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NOTES ON THE NONLINEAR DEPENDENCE OF A MULTISCALE COUPLED SYSTEM WITH RESPECT TO THE INTERFACE

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Abstract. This work studies the dependence of the solution with respect to interface geometric perturbations, in a multiscaled coupled Darcy flow system in direct variational formulation. A set of admissible perturbation functions and a sense of convergence is presented, as well as sufficient conditions on the forcing terms, in order to conclude strong convergence statements. For the rate of convergence of the solutions we start solving completely the one dimensional case, using orthogonal decompositions on the appropriate subspaces. Finally, the rate of convergence question is analyzed in a simple multi dimensional setting, by studying the nonlinear operators introduced due to the geometric perturbations.

Keywords: multiscale coupled systems, interface geometric perturbations, variational formulations, nonlinear dependence.

Mathematics Subject Classification: 58F15, 58F17, 53C35.

1. INTRODUCTION

The study of saturated flow in geological porous media frequently presents natural structures with a dense network of fissures nested in the rock matrix [18,22]. It is also frequent to observe vuggy porous media [1], which have the presence of cavities in the rock matrix significantly larger than the average pore size of the medium. This is a multi scale physics phenomenon, because there are regions of the medium where the flow velocity is significantly larger than the velocity on the other ones. The modeling of the interface between regions is subject to very active research: first, fluid transmission conditions across the interface are of great importance, see [3,20] for the discussion of governing laws; see [2,7] for a numerical point of view; see [1,5,12,16] for the analytic approach and [4,21] for a more general perspective. Second, the placement of the interface is debatable since a boundary layer phenomenon between regions occurs, see [10,19] for discussion. The interface couples regions of slow velocity (order $\mathcal{O}(1)$) and fast flow (order $\mathcal{O}(1/\epsilon)$), see Figure 1. Hence, its placement and geometric description become an important issue because perturbations of the interface are inevitable. On one hand the geological strata data available are always limited, on the other hand the numerical implementation of models involving curvy interfaces, in most of the cases can only approximate the real surface. Finally, on a very different line, in the analysis of saturated flow in deformable porous media, one of the aspects is understanding the geometric perturbations of an *interface of reference*.

Clearly, the continuity of the solution with respect to the geometry of the interface is an important issue, which has received very little treatment and mainly limited to flat interfaces. Most of the theoretical achievements in the field of multiscale coupled systems, concentrate their efforts in removing the singularities introduced by the scales using homogenization processes. These techniques can be either formal [13, 17], analytic [11] or numerical [14].

Here, we model the stationary problem with a coupled system of partial differential equations of Darcy flow in both regions, in direct variational formulation. We simulate the region of fast flow scaling by $\frac{1}{\epsilon}$ the ratio of permeability over viscosity, as in Figure 1. It will be assumed that the real interface Γ is horizontal flat and the perturbed one Γ^{ζ} is curved. Of course, a flat surface will not be perturbed when discretized and seems unrealistic to consider perturbations of it. Our choice is motivated by two reasons: first, for the sake of clarity in the notation, calculation and interpretation of the results. Second, when studying the phenomenon of saturated flow in *deformable porous media* perturbations of a flat surface are of interest. It is important to highlight that the mathematical essentials of the problem are captured in this framework.

The paper starts proving in Section 2, that the solutions depend continuously with respect to the interface, then it moves to the much deeper question of exploring the nature of the dependence itself. In Section 3, for the one dimensional case, the rate of convergence question is solved completely using orthogonal decomposition in adequate subspaces. Finally, Section 4 reveals the highly nonlinear dependence of the solutions with respect to the interface.

