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# ON MEAN-VALUE PROPERTIES FOR THE DUNKL POLYHARMONIC FUNCTIONS

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**Abstract.** We derive differential relations between the Dunkl spherical and solid means of continuous functions. Next we use the relations to give inductive proofs of mean-value properties for the Dunkl polyharmonic functions and their converses.

**Keywords:** Dunkl Laplacian, Dunkl polyharmonic functions, mean-values, Pizzetti formula.

**Mathematics Subject Classification:** 31A30, 31B30, 33C52.

## 1. INTRODUCTION

During the last few years there is a growing interest in the study of Dunkl harmonic functions, i.e., solutions to $\Delta_\kappa u = 0$, where $\Delta_\kappa$ is a second order differential-difference operator invariant under the action of a discrete reflection group, see (2.1). The operator $\Delta_\kappa$ was introduced by Dunkl in [2, 3], in the context of the theory of orthogonal polynomials in several variables. Afterwards the whole theory related to $\Delta_\kappa$ was elaborated including analogues of Fourier analysis, special functions connected with root systems, algebraic approaches and an application to the solution of quantum Calogero-Sutherland models (see [5] for an excellent survey). In particular, Mejjaoli and Triméche proved in [12] that the operator $\Delta_\kappa$ is hypoelliptic on $\mathbb{R}^n$ and that smooth Dunkl harmonic functions on $\mathbb{R}^n$ can be characterized by the Dunkl spherical mean value property. Furthermore, they derived a Pizzetti type formula for smooth functions on $\mathbb{R}^n$. Maslouhi and Yousffi solved in [10] the Dirichlet problem for $\Delta_\kappa$ on the unit ball $B$ and derived a characterization of $C^2$ Dunkl harmonic functions on $B$ by the Dunkl spherical mean value property. Recently, Hassine has obtained in [7] the characterization without smoothness assumptions. Maslouhi and Daher proved in [11] Weil's lemma for $\Delta_\kappa$ and gave a characterization of Dunkl harmonic functions in a class of tempered distributions in terms of invariance under Dunkl convolution with suitable kernels. The Pizzetti series associated with $\Delta_\kappa$ was studied by Salem and

 

Touahri in [15], they proved its convergence for a real analytic function and derived some Liouville type results for Dunkl polyharmonic functions. Some other results related to the Dunkl spherical mean value operator were also derived in [9, 14, 16].

In this paper we first derive differential relations between the Dunkl spherical and solid means of functions. Next, we use the relations to give a short proof of an analogue of the Beckenbach-Reade theorem stating that equality of the Dunkl spherical and solid means of a continuous function implies its Dunkl harmonicity. Taking full advantage of the relations we also give simple inductive proofs of the Dunkl solid and spherical mean-value properties for the Dunkl polyharmonic functions and their converses in arbitrary dimension. The paper is a continuation of [8], where analogous results were obtained for polyharmonic functions.

## 2. PRELIMINARIES

Recall that for a nonzero vector $\alpha \in \mathbb{R}^n \setminus \{0\}$ the reflection with respect to the orthogonal to the $\alpha$ hyperplane $H_\alpha$ is given by

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{\|\alpha\|^2}\,\alpha, \quad x \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ is the euclidian scalar product on $\mathbb{R}^n$ and $\|\cdot\|$ the associated norm. A finite set $R$ of nonzero vectors is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. The reflections $\sigma_\alpha$ with $\alpha$ in a given root system $R$ generate a finite group $W \subset O(n)$, called the reflection group associated with $R$. For a fixed $\beta \in \mathbb{R}^n \setminus \bigcup_{\alpha\in R} H_\alpha$ one can decompose $R = R_+ \cup R_-$ where $R_\pm = \{\alpha \in R : \pm\langle\alpha,\beta\rangle > 0\}$; vectors in $R_+$ are called positive roots. A function $\kappa : R \to \mathbb{R}$ is called a multiplicity function if it is invariant under the action of the associated reflection group $W$. Its index $\gamma$ is defined by

$$\gamma = \sum_{\alpha\in R_+} \kappa(\alpha).$$

Throughout the paper we shall assume that $\kappa \geq 0$ and $\gamma > 0$.

