_{j,\alpha}(t,z)$ ($(j,\alpha) \in \Lambda$) and $F(t,z)$ into the forms:

$$b_{j,\alpha}(t,z) = \sum_{k\ge 0} b_{j,\alpha,k}(z) t^k \quad ((j,\alpha) \in \Lambda),$$
$$F(t,z) = \sum_{k\ge 0} F_k(z) t^k.$$

We may suppose that $R > 0$ is sufficiently small. Therefore, we may suppose $0 < R < 1$, that $b_{j,\alpha,k}(z)$ and $F_k(z)$ are all holomorphic functions on D_R, and that there are $B > 0$ and $h > 0$ such that $|b_{j,\alpha,k}(z)| \le Bh^k$ ($(j,\alpha) \in \Lambda$) and $|F_k(z)| \le Bh^k$ ($k \ge 0$) hold on D_R.

Let

$$\hat{X}(t,z) = \sum_{n=0}^{\infty} X_n(z) t^n \in \mathcal{O}_R[[t]]$$

be a formal solution of (4.4). By a calculation we have the following recurrent formulas:

$$C(q^0, z) X_0 = F_0(z)$$

and for $n \ge 1$

$$C(q^n, z) X_n = F_n(z) - \sum_{(j,\alpha)\in\Lambda} \sum_{0\le k\le n-p_{j,\alpha}} b_{j,\alpha,k}(z) (q^j)^{n-k-p_{j,\alpha}} \partial_z^\alpha X_{n-k-p_{j,\alpha}}. \tag{4.7}$$

To prove Proposition 4.1 it is enough to show the following lemma.

Lemma 4.2. *There are* $A > 0$ *and* $H > 0$ *such that the estimate*

$$\|\partial_z^\alpha X_n\|_s \le \frac{AH^n q^{n(n-1)/2}}{(R-s)^{Ln}} \quad \text{for any } 0 < s < R \text{ and } |\alpha| \le L \tag{4.8}$$

holds for any $n = 0, 1, 2, \ldots$.

Proof of Lemma 4.2. Let μ be as in (4.2). Since $\partial_z^\alpha X_n(z)$ ($n = 0, 1, \ldots, \mu$ and $|\alpha| \le L$) are holomorphic functions on D_R, by taking $A > 0$ and $H > 0$ sufficiently large we have condition (4.8) for $n = 0, 1, \ldots, \mu$.

Let $n > \mu$, and suppose that (4.8) with n replaced by p is already proved for all $p < n$. Since (4.2) is known, X_n can be expressed in the form

$$X_n = X_{n,F} + \sum_{(j,\alpha)\in\Lambda} X_{n,j,\alpha}$$

where $X_{n,F}$ and $X_{n,j,\alpha}$ ($(j,\alpha) \in \Lambda$) are defined by $C(q^n,z) X_{n,F} = F_n(z)$ and

$$C(q^n, z) X_{n,j,\alpha} = -\sum_{0\le k\le n-p_{j,\alpha}} b_{j,\alpha,k}(z) (q^j)^{n-k-p_{j,\alpha}} \partial_z^\alpha X_{n-k-p_{j,\alpha}}. \tag{4.9}$$

Then, if $H \geq h$ we have

$$\|X_{n,F}\|_s \leq \frac{Bh^n}{c_0(q^n)^{m_0}} \leq \frac{AH^n}{c_0} \times \frac{B}{A}\Big(\frac{h}{H}\Big)^{\mu}, \tag{4.10}$$

and by (4.2), (4.9) and the induction hypothesis we have

$$\begin{aligned}\|X_{n,j,\alpha}\|_s \leq \frac{1}{c_0(q^n)^{m_0}} &\sum_{0\leq k\leq n-p_{j,\alpha}} Bh^k q^{(n-k-p_{j,\alpha})j} \\ &\times \frac{AH^{n-k-p_{j,\alpha}} q^{(n-k-p_{j,\alpha})(n-k-p_{j,\alpha}-1)/2}}{(R-s)^{L(n-k-p_{j,\alpha})}}.\end{aligned} \tag{4.11}$$

We recall that by (4.5) we have $p_{j,\alpha} - j + m_0 \geq |\alpha|/L$ and so

$$\begin{aligned}&-nm_0 + (n-k-p_{j,\alpha})j + (n-k-p_{j,\alpha})(n-k-p_{j,\alpha}-1)/2 \\ &\quad = n(n-1)/2 - (k+p_{j,\alpha}-j+m_0)(n-k-p_{j,\alpha}) \\ &\qquad - (k+p_{j,\alpha})(k+p_{j,\alpha}-1)/2 - m_0(k+p_{j,\alpha}) \\ &\quad \leq n(n-1)/2 - (p_{j,\alpha}-j+m_0)(n-k-p_{j,\alpha}) \\ &\quad \leq n(n-1)/2 - (|\alpha|/L)(n-k-p_{j,\alpha}).\end{aligned}$$

By applying this to (4.11), we have

$$\|X_{n,j,\alpha}\|_s \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)^{L(n-k-p_{j,\alpha})}} \frac{1}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \sum_{0\leq k\leq n-p_{j,\alpha}} B\Big(\frac{h}{H}\Big)^k \frac{1}{H^{p_{j,\alpha}}},$$

and if $H \geq 2h$ holds, we have

$$\|X_{n,j,\alpha}\|_s \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)^{L(n-k-p_{j,\alpha})}} \frac{1}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \frac{2B}{H^{p_{j,\alpha}}} \tag{4.12}$$

for any $0 < s < R$.