We close this section introducing the notation. Vectors are denoted by boldface letters, as are vector-valued functions and corresponding function spaces. We use $\widetilde{\mathbf{x}}$ to indicate a vector in $\mathbb{R}^{N-1}$; if $\mathbf{x} \in \mathbb{R}^N$, then the $\mathbb{R}^{N-1} \times \{0\}$ projection is identified with $\widetilde{\mathbf{x}} \overset{\mathbf{def}}{=} (x_1, x_2, \ldots, x_{N-1})$, so that $\mathbf{x} = (\widetilde{\mathbf{x}}, x_N)$. The symbol $\widehat{\mathbf{e}}_\ell$ indicates the unitary vector in the ℓ-th direction for $1 \leq \ell \leq N$. We denote by $\widehat{\boldsymbol{\nu}}$ the outwards normal vector to a smooth domain in $\mathbb{R}^N$ and $\widehat{\boldsymbol{n}}$ indicates the upwards normal vector, i.e., $\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{e}}_N \geq 0$. We write $\mathbb{1}_A$ for the indicator function of any given set A in $\mathbb{R}^N$ or $\mathbb{R}^{N-1}$, and the Lebesgue measure in $\mathbb{R}^N$ is denoted by λ_N. The symbol $\widetilde{\boldsymbol{\nabla}}$ represents the gradient in the first $N-1$ derivatives.

Given a function $f : \mathbb{R}^N \to \mathbb{R}$ then, $\int_M f\, dS$ is the notation for its surface integral on the $\mathbb{R}^{N-1}$ manifold $M \subseteq \mathbb{R}^N$. The notation $\int_A f\, d\mathbf{x}$ stands for the volume integral in the open set $A \subseteq \mathbb{R}^N$; whenever the context is clear we simply write $\int_A f$. In this work we restrict to domains, which are open, connected and with smooth boundary. The notation $\|\cdot\|_{0,A}$, $\|\cdot\|_{1,A}$, $\|\cdot\|_{\frac{1}{2},\partial A}$, $\|\cdot\|_{-\frac{1}{2},\partial A}$ indicates the $L^2(A)$, $H^1(A)$, $H^{\frac{1}{2}}(\partial A)$

and $H^{-\frac{1}{2}}(\partial A)$ norms on the domain $A \subseteq \mathbb{R}^N$. Recall that

$$H^{\frac{1}{2}}(\partial A) \overset{\mathbf{def}}{=} \{\gamma(r) : r \in H^1(A)\}, \tag{1.1a}$$

$$\|w\|_{\frac{1}{2},\partial A} \overset{\mathbf{def}}{=} \inf\{\|r\|_{1,A} : r \in H^1(A), \gamma(r) = w\}, \tag{1.1b}$$

where γ is the trace operator. Its dual space is given by

$$H^{-\frac{1}{2}}(\partial A) \overset{\mathbf{def}}{=} (H^{\frac{1}{2}}(\partial A))', \tag{1.1c}$$

$$\|\ell\|_{-\frac{1}{2},\partial A} \overset{\mathbf{def}}{=} \sup\{|\ell(w)| : w \in H^{\frac{1}{2}}(\partial A), \|w\|_{\frac{1}{2},\partial A} = 1\}.$$

2. FORMULATION AND CONVERGENCE

2.1. GEOMETRIC SETTING

In the following Γ denotes a connected set in $\mathbb{R}^{N-1} \times \{0\}$ whose projection onto $\mathbb{R}^{N-1}$ is open. From now on we make no distinction between these two domains. Similarly, Ω_1, Ω_2 denote smooth bounded open regions in $\mathbb{R}^N$ separated by Γ, i.e., $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma$, and such that $\operatorname{sgn}(\mathbf{x} \cdot \widehat{\boldsymbol{e}}_N) = (-1)^i$ for each $\mathbf{x} \in \Omega_i$, $i = 1, 2$ (see Figure 1).

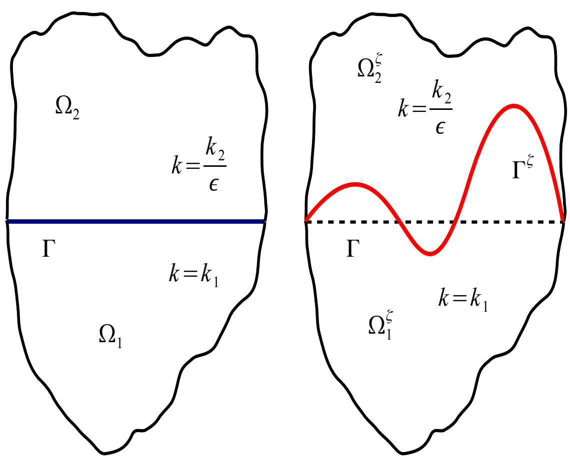

Fig. 1. Original domain and a perturbation

Next, we introduce the admissible perturbations of the interface Γ.