The Dunkl operators $T_j$, $j = 1,\ldots,n$, associated with a root system $R$ and a multiplicity function $\kappa$ were introduced by C. Dunkl [3] as

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in R_+} \kappa(\alpha)\frac{f(x) - f(\sigma_\alpha(x))}{\langle\alpha, x\rangle}\,\alpha_j \quad \text{for} \quad f \in C^1(\mathbb{R}^n).$$

Clearly, $T_j f$ is well defined for $f \in C^1(\Omega)$ where $\Omega$ is a $W$-invariant open subset of $\mathbb{R}^n$ and it reduces to $\frac{\partial}{\partial x_j} f$ if $f$ is $W$-invariant. The Dunkl Laplacian $\Delta_\kappa$ is defined as a sum of squares of the operators $T_j$, $j = 1,\ldots,n$, i.e.,

$$\Delta_\kappa f = \sum_{j=1}^{n} T_j^2 f \quad \text{for} \quad f \in C^2(\Omega).$$

A simple computation leads to

$$\Delta_\kappa f(x) = \Delta f(x) + \sum_{\alpha\in R_+} \kappa(\alpha) \left( \frac{2\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \|\alpha\| \, \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2} \right). \tag{2.1}$$

Here $\Delta$ and $\nabla$ denote the usual Laplacian and gradient, respectively.

The Dunkl intertwining operator $V_\kappa$ acting on polynomials was defined in [4] by

$$T_j V_\kappa f = V_\kappa \tfrac{\partial}{\partial x_j} f \quad \text{for} \quad j = 1, \ldots, n \quad \text{and} \quad V_\kappa 1 = 1.$$

The operator $V_\kappa$ extends to a topological isomorphism of $C^\infty(\mathbb{R}^n)$ onto itself [17]. In general there is no explicite description of $V_\kappa$, but Rösler has shown [13, Th. 1.2, Cor. 5.3] that for any $x \in \mathbb{R}^n$ there exists a unique probability measure $\mu_x$ such that

$$V_\kappa f(x) = \int_{\mathbb{R}^n} f(y) d\mu_x(y). \tag{2.2}$$

Moreover, the support of $\mu_x$ is contained in $\mathrm{ch}(Wx)$ – the convex hull of the set $\{gx : g \in W\}$, $\mu_{rx}(U) = \mu_x(r^{-1}U)$ and $\mu_{gx}(U) = \mu_x(g^{-1}U)$ for $r > 0$, $g \in W$ and a Borel set $U \subset \mathbb{R}^n$. Note that by (2.2), $V_\kappa$ can be extended to continuous functions and $|V_\kappa(f)(x)| \le \sup_{y\in \mathrm{ch}(Wx)} |f(y)|$; the extension is a topological isomorphism of $C(\mathbb{R}^n)$.

The Dunkl translation operators $\tau_x$, $x \in \mathbb{R}^n$, are defined on $C(\mathbb{R}^n)$ by

$$\tau_x f(y) = (V_\kappa)_x (V_\kappa)_y \big[ V_\kappa^{-1} f(x + y) \big] \quad \text{for} \quad y \in \mathbb{R}^n.$$

A more suggestive notation $f(x *_\kappa y) := \tau_x f(y)$ is also used. Note that $\tau_0 f = f$ and $\tau_y f(x) = \tau_x f(y)$ for $x, y \in \mathbb{R}^n$.