Now, let us apply Lemma 2.5 to these estimates (4.10) and (4.12). Namely, for any $|\alpha| \leq L$, we have

$$\|\partial_z^{\alpha} X_{n,F}\|_s \leq \frac{AH^n e^{|\alpha|}|\alpha|!}{c_0(R-s)^{|\alpha|}} \times \frac{B}{A}\Big(\frac{h}{H}\Big)^{\mu} \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)^{Ln}} \times \frac{e^L L! B}{A}\Big(\frac{h}{H}\Big)^{\mu} \tag{4.13}$$

and

$$\begin{aligned}\|\partial_z^{\alpha} X_{n,j,\alpha}\|_s \leq &\frac{AH^n q^{n(n-1)/2}}{c_0(R-s)^{L(n-k-p_{j,\alpha})+|\alpha|}} \times \frac{2B}{H^{p_{j,\alpha}}} \times \\ &\times \frac{e^{|\alpha|}(L(n-k-p_{j,\alpha})+1)\dots(L(n-k-p_{j,\alpha})+|\alpha|)}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}}.\end{aligned}$$

Since $(n+1)/(q^{1/L})^n \to 0$ (as $n \to \infty$) holds, we have the estimate $(n+1) \le c_1 (q^{1/L})^n$ $(n = 0, 1, 2, \dots)$ for some $c_1 > 0$. Then,

$$\begin{aligned} &\frac{e^{|\alpha|}(L(n-k-p_{j,\alpha})+1)\dots(L(n-k-p_{j,\alpha})+|\alpha|)}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \\ &\le \frac{e^{|\alpha|}(L(n-k-p_{j,\alpha}+1))^{|\alpha|}}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \le (eLc_1)^{|\alpha|}, \end{aligned}$$

and so we have

$$\|\partial_z^\alpha X_{n,j,\alpha}\|_s \le \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)^{L(n-k-p_{j,\alpha})+|\alpha|}} \times \frac{2B}{H^{p_{j,\alpha}}}(eLc_1)^{|\alpha|} \tag{4.14}$$

for any $0 < s < R$.

By (4.13) and (4.14), we have

$$\|\partial_z^\alpha X_n\|_s \le \frac{AH^n q^{n(n-1)/2}}{(R-s)^{Ln}} \times C_1 \quad \text{for any } 0 < s < R$$

with

$$C_1 = \frac{e^L L! B}{c_0 A}\Big(\frac{h}{H}\Big)^\mu + \sum_{(j,\alpha)\in\Lambda} \frac{2B}{c_0 H^{p_{j,\alpha}}}(eLc_1)^{|\alpha|}.$$

Thus, if $C_1 \le 1$ we can obtain the result (4.8). We note that if we take $A > 0$ and $H > 0$ sufficiently large, we have the condition $C_1 \le 1$. This completes the proof of Lemma 4.2. □

Thus, by (4.8) $(n = 0, 1, 2, \dots)$, we have the condition (4.6). This proves Proposition 4.1. □

Example 4.3. Let $A > 0$, $B > 0$, $p \in \mathbb{N}^*$ and $\alpha > 0$. The following equation is a particular case of (4.4) with $m_0 = 0$ and $m = 1$:

$$X = \frac{A}{1-z}t + t\sigma_q X + Bt^p \partial_z^\alpha X.$$

This equation has a unique formal power series solution and we can apply Proposition 4.1 to this case. In the case $p = 1$ the formal solution is given by

$$\hat{X}(t,z) = \frac{A}{1-z}t + \sum_{n\ge 2}\Big((q^1 + B\partial_z^\alpha)\dots(q^{n-1} + B\partial_z^\alpha)\frac{A}{1-z}\Big)t^n.$$

Since $q > 1$ holds, we have $(n\alpha)^\alpha \le cq^n$ $(n = 1, 2, \dots)$ for some $c > 0$. We have the following majorant relation:

$$\hat{X}(t,z) \ll \sum_{n\ge 1} \frac{A(1+Bc)^{n-1} q^{n(n-1)/2}}{(1-z)^{1+(n-1)\alpha}} t^n.$$

5. PROOF OF (2) OF THEOREM 2.3

We will consider the same equation

$$C(\sigma_q, z)X + \sum_{(j,\alpha)\in\Lambda} t^{p_{j,\alpha}} b_{j,\alpha}(t,z)(\sigma_q)^j \partial_z^\alpha X = F(t,z) \tag{5.1}$$

as (4.4) under the same conditions as in Section 4. In addition, as is supposed in Theorem 2.3, we assume here that $0 \le m_0 < m$, $a_{m_0,0}(0,0) \neq 0$, $b_{m,0}(0,0) \neq 0$, and

$$b_{j,\alpha}(0,z) \equiv 0 \quad \text{for } m_0 \le j < m \text{ and } |\alpha| > 0. \tag{5.2}$$

The last condition is equivalent to the condition that $\mathrm{ord}_t(a_{j,\alpha}) \ge j - m_0 + 2$ if $|\alpha| > 0$ and $m_0 \le j < m$. We set

$$P(\tau, z) = \sum_{m_0 < j \le m} \frac{b_{j,0}(0,z)}{q^{j(j-1)/2}} \tau^{j-m_0} + \frac{a_{m_0,0}(0,z)}{q^{m_0(m_0-1)/2}} \tag{5.3}$$

which is a polynomial of degree $m - m_0$ with respect to τ. Since $b_{m,0}(0,0) \neq 0$ and $a_{m_0,0}(0,0) \neq 0$ are supposed, the equation $P(\tau, 0) = 0$ in τ has $m - m_0$ non-zero roots. We denote them by $\tau_1, \dots, \tau_{m-m_0}$. We set

$$S = \bigcup_{i=1}^{m-m_0} \{t\tau_i \,;\, t > 0\}.$$

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\theta > 0$, we write $S_\theta(\lambda) = \{\xi \in \mathbb{C} \setminus \{0\} \,;\, |\arg \xi - \arg \lambda| < \theta\}$.