Definition 2.1. We say that the set $\mathscr{T}(\Gamma,\Omega)$ of piecewise C^1 perturbations of the interface Γ, contained in Ω is given by

$$\mathscr{T}(\Gamma,\Omega) \overset{\textbf{def}}{=} \{\zeta \in C\left(\overline{\Gamma}\right) : (\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})) \in \Omega \text{ for all } \widetilde{\mathbf{x}} \in \Gamma, \\ \zeta\,|_{\partial\Gamma} = 0 \text{ and } \zeta \text{ is a piecewise } C^1 \text{ function}\}. \tag{2.1}$$

The interface associated to $\zeta \in \mathscr{T}(\Gamma,\Omega)$ is given by the set

$$\Gamma^{\zeta} \overset{\textbf{def}}{=} \{(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})) \in \mathbb{R}^N : \widetilde{\mathbf{x}} \in \Gamma\}. \tag{2.2a}$$

The domains associated to $\zeta \in \mathscr{T}(\Gamma,\Omega)$ are defined by the sets

$$\Omega_1^{\zeta} \overset{\textbf{def}}{=} \{(\widetilde{\mathbf{x}}, x_N) \in \Omega : \widetilde{\mathbf{x}} \in \Gamma,\, x_N < \zeta(\widetilde{\mathbf{x}})\}, \tag{2.2b}$$

$$\Omega_2^{\zeta} \overset{\textbf{def}}{=} \{(\widetilde{\mathbf{x}}, x_N) \in \Omega : \widetilde{\mathbf{x}} \in \Gamma,\, \zeta(\widetilde{\mathbf{x}}) < x_N\}. \tag{2.2c}$$

Remark 2.2. Observe the following facts

$$\partial\Omega_1^{\zeta} \cap \partial\Omega_2^{\zeta} = \Gamma^{\zeta}, \tag{2.3a}$$

$$\Omega_1^{\zeta} \cup \Gamma^{\zeta} \cup \Omega_2^{\zeta} = \Omega, \tag{2.3b}$$

$$\partial\Omega_1^{\zeta} - \Gamma^{\zeta} = \partial\Omega_1 - \Gamma, \tag{2.3c}$$

$$\partial\Omega_2^{\zeta} - \Gamma^{\zeta} = \partial\Omega_2 - \Gamma. \tag{2.3d}$$

Definition 2.3. Define the space

$$V \overset{\textbf{def}}{=} \{u \in H^1(\Omega) : u|_{\partial\Omega_1 - \Gamma} = 0\}, \tag{2.4}$$

endowed with the inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$

$$\langle u, v\rangle_V \overset{\textbf{def}}{=} \int_{\Omega} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v, \tag{2.5}$$

and the norm $\|u\|_V \overset{\textbf{def}}{=} \sqrt{\langle u, u\rangle_V}$.

Remark 2.4. Recall that due to the boundary condition defining the space V and the Poincaré inequality, the $\|\cdot\|_V$-norm is equivalent to the standard H^1-norm.

2.2. THE PROBLEMS

Consider the strong problem

$$-\boldsymbol{\nabla}\cdot\frac{k_i}{\epsilon^{\,i-1}}\boldsymbol{\nabla}p_i = F \quad \text{in } \Omega_i,\ \ i=1,2, \tag{2.6a}$$

with the interface conditions

$$p_1 = p_2, \quad k_1\,\boldsymbol{\nabla}p_1\cdot\widehat{\boldsymbol{n}} - \frac{k_2}{\epsilon}\boldsymbol{\nabla}p_2\cdot\widehat{\boldsymbol{n}} = f \quad \text{on } \Gamma, \tag{2.6b}$$

and the boundary conditions

$$p_1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \quad \boldsymbol{\nabla}p_2\cdot\widehat{\boldsymbol{n}} = 0 \quad \text{on } \partial\Omega_2 - \Gamma. \tag{2.6c}$$