## 3. THE DUNKL MEAN VALUE PROPERTY

The Poisson kernel for the Dunkl Laplacian $\Delta_\kappa$ is defined in [6] by[1)]

$$P_\kappa(x, y) = V_\kappa \left[ \frac{1 - \|x\|^2}{(1 - 2\langle x, \cdot\rangle + \|x\|)^{\gamma + n/2}} \right](y) \quad \text{for} \quad \|x\| < 1,\ \|y\| \le 1.$$

The kernel $P_\kappa(x, y)$ is non-negative, bounded by 1 and it has the reproducing property for Dunkl harmonic functions on the unit ball $B = B(0, 1)$. Furthermore it is used as a tool to solve the Dirichlet problem for the Dunkl Laplacian. Namely it holds

**Theorem 3.1** ([10, Theorem A, Prop. 2.1])**.** *Let $u$ be a continuous function on the unit sphere $S(0, 1)$. Set*

$$P_\kappa[u](x) = \frac{1}{d_\kappa} \int_{S(0,1)} P_\kappa(x, y) u(y)\, \omega_\kappa(y) dS(y) \quad \textit{for} \quad \|x\| < 1,$$

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1) $P_\kappa(x, y) = P(h_\kappa^2; y, x)$ where $P(h_\kappa^2; \cdot, \cdot)$ is defined in [6, p. 190].

*where*

$$d_\kappa = \int\limits_{S(0,1)} \omega_\kappa(y)dS(y) \quad \text{and} \quad \omega_\kappa(y) = \prod_{\alpha\in R_+} |\langle\alpha, y\rangle|^{2\kappa(\alpha)}.$$

*Then* $P_\kappa[u]$ *is* $\Delta_\kappa$*-harmonic on the unit ball* $B$*, extends continuously to* $\overline{B}$ *and* $P_\kappa[u] = u$ *on* $S(0,1)$*. Furthermore,* $P_\kappa[u]$ *is the unique* $\Delta_\kappa$*-harmonic function on* $B$ *which extends continuously to* $u$ *on* $S(0,1)$*.*

Since $P_\kappa(0,y) = 1$ for $\|y\| \le 1$ for a function $u$ continuous on $\overline{B}$ and Dunkl harmonic in $B$ we get

$$u(0) = \frac{1}{d_\kappa} \int\limits_{S(0,1)} u(y)\,\omega_\kappa(y)dS(y).$$

More generally, if a function $u$ is continuous on $\overline{B}$ and $\Delta_\kappa$-harmonic in $B$, then for any $x \in B$ and $0 < r < 1 - \|x\|$ the spherical mean value formula holds (see [10, Theorem C])

$$u(x) = \frac{1}{d_\kappa} \int\limits_{S(0,1)} \tau_x u(ry)\,\omega_\kappa(y)dS(y) \tag{3.1}$$

The converse statement was also stated ([10, Theorem C]) under the assumption that $u$ is a $C^2$ function. Recently Hassine has proved it without that assumption.

**Theorem 3.2** ([7, Theorem 3.1])**.** *Let* $u$ *be a bounded function on the closed unit ball* $\overline{B}$*. If for any* $x \in B$ *and* $0 < r < 1 - \|x\|$ *the formula* (3.1) *holds, then* $u$ *is* $\Delta_\kappa$*-harmonic in* $B$*.*

## 4. RELATIONS BETWEEN THE DUNKL SPHERICAL AND SOLID MEANS

Let $u$ be a smooth function on the ball $\overline{B}$. For any $x \in B$ and $0 < R < 1 - \|x\|$ we denote by $N^D(u;\, x, R)$ the Dunkl spherical integral mean of $u$ over the sphere $S(x,R)$,

$$N^D(u;\, x, R) = \frac{1}{d_\kappa} \int\limits_{S(0,1)} \tau_x u(Ry)\,\omega_\kappa(y)dS(y). \tag{4.1}$$

It was proved in [14, Theorem 4.1] that the Dunkl spherical mean operator $u \mapsto N^D(u;\, x, R)$ can be represented in the form

$$N^D(u;\, x, R) = \int\limits_{\mathbb{R}^n} u(y)\, d\mu^\kappa_{x,R}(y),$$

where $\mu^\kappa_{x,R}$ is a probability measure with support in $\bigcup_{g\in W}\{y \in \mathbb{R}^n : \|y - gx\| \le R\}$. Hence $N^D(u;\, x, R)$ is well defined for a continuous function $u$. Since $\omega_\kappa$ is homogenous of degree $2\gamma$, we also have

$$N^D(u;\, x, R) = \frac{1}{d_\kappa R^{2\gamma+n-1}} \int\limits_{S(0,R)} \tau_x u(z)\,\omega_\kappa(z)dS(z).$$