Lemma 5.1. *For any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ we can find $c > 0$, $\theta > 0$, $r > 0$ and $R > 0$ such that $|P(\xi, z)| \ge c(|\xi| + 1)^{m-m_0}$ holds on $(S_\theta(\lambda) \cup D_r) \times D_R$.*

From now, we take any $\lambda \in \mathbb{C} \setminus (\{0\} \cup S)$ and fix it. Take also $c > 0$, $\theta > 0$, $r > 0$ and $R > 0$ so that Lemma 5.1 holds, and fix them. We may suppose that r and R are sufficiently small. Set $\Omega = (S_\theta(\lambda) \cup D_r) \times D_R$. Under these settings, we take a sufficiently large $\mu \in \mathbb{N}^*$ so that

$$\beta = \sum_{j<m_0} \frac{\|a_{j,0}(0)\|_R}{c q^{m_0(m_0-1)/2} (q^{m_0-j})^\mu} < 1. \tag{5.4}$$

This is possible, because $(q^{m_0-j})^\mu \to \infty$ (as $\mu \to \infty$).

5.1. FORMAL q-BOREL TRANSFORMS

Let us recall the definition of formal q-Borel transforms introduced by Zhang [14]. For a formal series

$$\hat{V}(t,z) = \sum_{n\ge0} V_n(z) t^n \in \mathcal{O}_R[[t]],$$

the formal q-Borel transform $\hat{\mathcal{B}}_{q;1}[\hat{V}](\xi, z)$ of $\hat{V}(t, z)$ is defined by

$$\hat{\mathcal{B}}_{q;1}[\hat{V}](\xi, z) = \sum_{n \geq 0} \frac{V_n(z)}{q^{n(n-1)/2}} \xi^n \in \mathcal{O}_R[[\xi]].$$

The following property is known (see Statement 1.3.3 in [14]).

Lemma 5.2. *Let* $\hat{a}(t, z) = \sum_{k \geq 0} a_k(z) t^k \in \mathcal{O}_R[[t]]$, *and let* $\hat{V}(t, z) \in \mathcal{O}_R[[t]]$. *Set* $v(\xi, z) = \hat{\mathcal{B}}_{q;1}[V](\xi, z)$. *Then, for any* $m \in \mathbb{N}$ *we have*

$$\hat{\mathcal{B}}_{q;1}\big[\hat{a} \times (\sigma_q)^m \hat{V}\big](\xi, z) = \sum_{k \geq 0} \frac{a_k(z)}{q^{k(k-1)/2}} \xi^k v(q^{m-k}\xi, z).$$

Corollary 5.3. *For any* $m \in \mathbb{N}^*$ *and* $k \in \mathbb{N}^*$, *we have*

(1) $\hat{\mathcal{B}}_{q;1}[t^m (\sigma_q)^m \hat{V}](\xi, z) = \dfrac{\xi^m}{q^{m(m-1)/2}} v(\xi, z)$,

(2) $\hat{\mathcal{B}}_{q;1}[t^{m+k} (\sigma_q)^m \hat{V}](\xi, z) = \dfrac{\xi^{m+k}}{q^{(m+k)(m+k-1)/2}} (\sigma_{q^{-1}})^k v(\xi, z)$,

(3) $\hat{\mathcal{B}}_{q;1}[t^m (\sigma_q)^{m+k} \hat{V}](\xi, z) = \dfrac{\xi^m}{q^{m(m-1)/2}} (\sigma_q)^k v(\xi, z)$.

5.2. EQUATION IN THE q-BOREL PLANE

Let

$$\hat{X}(t, z) = \sum_{n \geq 0} X_n(z) t^k \in \mathcal{O}_R[[t]]$$

be a formal solution of (5.1), and let μ be as in (5.4). We set

$$X^*(t, z) = \sum_{n \geq \mu} X_n(z) t^n.$$

Then, $X^*(t, z)$ is a formal solution of the equation

$$C(\sigma_q, z) X^* + \sum_{(j,\alpha) \in \Lambda} t^{p_{j,\alpha}} b_{j,\alpha}(t, z) (\sigma_q)^j \partial_z^\alpha X^* = F^*(t, z) \tag{5.5}$$

for some holomorphic function $F^*(t, z)$ on $D_r \times D_R$ with $\mathrm{ord}_t(F^*) \geq \mu$.

Lemma 5.4. *By multiplying equation* (5.5) *by* t^{m_0} *we have the expression*

$$\begin{aligned} &\sum_{j \leq m_0} t^{m_0} a_{j,0}(0, z) (\sigma_q)^j X^* + \sum_{m_0 < j \leq m} t^j b_{j,0}(0, z) (\sigma_q)^j X^* \\ &+ \sum_{j \leq m_0} t^{m_0+1} b^*_{j,0}(t, z) (\sigma_q)^j X^* + \sum_{m_0 < j \leq m} t^{j+1} b^*_{j,0}(t, z) (\sigma_q)^j X^* \\ &+ \sum_{j < m_0, |\alpha| > 0} t^{m_0+1} b^*_{j,\alpha}(t, z) (\sigma_q)^j \partial_z^\alpha X^* \\ &+ \sum_{m_0 \leq j < m, |\alpha| > 0} t^{j+2} b^*_{j,\alpha}(t, z) (\sigma_q)^j \partial_z^\alpha X^* = t^{m_0} F^*(t, z) \end{aligned} \tag{5.6}$$

for some holomorphic functions $b^*_{j,\alpha}(t, z)$ $((j, \alpha) \in \Lambda)$ *on* $D_r \times D_R$.

Proof. By the definition of $p_{j,\alpha}$, we have

$$\sum_{j\le m_0} t^{m_0} a_{j,0}(0,z)(\sigma_q)^j X^* + \sum_{j\le m_0} t^{m_0+1} b_{j,0}(t,z)(\sigma_q)^j X^*$$

$$+ \sum_{m_0<j\le m} t^j b_{j,0}(t,z)(\sigma_q)^j X^*$$

$$+ \sum_{(j,\alpha)\in\Lambda,|\alpha|>0} t^{\max\{1+m_0,j+1\}} b_{j,\alpha}(t,z)(\sigma_q)^j \partial_z^\alpha X^* = t^{m_0} F^*(t,z).$$