Now, its perturbation in strong form is given by

$$-\boldsymbol{\nabla}\cdot\frac{k_i}{\epsilon^{\,i-1}}\boldsymbol{\nabla}q_i = F \quad \text{in } \Omega_i^{\zeta},\ \ i=1,2, \tag{2.7a}$$

with the interface conditions

$$q_1 = q_2, \quad k_1\,\boldsymbol{\nabla}q_1\cdot\widehat{\boldsymbol{n}} - \frac{k_2}{\epsilon}\boldsymbol{\nabla}q_2\cdot\widehat{\boldsymbol{n}} = f \quad \text{on } \Gamma^{\zeta}, \tag{2.7b}$$

and the boundary conditions

$$q_1 = 0 \quad \text{on } \partial\Omega_1^{\zeta} - \Gamma^{\zeta}, \quad \boldsymbol{\nabla}q_2\cdot\widehat{\boldsymbol{n}} = 0 \quad \text{on } \partial\Omega_2^{\zeta} - \Gamma^{\zeta}. \tag{2.7c}$$

Both systems above model stationary Darcy flow, coupling the regions depicted in the left and right hand side of Figure 1, respectively. The coefficients k_1, k_2 indicate the permeability in the corresponding domain; for simplicity they will be omitted in the following. The scaling factor $\frac{1}{\epsilon}$ ensures a much higher velocity $\mathcal{O}(\frac{1}{\epsilon})$ in the upper region with respect to the lower region fluid velocity $\mathcal{O}(1)$. The term F stands for fluid sources and f for a normal flux forcing term on the interface. It is assumed that f is a well defined L^2-function on both manifolds Γ and Γ^{ζ}. Hence, the weak problems in direct formulation are given by

$$p\in V:\quad \int_{\Omega_1}\boldsymbol{\nabla}p\cdot\boldsymbol{\nabla}r + \frac{1}{\epsilon}\int_{\Omega_2}\boldsymbol{\nabla}p\cdot\boldsymbol{\nabla}r = \int_{\Omega}F\,r + \int_{\Gamma}f\,r\,dS \quad \text{for all} \quad r\in V, \tag{2.8a}$$

$$q^{\zeta}\in V:\quad \int_{\Omega_1^{\zeta}}\boldsymbol{\nabla}q^{\zeta}\cdot\boldsymbol{\nabla}r + \frac{1}{\epsilon}\int_{\Omega_2^{\zeta}}\boldsymbol{\nabla}q^{\zeta}\cdot\boldsymbol{\nabla}r = \int_{\Omega}F\,r + \int_{\Gamma^{\zeta}}f\,r\,dS \quad \text{for all} \quad r\in V. \tag{2.8b}$$

Theorem 2.5. *The problems* (2.8a) *and* (2.8b) *are well-posed.*

Proof. The result follows by a direct application of Lax-Milgram's lemma and the Poincaré inequality, see [15] for details. □

2.3. A-PRIORI ESTIMATES AND WEAK CONVERGENCE

In this section, under reasonable conditions on the forcing terms and the appropriate type of convergence for the perturbations ζ, a-priori estimates on the solutions of problems (2.8b) as well as weak convergence statements to the solution of problem (2.8a) are attained. For test equation (2.8b) with the solution q^{ζ}, due to the boundary conditions of V and the Poincaré constant C_{Ω} we get

$$\begin{split}\frac{1}{1+C_{\Omega}^{2}}\,\|\,q^{\zeta}\|_{1,\Omega}^{2} \leq \|\boldsymbol{\nabla} q^{\zeta}\|_{0,\Omega}^{2} \leq \|\boldsymbol{\nabla} q^{\zeta}\|_{0,\Omega_1^{\zeta}}^{2} + \frac{1}{\epsilon}\,\|\boldsymbol{\nabla} q^{\zeta}\|_{0,\Omega_2^{\zeta}}^{2} \\ \leq \|F\|_{0,\Omega}\,\|q^{\zeta}\|_{0,\Omega} + \int\limits_{\Gamma^{\zeta}} f\, q^{\zeta}\, dS.\end{split} \tag{2.9}$$