Note that using the spherical coordinates by homogeneity of $\omega_\kappa$ we get

$$\int\limits_{B(0,1)} \omega_\kappa(x)\,dx = \int\limits_0^1 \Big( \int\limits_{S(0,1)} \omega_\kappa(y)dS(y)\Big)\, t^{2\gamma+n-1}dt = \frac{d_\kappa}{2\gamma+n}.$$

So we can define the Dunkl solid integral mean of $u$ over the ball $\overline{B}(x,R)$ by

$$\begin{aligned} M^D(u;\,x,R) &= \frac{2\gamma+n}{d_\kappa} \int\limits_{B(0,1)} \tau_x u(Ry)\,\omega_\kappa(y)dy \\ &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int\limits_{B(0,R)} \tau_x u(z)\,\omega_\kappa(z)dz. \end{aligned} \tag{4.2}$$

For the convenience of the reader recall the Green formula for the Dunkl Laplacian.

**Theorem 4.1** (Green formula for $\Delta_\kappa$, [12, Theorem 4.11])**.** *Let $\Omega$ be a bounded $W$-invariant regular open set in $\mathbb{R}^n$ containing the origin and $u \in C^2(\Omega)$. Then for any closed ball $\overline{B}(0,R) \subset \Omega$ it holds*

$$\int\limits_{B(0,R)} \Delta_\kappa u(z)\,\omega_\kappa(z)dz = \int\limits_{S(0,R)} \frac{\partial u(z)}{\partial \eta}\,\omega_\kappa(z)dS(z), \tag{4.3}$$

*where $\frac{\partial u}{\partial \eta}$ denotes the external normal derivative of $u$.*

The relations between $M^D(u;\,x,R)$ and $N^D(u;\,x,R)$ are given in the following lemma.

**Lemma 4.2.** *Let $u$ be a continuous function on the ball $\overline{B}$. Then for any $x \in B$ and $0 < R < 1 - \|x\|$ it holds*

$$\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R} + 1\Big) M^D(u;\,x,R) = N^D(u;\,x,R). \tag{4.4}$$

*If we further assume that $u$ has continuous derivatives up to second order, then*

$$\frac{2\gamma+n}{R}\frac{\partial}{\partial R} N^D(u;\,x,R) = M^D(\Delta_\kappa u;\,x,R). \tag{4.5}$$

*Proof.* By (4.2) using the spherical coordinates, homogeneity of $\omega_\kappa$ and (4.1) we compute

$$\begin{aligned} M^D(u;\,x,R) &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}} \int\limits_0^R \Big( \int\limits_{S(0,s)} \tau_x u(z)\,\omega_\kappa(z)dS(z)\Big) ds \\ &= \frac{2\gamma+n}{R^{2\gamma+n}} \int\limits_0^R N^D(u;x,s)\,s^{2\gamma+n-1}ds. \end{aligned}$$

Hence, by the Leibniz rule

$$\frac{\partial}{\partial R}M^D(u;\, x, R) = \frac{2\gamma+n}{R}\Big(N^D(u;\, x, R) - M^D(u;\, x, R)\Big),$$

which proves (4.4).

To show (4.5) we differentiate (4.1) under the integral sign to get

$$\begin{aligned}\frac{\partial}{\partial R}N^D(u;\, x, R) &= \frac{1}{d_\kappa}\int\limits_{S(0,1)} \langle \nabla(\tau_x u)(Ry), y\rangle\, \omega_\kappa(y) dS(y)\\ &= \frac{1}{d_\kappa R^{2\gamma+n-1}}\int\limits_{S(0,R)} \langle \nabla(\tau_x u)(z), \tfrac{z}{R}\rangle\, \omega_\kappa(z) dS(z).\end{aligned}$$