Therefore, by setting

$$\begin{cases} b^*_{j,0}(t,z) = (b_{j,0}(t,z) - b_{j,0}(0,z))/t, & \text{if } m_0 < j \le m, \\ b^*_{j,\alpha}(t,z) = b_{j,\alpha}(t,z)/t, & \text{if } m_0 \le j < m \text{ and } |\alpha| > 0, \\ b^*_{j,\alpha}(t,z) = b_{j,\alpha}(t,z), & \text{in the other case} \end{cases}$$

we obtain (5.6). In the case $|\alpha| > 0$ and $m_0 \le j < m$, we have used condition (5.2). $\square$

Now, let us apply formal q-Borel transform to equation (5.6). Under the setting

$$u(\xi,z) = \hat{\mathcal{B}}_{q;1}[X^*](\xi,z), \quad F^*(t,z) = \sum_{n\ge\mu} F^*_n(z)t^n,$$

$$t^{m_0+1} b^*_{j,0}(t,z) = \sum_{k\ge m_0+1} c_{j,0,k}(z)t^k \quad (|\alpha| = 0 \text{ and } j \le m_0),$$

$$t^{j+1} b^*_{j,0}(t,z) = \sum_{k\ge j+1} c_{j,0,k}(z)t^k \quad (|\alpha| = 0 \text{ and } m_0 \le j \le m),$$

$$t^{m_0+1} b^*_{j,\alpha}(t,z) = \sum_{k\ge m_0+1} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0 \text{ and } j < m_0),$$

$$t^{j+2} b^*_{j,\alpha}(t,z) = \sum_{k\ge j+2} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0 \text{ and } m_0 \le j < m)$$

we have the equation

$$\begin{aligned}
&\sum_{j\le m_0}\frac{a_{j,0}(0,z)}{q^{m_0(m_0-1)/2}}\xi^{m_0}(\sigma_{q^{-1}})^{m_0-j}u+\sum_{m_0<j\le m}\frac{b_{j,0}(0,z)}{q^{j(j-1)/2}}\xi^{j}u\\
&\quad+\sum_{j\le m_0}\sum_{k\ge m_0+1}\frac{c_{j,0,k}(z)}{q^{k(k-1)/2}}\xi^{k}(\sigma_{q^{-1}})^{k-j}u+\sum_{m_0<j\le m}\sum_{k\ge j+1}\frac{c_{j,0,k}(z)}{q^{k(k-1)/2}}\xi^{k}(\sigma_{q^{-1}})^{k-j}u\\
&\quad+\sum_{j<m_0,|\alpha|>0}\sum_{k\ge m_0+1}\frac{c_{j,\alpha,k}(z)}{q^{k(k-1)/2}}\xi^{k}(\sigma_{q^{-1}})^{k-j}\partial_z^{\alpha}u \qquad (5.7)\\
&\quad+\sum_{m_0\le j<m,|\alpha|>0}\sum_{k\ge j+2}\frac{c_{j,\alpha,k}(z)}{q^{k(k-1)/2}}\xi^{k}(\sigma_{q^{-1}})^{k-j}\partial_z^{\alpha}u\\
&=\sum_{n\ge\mu}\frac{F_n^*(z)}{q^{(n+m_0)(n+m_0-1)/2}}\xi^{n+m_0}.
\end{aligned}$$

Therefore, by canceling ξ^{m_0} from both sides of this equation, and then by using $P(\xi,z)$ in (5.3) and the notations

$$a^0_{m_0-i}(z)=\frac{a_{m_0-i,0}(0,z)}{q^{m_0(m_0-1)/2}} \quad (i=1,\dots,m_0),$$

$$c^0_{j,0,k}(z)=\frac{c_{j,0,k+m_0}(z)}{q^{m_0(m_0-1)/2}q^{m_0k}} \quad (j\le m_0 \text{ and } k\ge 1),$$

$$c^0_{j,0,k}(z)=\frac{c_{j,0,k+j}(z)}{q^{j(j-1)/2}q^{jk}} \quad (m_0<j\le m \text{ and } k\ge 1),$$

$$c^0_{j,\alpha,k}(z)=\frac{c_{j,\alpha,k+m_0}(z)}{q^{m_0(m_0-1)/2}q^{m_0k}} \quad (|\alpha|>0,\ j<m_0 \text{ and } k\ge 1),$$

$$c^0_{j,\alpha,k}(z)=\frac{c_{j,\alpha,k+j+1}(z)}{q^{j(j+1)/2}q^{(j+1)k}} \quad (|\alpha|>0,\ m_0\le j<m \text{ and } k\ge 1),$$

$$f_n(z)=\frac{F_n^*(z)}{q^{m_0(m_0-1)/2}q^{m_0n}}, \quad n\ge\mu,$$

we can reduce our equation (5.7) into the form

$$\begin{aligned} P(\xi,z)u &+ \sum_{i=1}^{m_0} a^0_{m_0-i}(z)(\sigma_{q^{-1}})^i u \\ &+ \sum_{j\le m_0}\sum_{k\ge 1}\frac{c^0_{j,0,k}(z)}{q^{k(k-1)/2}}\xi^k(\sigma_{q^{-1}})^{k+(m_0-j)}u \\ &+ \sum_{m_0<j\le m}\sum_{k\ge 1}\frac{c^0_{j,0,k}(z)}{q^{k(k-1)/2}}\xi^{k+(j-m_0)}(\sigma_{q^{-1}})^k u \\ &+ \sum_{0\le j<m_0,|\alpha|>0}\sum_{k\ge 1}\frac{c^0_{j,\alpha,k}(z)}{q^{k(k-1)/2}}\xi^k(\sigma_{q^{-1}})^{k+(m_0-j)}\partial_z^\alpha u \\ &+ \sum_{m_0\le j<m,|\alpha|>0}\sum_{k\ge 1}\frac{c^0_{j,\alpha,k}(z)}{q^{k(k-1)/2}}\xi^{k+(j+1-m_0)}(\sigma_{q^{-1}})^{k+1}\partial_z^\alpha u \\ =& \sum_{n\ge\mu}\frac{f_n(z)}{q^{n(n-1)/2}}\xi^n. \end{aligned} \tag{5.8}$$

The meaning of this equation is as follows:

Lemma 5.5.