If a sequence of perturbations $\{\zeta_n\} \subseteq \mathscr{T}(\Gamma,\Omega)$ is to be analyzed, conditions on the type of convergence must be specified. For the perturbations, we assume that

$$\operatorname{ess\,sup}\left\{|(-\widetilde{\boldsymbol{\nabla}}\zeta_n(\widetilde{\mathbf{x}}),1)| : \widetilde{\mathbf{x}} \in \Gamma\right\} \leq C_0, \quad n \in \mathbb{N}, \tag{2.10a}$$

i.e., the gradients are globally bounded. Additionally assume uniform convergence

$$\|\zeta_n\|_{C(\Gamma)} \xrightarrow[n\to\infty]{} 0. \tag{2.10b}$$

From now on, we denote by $\Gamma^n = \Gamma^{\zeta_n}$ and $q^n = q^{\zeta_n}$.

For the forcing terms we assume there exists an open set G containing $\{\Gamma^n\}$ and an element $\Phi \in \mathbf{H}_{\text{div}}(G)$; with G an open region such that $\Phi \cdot \widehat{\boldsymbol{n}}|_{\Gamma^n} = f$ for all n. Here $\widehat{\boldsymbol{n}}$ denotes the upwards normal vector to Γ^n and

$$\mathbf{H}_{\text{div}}(G) = \{\mathbf{v} \in \mathbf{L}^2(G) : \boldsymbol{\nabla}\cdot\mathbf{v} \in L^2(G)\},$$

where $\mathbf{L}^2(G) = (L^2(G))^N$ is the Lebesgue space of vector-valued functions. Given $\widetilde{\mathbf{x}} \in \Gamma$ we denote by $\widetilde{\mathbf{x}} \times (0,\zeta_n(\widetilde{\mathbf{x}})) \cup \widetilde{\mathbf{x}} \times (\zeta_n(\widetilde{\mathbf{x}}),0)$ the sections in $\mathbb{R}^N$ understanding that

$$\widetilde{\mathbf{x}} \times (0,\zeta_n(\widetilde{\mathbf{x}})) \cup \widetilde{\mathbf{x}} \times (\zeta_n(\widetilde{\mathbf{x}}),0) = \begin{cases} \widetilde{\mathbf{x}} \times (0,\zeta_n(\widetilde{\mathbf{x}})), & \text{if } \zeta_n(\widetilde{\mathbf{x}})) < 0, \\ \widetilde{\mathbf{x}} \times (\zeta_n(\widetilde{\mathbf{x}}),0), & \text{if } \zeta_n(\widetilde{\mathbf{x}})) > 0, \\ \emptyset, & \text{if } \zeta_n(\widetilde{\mathbf{x}})) = 0. \end{cases} \tag{2.11}$$

Define

$$U^n \overset{\mathbf{def}}{=} \bigcup_{\widetilde{\mathbf{x}}\in\Gamma} \widetilde{\mathbf{x}} \times (0,\zeta_n(\widetilde{\mathbf{x}})) \cup \widetilde{\mathbf{x}} \times (\zeta_n(\widetilde{\mathbf{x}}),0). \tag{2.12}$$

Hence, $\partial U^n = \Gamma^n \cup \Gamma$. Due to the condition (2.10a) the domain U^n has Lipschitz boundary, then the classical duality relationship [23] holds, i.e.,

$$\int\limits_{\Gamma^n} f\, q^n\, dS - \int\limits_{\Gamma} f\, q^n\, dS = \int\limits_{\partial U^n} \Phi\cdot\widehat{\boldsymbol{n}}\, q^n\, dS = \int\limits_{U^n} \boldsymbol{\nabla}\cdot\Phi\, q^n + \Phi\cdot\boldsymbol{\nabla} q^n.$$

Therefore

$$\begin{aligned} \int_{\Gamma^n} f\, q^n \, dS &\leq \|f\|_{-\frac{1}{2},\Gamma} \, \|q^n\|_{\frac{1}{2},\Gamma} + \|\Phi\|_{\mathbf{H}_{\mathrm{div}}(\Gamma)} \|q^n\|_{1,\Omega} \\ &\leq \{\|f\|_{-\frac{1}{2},\Gamma} + \|\Phi\|_{\mathbf{H}_{\mathrm{div}}(\Gamma)}\} \|q^n\|_{1,\Omega}. \end{aligned} \tag{2.13}$$