Note that the external normal vector to $S(0,R)$ at a point $z \in S(0,R)$ is $\eta = \frac{z}{R}$ and $\langle\nabla(\tau_x u), \eta\rangle = \frac{\partial(\tau_x u)}{\partial\eta}$. So applying the Green formula (4.3) we get

$$\frac{\partial}{\partial R}N^D(u;\, x, R) = \frac{1}{d_\kappa R^{2\gamma+n-1}}\int\limits_{B(0,R)} \Delta_\kappa(\tau_x u)(z)\, \omega_\kappa(z) dz = \frac{R}{2\gamma+n}M^D(\Delta_\kappa u;\, x, R),$$

since $\Delta_\kappa \tau_x u(z) = \tau_x \Delta_\kappa u(z)$, which implies (4.5). □

By (4.4) and (4.5), we obtain the following corollary.

**Corollary 4.3.** *Let $u \in C^2(B)$. Then for any $x \in B$ and $0 < R < 1 - \|x\|$ it holds that*

$$M^D(\Delta_\kappa u;\, x, R) = \Big(\frac{\partial^2}{\partial R^2} + \frac{2\gamma+n+1}{R}\frac{\partial}{\partial R}\Big)M^D(u;\, x, R) \tag{4.6}$$

*and*

$$N^D(\Delta_\kappa u;\, x, R) = \Big(\frac{\partial^2}{\partial R^2} + \frac{2\gamma+n-1}{R}\frac{\partial}{\partial R}\Big)N^D(u;\, x, R). \tag{4.7}$$

Let us point out that formula (4.7) was established in [12, Proposition 4.16].

By the first part of Lemma 4.2 we get an analogue of the Beckenbach-Reade theorem ([1]) for the Dunkl harmonic functions.

**Corollary 4.4.** *Let $u \in C^2(B)$. If for any $x \in B$ and $0 < R < 1 - \|x\|$ it holds*

$$M^D(u;\, x, R) = N^D(u;\, x, R), \tag{4.8}$$

*then $u$ is Dunkl harmonic on $B$.*

*Proof.* The assumption (4.8) and (4.4) imply that $\frac{\partial}{\partial R}M^D(u;\, x, R) = 0$. So for any $x \in B$, $M^D(u;\, x, R)$ is a constant equal to $u(x)$ and the converse to the mean-value property for Dunkl harmonic functions ([10, Theorem C]) implies that $u$ is Dunkl harmonic on $B$. □

## 5. MEAN-VALUE PROPERTIES FOR DUNKL POLYHARMONIC FUNCTIONS

Let $m \in \mathbb{N}$. A function $u \in C^{2m}(\Omega)$ defined on a $W$-invariant open set $\Omega \subset \mathbb{R}^n$ is called an $m$-Dunkl harmonic if it is a solution of the $m$-times iteration of the Dunkl operator, i.e., $\Delta_\kappa^m u = 0$. One of the most trivial examples is given by an even power of the Euclidean distance from the origin.

**Example 5.1.** Let $u(x) = r^{2m}(x)$ with $m \in \mathbb{N}_0$, where $r(x) = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$ is the radius function. Since $u$ is $W$-invariant $\Delta_\kappa u$ reduces to

$$\Delta_\kappa u(x) = \Delta u(x) + 2\sum_{\alpha\in R_+} \kappa(\alpha)\,\frac{\langle \nabla u(x), \alpha\rangle}{\langle \alpha, x\rangle}.$$

Since $\Delta u = 2m(n+2m-2)\,r^{2m-2}$ and $\nabla u = 2m\,x\cdot r^{2m-2}$, we get $\Delta_\kappa u = 2m(n+2m+2\gamma-2)\,r^{2m-2}$. So $u$ is $(m+1)$-Dunkl harmonic, $\Delta^i u(0) = 0$ for $i = 0,1,\ldots,m-1$ and

$$\begin{aligned}\Delta_\kappa^m u(0) &= 2m(2m-2)\cdots 2\times(n+2m+2\gamma-2)\cdots(n+2\gamma)\,r^0(0)\\ &= 4^m\Big(\gamma+\frac{n}{2}\Big)_m m!,\end{aligned}$$