(1) *By taking $r>0$ and $R>0$ sufficiently small, we may assume that $u(\xi,z)=\hat{\mathcal{B}}_{q;1}[X^*](\xi,z)$ is a holomorphic function on $D_r\times D_R$.*
(2) *Each sum in (5.8) is a holomorphic function on $D_r\times D_R$ in the following sense: if $c_k(z)\in\mathcal{O}_R$ $(k\ge 1)$ satisfy the estimates $|c_k(z)|\le Ch^k$ on D_R $(k\ge 1)$ for some $C>0$ and $h>0$, the sum*

$$\sum_{k\ge 1}\frac{c_k(z)}{q^{k(k-1)/2}}\xi^{k+i}(\sigma_{q^{-1}})^{k+e}u \quad (\text{with } i\in\mathbb{N},\ e\in\mathbb{N})$$

is a holomorphic function on $D_{r'}\times D_R$ with $r'=rq^{1+e}$.

Proof. By Proposition 4.1, we have the estimates $\|X_n\|_R\le AH^nq^{n(n-1)/2}$ $(n=0,1,2,\dots)$ for some $A>0$ and $H>0$. By taking $0<r<1/H$ we have the result (1). We note that

$$\sum_{k\ge 1}\frac{|c_k(z)|}{q^{k(k-1)/2}}|\xi|^{k+i}|(\sigma_{q^{-1}})^{k+e}u|\le C(|\xi|)W(|\xi|),\quad z\in D_R,$$

where

$$C(\xi)=\sum_{k\ge 1}\frac{Ch^k}{q^{k(k-1)/2}}\xi^{k+i}\quad\text{and}\quad W(\xi)=\sum_{n\ge\mu}AH^n\Big(\frac{\xi}{q^{1+e}}\Big)^n.$$

Since $C(\xi)$ is an entire function in ξ and $W(\xi)$ is a holomorphic function on $\{\xi\,;\,|\xi|<q^{1+e}/H\}$, we have the result (2). □

5.3. HOLOMORPHIC EXTENSION OF $u(\xi, z)$

As is seen above, the formal q-Borel transform $u(\xi, z) = \hat{\mathcal{B}}_{q;1}[X^*](\xi, z)$ is a holomorphic solution of (5.8) on $D_r \times D_R$. The following is the main result on equation (5.8).

Proposition 5.6. *The local solution $u(\xi, z)$ has a holomorphic extension $u^*(\xi, z)$ to a domain $(S_\theta(\lambda) \cup D_{r_1}) \times D_R$ for some $r_1 > 0$ that satisfies the following properties:*

(1) *$u^*(\xi, z)$ is also a solution of* (5.8).
(2) *For any $0 < R_1 < R$ there are $A > 0$ and $H > 0$ such that*

$$|u(\lambda q^m, z)| \leq AH^m q^{m(m+1)/2} \quad \text{on } D_{R_1} \text{ for any } m = 0, 1, 2, \ldots.$$

The proof of this result will be given in Section 6. We will admit this result for a while.

5.4. q-ANALOGUE OF THE SUMMABILITY OF $\hat{X}(t, z)$

Now, let us return to the situation in Theorem 2.3. Let $u^*(\xi, z)$ be the holomorphic extension of $u(\xi, z)$ to the domain $\Omega_1 = (S_\theta(\lambda) \cup D_{r_1}) \times D_R$. Let $\vartheta_q(x)$ be the Jacobi theta function defined by

$$\vartheta_q(x) = \sum_{m \in \mathbb{Z}} \frac{x^m}{q^{m(m-1)/2}}$$

which is a holomorphic function on $\mathbb{C} \setminus \{0\}$. We set

$$W^*(t, z) = \mathcal{L}^\lambda_{q;1}[u^*](t, z) = \sum_{n \in \mathbb{Z}} \frac{u^*(\lambda q^n, z)}{\vartheta_q(\lambda q^n / t)}$$

which is the q-Laplace transform of $u^*(\xi, z)$ in the direction λ (introduced by Ramis-Zhang [11]). Then, by combining the above Proposition 5.6 with Théorème 1.3.2 in [15] (or Proposition 1 in [4]) we get the following theorem.

Theorem 5.7.

(1) *$W^*(t, z)$ is a holomorphic solution of equation* (5.5) *on $(D_{r_2} \setminus (\{0\} \cup \mathscr{Z}_\lambda)) \times D_{R_1}$ for some $r_2 > 0$.*
(2) *Moreover, there are $M_1 > 0$ and $H_1 > 0$ such that the following estimate holds*

$$\left| W^*(t, z) - \sum_{n=\mu}^{N-1} X_n(z) t^n \right| \leq \frac{M_1 {H_1}^N}{\epsilon} q^{N(N-1)/2} |t|^N \quad \text{for } t \in U_\epsilon \text{ and } z \in D_{R_1}$$

for any sufficiently small $\epsilon > 0$ and any $N \geq \mu$, where $U_\epsilon = D_{r_2} \setminus (\{0\} \cup \mathscr{Z}_{\lambda,\epsilon})$.

By setting

$$W(t, z) = \sum_{n=0}^{\mu-1} X_n(z) t^n + W^*(t, z)$$

we have a true holomorphic solution of (2.1) which admits $\hat{X}(t, z)$ as a q-Gevrey asymptotic expansion of order 1. This proves (2) of Theorem 2.3.

6. PROOF OF PROPOSITION 5.6

Let $\lambda \in \mathbb{C} \setminus \{0\}$, $\theta > 0$, $r > 0$, and $R > 0$, set $\Omega = (D_r \cup S_\theta(\lambda)) \times D_R \subset \mathbb{C}_\xi \times \mathbb{C}_z^d$, and set $N = m - m_0$. In this section, as a model of (5.8) we will consider the equation

$$\begin{aligned} &P(\xi,z)u + \sum_{i=1}^{K} a_i(z)(\sigma_{q^{-1}})^i u + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \sum_{k\geq 1} \frac{c_{i,j,\alpha,k}(z)}{q^{k(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_z^\alpha u \\ &= \sum_{n\geq\mu} \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n \end{aligned} \tag{6.1}$$

on Ω. We suppose that $0 < R < 1$ and the following conditions (c_1)–(c_5) hold:

(c_1) $P(\xi,z) = \xi^N + c_1(z)\xi^{N-1} + \ldots + c_N(z) \in \mathcal{O}_R[\xi]$ for some $N \in \mathbb{N}$. Moreover, $|P(\xi,z)| \geq c(|\xi|+1)^N$ holds on Ω for some $c > 0$.