Combining (2.13) with (2.9) gives

$$\| q^n\|_{1,\,\Omega} \leq \{C_\Omega \|F\|_{0,\Omega} + \|f\|_{-\frac{1}{2},\Gamma} + \|\Phi\|_{\mathbf{H}_{\mathrm{div}}(\Gamma)}\}. \tag{2.14}$$

Due to the Rellich-Kondrachov theorem, there must exist a subsequence, denoted by $\{q^k\}$, and an element $q^* \in H^1(\Omega)$ such that

$$q^k \to q^* \text{ weekly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega).$$

Denoting $\Omega_i^k = \Omega_i^{\zeta_k}$ for $i = 1, 2$, the variational statement can be written as

$$\int_\Omega \boldsymbol{\nabla} q^k \cdot \boldsymbol{\nabla} r \, \mathbb{1}_{\Omega_1^k} + \frac{1}{\epsilon} \int_\Omega \boldsymbol{\nabla} q^k \cdot \boldsymbol{\nabla} r \, \mathbb{1}_{\Omega_2^k} = \int_\Omega F\, r + \int_{\Gamma^k} f\, r \, d\, S \tag{2.15}$$

for $r \in V$ arbitrary. Let $\zeta_k \to 0$ in $C(\Gamma)$, first observe that for any $r \in H^1(\Omega)$ holds

$$\left| \int_{\Gamma^k} f\, r\, dS - \int_\Gamma f\, r\, dS \right| = \left| \int_{U^k} \boldsymbol{\nabla} \cdot \Phi\, r + \Phi \cdot \boldsymbol{\nabla} r \right| \leq \|\Phi\|_{\mathbf{H}_{\mathrm{div}}(U^k)} \|r\|_{H^1(U^k)}.$$

Since the right hand side converges to 0 as $k \to \infty$ it follows that $\int_{\Gamma^k} f\, r\, dS \to \int_\Gamma f\, r\, dS$. Next observe that $\{\boldsymbol{\nabla} r \, \mathbb{1}_{\Omega_i^k}\}$ converges strongly in $\mathbf{L}^2(\Omega)$; together with the convergence of the surface forcing terms previously discussed, the expression (2.15) converges to

$$\int_\Omega \boldsymbol{\nabla} q^* \cdot \boldsymbol{\nabla} r \, \mathbb{1}_{\Omega_1} + \frac{1}{\epsilon} \int_\Omega \boldsymbol{\nabla} q^* \cdot \boldsymbol{\nabla} r \, \mathbb{1}_{\Omega_2} = \int_\Omega F\, r + \int_\Gamma f\, r\, dS.$$

Since q^* is in V and the variational statement above holds for all $r \in V$, the uniqueness of the solution of the problem (2.8a), implies that $q^* = p$. The reasoning above holds for any subsequence of $\{q^n\}$ and the solution of (2.8a) is unique, then it follows that the whole sequence converges to p, i.e.,

$$q^n \to p \text{ weekly in } H^1(\Omega), \text{ strongly in } L^2(\Omega). \tag{2.16}$$

We close the section with an important observation. Testing the statements (2.8b) on the diagonal q^n and letting $n \to \infty$ yields

$$\begin{aligned} \lim_{n\to\infty} \left\{ \int_{\Omega_1^n} |\boldsymbol{\nabla} q^n|^2 + \frac{1}{\epsilon} \int_{\Omega_2^n} |\boldsymbol{\nabla} q^n|^2 \right\} &= \int_\Omega F\, p + \int_\Gamma f\, p\, dS \\ &= \int_{\Omega_1} |\boldsymbol{\nabla} p|^2 + \frac{1}{\epsilon} \int_{\Omega_2} |\boldsymbol{\nabla} p|^2. \end{aligned} \tag{2.17}$$