where for $a \in \mathbb{R}$, $(a)_0 = 1$ and $(a)_i = a(a+1)\cdots(a+i-1)$ for $i \in \mathbb{N}$. On the other hand using the spherical coordinates and the fact that $\omega$ is homogeneous of degree $2\gamma$ we get

$$\begin{aligned}M^D(u;\,0,R) &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}}\int_0^R\int_{S(0,s)} \|y\|^{2m}\,\omega_\kappa(y)dS(y)ds\\ &= \frac{2\gamma+n}{d_\kappa R^{2\gamma+n}}\int_0^R d_\kappa s^{2m+2\gamma+n-1}\,ds = \frac{2\gamma+n}{2m+2\gamma+n}\,R^{2m}.\end{aligned}\tag{5.1}$$

Hence

$$M^D(u;\,0,R) = \frac{\Delta_\kappa^m u(0)}{4^m\big(\gamma+\frac{n}{2}+1\big)_m m!}\cdot R^{2m}.$$

The above example suggests a form of an expansion of $M(u;\,x,R)$ for a polyharmonic function $u$ into powers of the radius $R$ of the ball $B(x,R)$.

**Theorem 5.2** (Mean-value property for solid means, [16, formula (1.1)])**.** *Let $m \in \mathbb{N}_0$. If $u \in C^{2m+2}(B)$ and $\Delta_\kappa^{m+1} u = 0$ in $B$, then for any $x \in B$ and $0 < R < 1 - \|x\|$ it holds*

$$M^D(u;\,x,R) = \sum_{k=0}^m \frac{\Delta_\kappa^k u(x)}{4^k\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot R^{2k}.\tag{5.2}$$

*Proof.* It was pointed out in [16, p. 120] that the mean value formula (5.2) for solid means can be derived from an analogous one for spherical means by integration. Here we give a proof based on a simple inductive arguments. Clearly, by the mean-value property for the Dunkl harmonic functions, which follows from [10, Theorem C], the formula (5.2) holds for $m = 0$. Inductively assume that Theorem 5.2 holds for a fixed $m \in \mathbb{N}_0$. Let $v \in C^{2m+4}(B)$ and $\Delta_\kappa^{m+2} v = 0$. Then $u = \Delta_\kappa v \in C^{2m+2}(B)$ satisfies $\Delta_\kappa^{m+1} u = 0$ and so (5.2) holds. But, by (4.6),

$$\frac{2\gamma+n}{R}\frac{\partial}{\partial R}\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;\, x,R) = M^D(\Delta_\kappa v;\, x,R) = M^D(u;\, x,R).$$

So after one integration

$$\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;\, x,R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(x)}{4^k(2\gamma+n)\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot\frac{R^{2k+2}}{2k+2}+c. \tag{5.3}$$

Note that the general solution of $\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;\, x,R) = 0$ is $CR^{-2\gamma-n}$ and a particular solution of

$$\Big(\frac{R}{2\gamma+n}\frac{\partial}{\partial R}+1\Big)M^D(v;\, x,R) = \frac{\Delta_\kappa^k u(x)}{4^k(2\gamma+n)\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot\frac{R^{2k+2}}{2k+2}$$

is $A_k R^{2k+2}$, where

$$A_k\Big(\frac{2k+2}{2\gamma+n}+1\Big) = \Delta_\kappa^k u(x)\cdot\Big[4^k(2\gamma+n)(2k+2)\big(\gamma+\frac{n}{2}+1\big)_k k!\Big]^{-1}.$$

So

$$\begin{aligned} A_k &= \frac{\Delta_\kappa^k u(x)}{4^k(2k+2)(2\gamma+n+2k+2)\big(\gamma+\frac{n}{2}+1\big)_k k!} \\ &= \frac{\Delta_\kappa^k u(x)}{4^{k+1}\big(\gamma+\frac{n}{2}+1\big)_{k+1}(k+1)!}. \end{aligned}$$