(c_2) K and μ are positive integers, and Λ^* is a finite subset of $\mathbb{N} \times \{\alpha \in \mathbb{N}^d\,;\, |\alpha| \leq L\}$ (where $L \in \mathbb{N}^*$).

(c_3) $e_{j,\alpha}$ $((j,\alpha) \in \Lambda^*)$ are integers satisfying

$$\begin{cases} e_{j,\alpha} \geq 0, & \text{if } |\alpha| = 0, \\ e_{j,\alpha} \geq 1, & \text{if } |\alpha| > 0. \end{cases}$$

(c_4) $a_i(z) \in \mathcal{O}_R$ $(i = 1,\ldots,K)$ and satisfy

$$\beta = \sum_{i=1}^{K} \frac{\|a_i\|_R}{c(q^i)^\mu} < 1 \quad \text{(this corresponds to (5.4)).}$$

(c_5) $c_{i,j,\alpha,k}(z) \in \mathcal{O}_R$ $(0 \leq i \leq N$, $(j,\alpha) \in \Lambda^*$ and $k \geq 1)$ and $f_n(z) \in \mathcal{O}_R$ $(n \geq \mu)$. Moreover, there are $B > 0$ and $h > 0$ such that $\|c_{i,j,\alpha,k}\|_R \leq Bh^k$ $(0 \leq i \leq N$, $(j,\alpha) \in \Lambda^*$, $k \geq 1)$ and $\|f_n\|_R \leq Bh^n$ $(n \geq \mu)$ hold.

Then, we have the following result which yields Proposition 5.6.

Proposition 6.1.

(1) *Equation* (6.1) *has a unique formal solution of the form* $\hat{u}(\xi,z) \in \xi^\mu \times \mathcal{O}_R[[\xi]]$.

(2) *Equation* (6.1) *has a unique holomorphic solution* $u(\xi,z)$ *on* Ω. *Moreover, for any* $0 < R_1 < R$ *there are* $A_0 > 0$ *and* $H_0 > 0$ *such that*

$$|u(\lambda q^m, z)| \leq A_0 {H_0}^m q^{m(m+1)/2} \quad \text{on } D_{R_1} \text{ for any } m = 0,1,2,\ldots. \tag{6.2}$$

The part (1) is verified by a simple calculation and the following lemma:

Lemma 6.2. *For any* $n \geq \mu$ *and* $g_n(z) \in \mathcal{O}_R$, *the equation*

$$P(0,z)w_n + \sum_{i=1}^{K} a_i(z) \frac{w_n}{(q^i)^n} = g_n(z)$$

has a unique solution $w_n(z) \in \mathcal{O}_R$.

Proof. Since $|P(0,z)| \geq c$ holds on D_R, by the assumption (c_4) we have

$$\left| P(0,z) + \sum_{i=1}^{K} \frac{a_i(z)}{(q^i)^n} \right| \geq |P(0,z)| - \sum_{i=1}^{K} \frac{\|a_i\|_R}{(q^i)^n} \geq c(1-\beta) > 0,$$

and so we have the result. □

The proof of the part (2) will be done in Subsections 6.1–6.3.

6.1. ON EQUATION $\mathscr{L}w = g$

We set

$$\mathscr{L} = P(\xi,z) + \sum_{i=1}^{K} a_i(z)(\sigma_{q^{-1}})^i$$

and consider the equation

$$\mathscr{L}w = g(\xi,z) \quad \text{on } \Omega. \tag{6.3}$$

We denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω.

Lemma 6.3.

(1) *Let $g(\xi,z) \in \mathcal{O}(\Omega)$. If $|g(\xi,z)| \leq A|\xi|^b$ holds on Ω for some $A > 0$ and $b \geq \mu$, equation (6.3) has a unique holomorphic solution $w(\xi,z) \in \mathcal{O}(\Omega)$ satisfying*

$$|w(\xi,z)| \leq \frac{A|\xi|^b}{c(1-\beta)(|\xi|+1)^N} \quad \text{on } \Omega. \tag{6.4}$$

(2) *Let $g(\xi,z) \in \mathcal{O}(\Omega)$. If it satisfies*

$$\|g(\xi)\|_s \leq \frac{A|\xi|^b}{(R-s)^a} \quad \text{on } D_r \cup S_\theta(\lambda) \text{ for any } 0 < s < R$$

for some $A > 0$, $a \geq 0$ and $b \geq \mu$, equation (6.3) has a unique holomorphic solution $w(\xi,z) \in \mathcal{O}(\Omega)$ satisfying

$$\|w(\xi)\|_s \leq \frac{1}{c(1-\beta)} \frac{A|\xi|^b}{(R-s)^a(|\xi|+1)^N} \quad \text{on } D_r \cup S_\theta(\lambda) \text{ for any } 0 < s < R.$$

Proof. Let us show (1). We construct a solution in the form

$$w(\xi,z) = \sum_{n\geq 0} w_n(\xi,z), \tag{6.5}$$

where $w_n(\xi,z)$ $(n = 0,1,2,\ldots)$ are solutions of the following recurrent formulas:

$$P(\xi,z)w_0 = g(\xi,z) \tag{6.6}$$

and for $n \geq 1$

$$P(\xi,z)w_n = -\sum_{1\leq i\leq K} a_i(z)(\sigma_{q^{-1}})^i w_{n-1}. \tag{6.7}$$

Since $|P(\xi, z)| \geq c(|\xi|+1)^N$ on Ω is supposed, by (6.6) and (6.7) we can uniquely determine $w_n(\xi, z) \in \mathcal{O}(\Omega)$ $(n = 0, 1, 2, \dots)$ inductively on n.