The maps

$$r \mapsto \left\{ \int_{\Omega_1} |\boldsymbol{\nabla} r|^2 + \frac{1}{\epsilon} \int_{\Omega_2} |\boldsymbol{\nabla} r|^2 \right\}^{\frac{1}{2}}$$

and

$$r \mapsto \left\{ \int_{\Omega_1^n} |\boldsymbol{\nabla} r|^2 + \frac{1}{\epsilon} \int_{\Omega_2^n} |\boldsymbol{\nabla} r|^2 \right\}^{\frac{1}{2}},$$

for any $n \in \mathbb{N}$, are norms equivalent to the norm $\|\cdot\|_V$. However, due to the presence of the domains Ω_i^n, $i = 1, 2$ the equality (2.17) is not a statement of norms convergence which, together with the weak convergence, would allow us to conclude strong convergence. However, due to the weak convergence and the equivalence of the norms $r \mapsto \{\int_{\Omega_1} |\boldsymbol{\nabla} r|^2 + \frac{1}{\epsilon}\int_{\Omega_2} |\boldsymbol{\nabla} r|^2\}^{\frac{1}{2}}$ and $\|\cdot\|_V$, we can conclude that

$$\int_{\Omega_1} |\boldsymbol{\nabla} p|^2 + \frac{1}{\epsilon} \int_{\Omega_2} |\boldsymbol{\nabla} p|^2 \leq \liminf_n \left\{ \int_{\Omega_1} |\boldsymbol{\nabla} q^n|^2 + \frac{1}{\epsilon} \int_{\Omega_2} |\boldsymbol{\nabla} q^n|^2 \right\}. \tag{2.18}$$

2.4. THE STRONG CONVERGENCE

Given a function $r \in V$ consider the following identities

$$\int_{\Omega_1^\zeta} |\boldsymbol{\nabla} r|^2 = \int_{\Omega_1^\zeta - \Omega_1} |\boldsymbol{\nabla} r|^2 - \int_{\Omega_2^\zeta - \Omega_2} |\boldsymbol{\nabla} r|^2 + \int_{\Omega_1} |\boldsymbol{\nabla} r|^2, \tag{2.19a}$$

$$\int_{\Omega_2^\zeta} |\boldsymbol{\nabla} r|^2 = \int_{\Omega_2^\zeta - \Omega_2} |\boldsymbol{\nabla} r|^2 - \int_{\Omega_1^\zeta - \Omega_1} |\boldsymbol{\nabla} r|^2 + \int_{\Omega_2} |\boldsymbol{\nabla} r|^2. \tag{2.19b}$$

We define the perturbation term as

$$\Xi_\zeta(r) \overset{\textbf{def}}{=} \int_{\Omega_2^\zeta - \Omega_2} |\boldsymbol{\nabla} r|^2 - \int_{\Omega_1^\zeta - \Omega_1} |\boldsymbol{\nabla} r|^2. \tag{2.20}$$

Moreover, the perturbation term satisfies

$$\begin{aligned} |\Xi_\zeta(r)| = \left| \int_{\Omega_2^\zeta - \Omega_2} |\boldsymbol{\nabla} r|^2 - \int_{\Omega_1^\zeta - \Omega_1} |\boldsymbol{\nabla} r|^2 \right| &\leq \left| \int_{\Omega_2^\zeta - \Omega_2} |\boldsymbol{\nabla} r|^2 \right| + \left| \int_{\Omega_1^\zeta - \Omega_1} |\boldsymbol{\nabla} r|^2 \right| \\ &\leq \|r\|_V^2 \left[\lambda_N(\Omega_1^\zeta - \Omega_1) + \lambda_N(\Omega_2^\zeta - \Omega_2) \right]. \end{aligned} \tag{2.21}$$

Therefore, the following estimate holds

$$\begin{aligned}\int\limits_{\Omega_1^\zeta} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2^\zeta} |\nabla r|^2 &= \int\limits_{\Omega_1} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla r|^2 + \left(1-\frac{1}{\epsilon}\right)\Xi_\zeta(r)\\ &\geq \int\limits_{\Omega_1} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla r|^2\\ &\quad - \left|1-\frac{1}{\epsilon}\right|\left[\lambda_N(\Omega_1^\zeta - \Omega_1) + \lambda_N(\Omega_2^\zeta - \Omega_2)\right]\|r\|_V^2.\end{aligned}$$