Hence, the general solution of (5.3) is

$$M^D(v;\, x,R) = CR^{-2\gamma-n} + \sum_{k=0}^{m}\frac{\Delta_\kappa^k u(x)}{4^{k+1}\big(\gamma+\frac{n}{2}+1\big)_{k+1}(k+1)!}\cdot R^{2k+2}+c.$$

Finally, note that $\lim_{R\to 0} M^D(v;\, x,R) = v(x)$ and $\lim_{R\to 0} R^{2\gamma+n} M^D(v;\, x,R) = 0$. So $c = v(x)$, $C = 0$ and

$$\begin{aligned} M^D(v;\, x,R) &= v(x) + \sum_{k=0}^{m}\frac{\Delta_\kappa^{k+1} v(x)}{4^{k+1}\big(\gamma+\frac{n}{2}+1\big)_{k+1}(k+1)!}\cdot R^{2k+2} \\ &= \sum_{k=0}^{m+1}\frac{\Delta_\kappa^k v(x)}{4^k\big(\gamma+\frac{n}{2}+1\big)_k k!}\cdot R^{2k} \end{aligned}$$

which proves Theorem 5.2 □

By Theorem 5.2 and the relation (4.4), we get the following corollary.

**Corollary 5.3** (Mean-value property for spherical means, [15, Proposition 3.1] and [12, Theorem 4.17]). *Under the assumptions of Theorem 5.2 for any $x \in B$ and $0 < R < 1 - \|x\|$ it holds*

$$N^D(u;\, x, R) = \sum_{k=0}^{m} \frac{\Delta_\kappa^k u(x)}{4^k \left(\gamma + \frac{n}{2}\right)_k k!} \cdot R^{2k}. \tag{5.4}$$

**Theorem 5.4** (Converse to the mean value property for spherical means). *Let $m \in \mathbb{N}_0$. If $u \in C^{2m}(B)$ and the formula* (5.4) *holds for any $x \in B$ and $0 < R < 1 - \|x\|$, then $\Delta_\kappa^{m+1} u = 0$ in $B$.*

*Proof.* Clearly, if $m = 0$, Theorem 5.4 follows from Theorem 3.2. Fix $p \in \mathbb{N}$ and assume that Theorem 5.4 holds for $m < p$. We shall prove that it holds for $m = p$. To this end take $v \in C^{2p}(B)$ and assume that for any $x \in B$ and $R$ small enough (5.4) holds with $m = p$ and $u = v$. Set $u = \Delta_\kappa v$. Then $u \in C^{2p-2}(B)$. By (4.5) and (5.4) with $m = p$ and $u = v$, we get

$$M^D(u;\, x, R) = \sum_{k=1}^{p} \frac{2k(2\gamma + n)\Delta_\kappa^k v(x)}{4^k \left(\gamma + \frac{n}{2}\right)_k k!} \cdot R^{2k-2} = \sum_{k=0}^{p-1} \frac{\Delta_\kappa^k u(x)}{4^k \left(\gamma + \frac{n}{2} + 1\right)_k k!} \cdot R^{2k}.$$

So for any $x \in B$ and $R$ small enough, by (4.4) we derive

$$N^D(u;\, x, R) = \sum_{k=0}^{p-1} \left(\frac{2k}{2\gamma + n} + 1\right) \frac{\Delta_\kappa^k u(x)}{4^k \left(\gamma + \frac{n}{2} + 1\right)_k k!} \cdot R^{2k} = \sum_{k=0}^{p-1} \frac{\Delta_\kappa^k u(x)}{4^k \left(\gamma + \frac{n}{2}\right)_k k!} \cdot R^{2k}.$$

Hence, by the inductive assumption, $\Delta_\kappa^p u = \Delta_\kappa^{p+1} v = 0$. □

By Theorem 5.4 and the relation (4.4), we get the following corollary.

**Corollary 5.5** (Converse to the mean value property for solid means). *Under the assumptions of Theorem 5.2 if $u \in C^{2m}(B)$ and for all $x \in B$ and $R$ small enough formula* (5.2) *holds, then $\Delta_\kappa^{m+1} u = 0$ in $B$.*

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