By (6.6) and the assumption, we have

$$|w_0(\xi, z)| \leq \frac{A|\xi|^b}{c(|\xi|+1)^N} \quad \text{on } \Omega.$$

Then, we have

$$\begin{aligned}\Big|\sum_{1\leq i\leq K} a_i(z)(\sigma_{q^{-1}})^i w_0\Big| &\leq \sum_{1\leq i\leq K} \|a_i\|_R \times |w_0(\xi/q^i, z)| \\ &\leq \sum_{1\leq i\leq K} \|a_i\|_R \times \frac{A|\xi/q^i|^b}{c(|\xi/q^i|+1)^N} \leq \sum_{1\leq i\leq K} \frac{\|a_i\|_R}{c(q^i)^b} \times A|\xi|^b \leq \beta A|\xi|^b.\end{aligned}$$

Therefore, by (6.7) with $n = 1$, we have the estimate

$$|w_1(\xi, z)| \leq \frac{\beta A|\xi|^b}{c(|\xi|+1)^N} \quad \text{on } \Omega.$$

By repeating the same argument we have the estimates

$$|w_n(\xi, z)| \leq \frac{\beta^n A|\xi|^b}{c(|\xi|+1)^N} \quad \text{on } \Omega, \quad n = 0, 1, 2, \dots. \tag{6.8}$$

Thus, we can see that the formal solution $w(\xi, z)$ in (6.5) is convergent and it defines a holomorphic solution of (6.3) on Ω. The estimate (6.4) is clear from the estimates (6.8).

As is seen in (1) of Proposition 6.1, it is clear that equation (6.3) has a unique formal solution $\hat{w}(t, z) \in \xi^\mu \times \mathcal{O}_R[[\xi]]$. This shows the uniqueness of the solution in $\mathcal{O}(\Omega)$.

Thus, part (1) is proved. The result (2) is a consequence of (1). □

6.2. ON EQUATION (6.1)

Next, let us solve equation (6.1), that is,

$$\mathscr{L}u + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \sum_{k\geq 1} \frac{c_{i,j,\alpha,k}(z)}{q^{k(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_z^\alpha u = \sum_{n\geq\mu} \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n$$

on Ω. To do so, we set the formal solution $u(\xi, z)$ in the form

$$u(\xi, z) = \sum_{n\geq\mu} u_n(\xi, z)$$

and we solve the following recurrent formulas:

$$\mathscr{L}u_\mu = \frac{f_\mu(z)}{q^{\mu(\mu-1)/2}} \xi^\mu \tag{6.9}$$

and for $n \geq \mu + 1$

$$\mathscr{L} u_n = \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n - \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \sum_{1\leq k\leq n-\mu} \frac{c_{i,j,\alpha,k}(z)}{q^{k(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_z^\alpha u_{n-k}. \quad (6.10)$$

Lemma 6.4. *We have a unique solution $u_n(\xi, z) \in \mathcal{O}(\Omega)$ $(n \geq \mu)$ of the system* (6.9) *and* (6.10) *that satisfies the following: there are $A > 0$ and $H > 0$ such that*

$$\begin{aligned} &\|\partial_z^\alpha u_n(\xi)\|_s \leq \tfrac{AH^n n^{|\alpha|}}{q^{n(n-1)/2}(R-s)^{Ln}} |\xi|^n \quad \text{on } D_r \cup S_\theta(\lambda) \\ &\text{for any } 0 < s < R \text{ and any } |\alpha| \leq L \end{aligned} \quad (6.11)$$

holds for any $n \geq \mu$.

Proof. Since $\|f_\mu\|_R \leq Bh^\mu$ is supposed, by applying (1) of Lemma 6.3 to equation (6.9) we have a unique solution $u_\mu(\xi, z) \in \mathcal{O}(\Omega)$ satisfying the estimate

$$|u_\mu(\xi, z)| \leq \frac{1}{c(1-\beta)(|\xi|+1)^N} \times \frac{Bh^\mu |\xi|^\mu}{q^{\mu(\mu-1)/2}} \leq \frac{1}{c(1-\beta)} \times \frac{Bh^\mu |\xi|^\mu}{q^{\mu(\mu-1)/2}} \quad \text{on } \Omega.$$

By applying Lemma 2.5 to this estimate and by using the condition $0 < R < 1$ we have

$$\begin{aligned} \|\partial_z^\alpha u_\mu(\xi)\|_s &\leq \frac{1}{c(1-\beta)} \times \frac{Bh^\mu |\xi|^\mu}{q^{\mu(\mu-1)/2}} \frac{|\alpha|! e^{|\alpha|}}{(R-s)^{|\alpha|}} \\ &\leq \frac{L! e^L}{c(1-\beta)} \times \frac{Bh^\mu |\xi|^\mu}{q^{\mu(\mu-1)/2}(R-s)^L} \quad \text{on } D_r \cup S_\theta(\lambda) \end{aligned}$$

for any $0 < s < R$ and $|\alpha| \leq L$. Hence, if we take $A > 0$ and $H > 0$ so that

$$AH^\mu \geq \frac{L! e^L}{c(1-\beta)} \times Bh^\mu, \quad (6.12)$$

by the condition $\mu \geq 1$ we have the estimate (6.11) for $n = \mu$. Let us show the general case by induction on n.

Let $n \geq \mu + 1$, and suppose that we already have $u_p(\xi, z) \in \mathcal{O}(\Omega)$ $(\mu \leq p < n)$ which satisfy estimate (6.11) with n replaced by p for all $\mu \leq p < n$. We set

$$g_n(\xi, z) = \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n - \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \sum_{1\leq k\leq n-\mu} \frac{c_{i,j,\alpha,k}(z)}{q^{k(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_z^\alpha u_{n-k}.$$

Then our equation (6.10) is written as $\mathscr{L} u_n = g_n(\xi, z)$. By assumption (c_5) and the induction hypothesis, we can see that $g_n(\xi, z) \in \mathcal{O}(\Omega)$ is known and it satisfies the estimate

$$\begin{aligned} \|g_n(\xi)\|_s \leq \frac{Bh^n}{q^{n(n-1)/2}} |\xi|^n &+ \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \sum_{1\leq k\leq n-\mu} \frac{Bh^k}{q^{k(k-1)/2}} |\xi|^{k+i} \times \\ &\times \frac{AH^{n-k}(n-k)^{|\alpha|}}{q^{(n-k)(n-k-1)/2}(R-s)^{L(n-k)}} \left(\frac{|\xi|}{q^{k+e_{j,\alpha}}}\right)^{n-k} \end{aligned} \quad (6.13)$$

on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$. Since $0 < R < 1$ is supposed and

$$\frac{n(n-1)}{2} = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} + k(n-k)$$

holds, from (6.13) we have

$$\begin{aligned}\|g_n(\xi)\|_s \le \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)^{L(n-1)}} \Bigg[\frac{B}{A}\Big(\frac{h}{H}\Big)^n \\ + \sum_{i=0}^{N} \sum_{(j,0)\in\Lambda^*} \sum_{1\le k\le n-\mu} B\Big(\frac{h}{H}\Big)^k \frac{1}{q^{e_{j,0}(n-k)}} \times |\xi|^i \\ + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*, |\alpha|>0} \sum_{1\le k\le n-\mu} B\Big(\frac{h}{H}\Big)^k \frac{(n-k)^{|\alpha|}}{q^{e_{j,\alpha}(n-k)}} \times |\xi|^i \Bigg].\end{aligned}$$

Since $e_{j,0} \ge 0$, we have $1/q^{e_{j,0}(n-k)} \le 1$. Since $m^L/q^m \to 0$ (as $m \to \infty$), we have $m^L/q^m \le c_0$ for some c_0 (we may assume that $c_0 > 1$ holds). Then for $0 < |\alpha| \le L$, we have $e_{j,\alpha} \ge 1$ and so

$$\frac{(n-k)^{|\alpha|}}{q^{e_{j,\alpha}(n-k)}} \le \frac{(n-k)^L}{q^{(n-k)}} \le c_0.$$

Therefore, if we assume the conditions $A > B$ and $H > h$, we have the estimate

$$\|g_n(\xi)\|_s \le \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)^{L(n-1)}} \Bigg[\Big(\frac{h}{H}\Big)^n + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \frac{c_0 B(h/H)}{1-h/H} \times |\xi|^i \Bigg]$$

for any $0 < s < R$. Thus, by applying Lemma 6.3 to equation $\mathscr{L} u_n = g_n(\xi, z)$ and by using the estimates $|\xi|^i/(|\xi|+1)^N \le 1$ $(0 \le i \le N)$ we have

$$\|u_n(\xi)\|_s \le \frac{1}{c(1-\beta)} \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)^{L(n-1)}} \Bigg[\Big(\frac{h}{H}\Big)^n + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \frac{c_0 B(h/H)}{1-h/H} \Bigg]$$

on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$.

Now, let us apply Lemma 2.5. We get

$$\begin{aligned}&\|\partial_z^\alpha u_n(\xi)\|_s \\ &\le \frac{1}{c(1-\beta)} \frac{e^{|\alpha|}(L(n-1)+1)\dots(L(n-1)+|\alpha|)AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)^{L(n-1)+|\alpha|}} \times \\ &\quad \times \Bigg[\Big(\frac{h}{H}\Big)^n + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \frac{c_0 B(h/H)}{1-h/H} \Bigg] \\ &\le \frac{1}{c(1-\beta)} \frac{e^L L^L n^{|\alpha|} AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)^{Ln}} \times \Bigg[\Big(\frac{h}{H}\Big)^\mu + \sum_{i=0}^{N} \sum_{(j,\alpha)\in\Lambda^*} \frac{c_0 B(h/H)}{1-h/H} \Bigg]\end{aligned}$$

on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$. If

$$\frac{(eL)^L}{c(1-\beta)}\Big[\Big(\frac{h}{H}\Big)^{\mu} + \sum_{i=0}^{N}\sum_{(j,\alpha)\in\Lambda^*}\frac{c_0 B(h/H)}{1-h/H}\Big] \le 1 \tag{6.14}$$

holds, we have the result (6.10).

Thus, by taking A and H so that $A > B$, $H > h$, (6.12) and (6.14) are satisfied we have the result in Lemma 6.4. □

6.3. COMPLETION OF THE PROOF OF PART (2)

By Lemma 6.4, we can easily see that the formal solution

$$u(\xi, z) = \sum_{n\ge\mu} u_n(\xi, z)$$

is convergent on Ω and it defines a holomorphic solution of (6.1). Let us show the estimate (6.2).

Take any $0 < R_1 < R$. By Lemma 6.4, we have

$$|u_n(\xi, z)| \le \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-R_1)^{Ln}}$$

on $\Omega_1 = (D_r \cup S_\theta(\lambda)) \times D_{R_1}$ for any $n \ge \mu$. We set $H_2 = H|\lambda|/(R-R_1)^L$: we obtain

$$\begin{aligned}
|u(\lambda q^m, z)| &\le \sum_{n\ge\mu}|u_n(\lambda q^m, z)| \le \sum_{n\ge\mu}\frac{AH^n(|\lambda|q^m)^n}{q^{n(n-1)/2}(R-R_1)^{Ln}} \\
&\le A\sum_{n\ge\mu}\frac{(H|\lambda|/(R-R_1)^L)^n q^{mn}}{q^{n(n-1)/2}} \\
&= A{H_2}^m q^{m(m+1)/2}\sum_{n\ge\mu}\frac{(H_2)^{n-m}}{q^{(n-m)(n-m-1)/2}} \\
&\le \vartheta_q(H_2)A{H_2}^m q^{m(m+1)/2}, \quad m = 0, 1, 2, \dots
\end{aligned}$$

where $\vartheta_q(x)$ is the Jacobi theta function. This proves (6.2).

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