We know that $\|r\|_V^2 \leq \{\int_{\Omega_1}|\nabla r|^2 + \frac{1}{\epsilon}\int_{\Omega_2}|\nabla r|^2\}$. This, combined with the expression above gives

$$\begin{aligned}&\int\limits_{\Omega_1^\zeta} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2^\zeta} |\nabla r|^2\\ &\geq \left(1-\left|1-\frac{1}{\epsilon}\right|\left[\lambda_N(\Omega_1^\zeta - \Omega_1) + \lambda_N(\Omega_2^\zeta - \Omega_2)\right]\right)\left\{\int\limits_{\Omega_1} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla r|^2\right\}.\end{aligned}$$

Defining

$$C_\zeta \overset{\mathbf{def}}{=} \left|1-\frac{1}{\epsilon}\right|\left[\lambda_N(\Omega_1^\zeta - \Omega_1) + \lambda_N(\Omega_2^\zeta - \Omega_2)\right], \tag{2.22}$$

we get the estimate

$$\int\limits_{\Omega_1^\zeta} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2^\zeta} |\nabla r|^2 \geq (1-C_\zeta)\left\{\int\limits_{\Omega_1} |\nabla r|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla r|^2\right\} \quad \text{for all} \quad r\in V. \tag{2.23}$$

In particular, for the sequence of solutions $\{q^n : n\in\mathbb{N}\}\subseteq V$, it holds that

$$(1-C_{\zeta_n})\left\{\int\limits_{\Omega_1} |\nabla q^n|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla q^n|^2\right\} \leq \int\limits_{\Omega_1^n} |\nabla q^n|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2^n} |\nabla q^n|^2.$$

Letting $n\to\infty$ it follows that $\|\zeta_n\|_{C(\Gamma)}\to 0$. Consequently $C_{\zeta_n}\to 0$. Therefore, taking the limit superior in the expression above yields

$$\begin{aligned}\limsup_{n\to\infty}\left\{\int\limits_{\Omega_1} |\nabla q^n|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla q^n|^2\right\} &\leq \limsup_{n\to\infty}\left\{\int\limits_{\Omega_1^n} |\nabla q^n|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2^n} |\nabla q^n|^2\right\}\\ &= \int\limits_{\Omega_1} |\nabla p|^2 + \frac{1}{\epsilon}\int\limits_{\Omega_2} |\nabla p|^2,\end{aligned} \tag{2.24}$$

where the last equality holds due to (2.17). Putting together (2.18) and (2.24) we conclude that

$$\lim_{n\to\infty}\left\{\int_{\Omega_1}|\boldsymbol{\nabla} q^n|^2+\frac{1}{\epsilon}\int_{\Omega_2}|\boldsymbol{\nabla} q^n|^2\right\}=\int_{\Omega_1}|\boldsymbol{\nabla} p|^2+\frac{1}{\epsilon}\int_{\Omega_2}|\boldsymbol{\nabla} p|^2, \tag{2.25}$$

i.e., the norms $r\mapsto\{\int_{\Omega_1}|\boldsymbol{\nabla} r|^2+\frac{1}{\epsilon}\int_{\Omega_2}|\boldsymbol{\nabla} r|^2\}^{\frac{1}{2}}$ converge and, due to the equivalence with the V-norm it follows that $\|q^n\|_V\to\|p\|_V$. Finally, since $\{q^n\}$ converges weakly to p in V it follows that

$$\|q^n-p\|_V^2\to 0 \quad \text{as } \; n\to\infty. \tag{2.26}$$

3. THE ONE DIMENSIONAL CASE

Here, we restrict our attention to the one dimensional problem in order to gain deep insight on the phenomenon. An example of how valuable this approach is, can be found in [6]. For the problem in one dimensional setting we choose $\Omega_1\overset{\mathbf{def}}{=}(-1,0)$, $\Omega_2\overset{\mathbf{def}}{=}(0,1)$ and the interface $\Gamma=\{0\}$. In this context a perturbation is given by a single point $\zeta\in(-1,1)$; the perturbed domains are given by $\Omega_1^\zeta\overset{\mathbf{def}}{=}(-1,\zeta)$, $\Omega_2^\zeta\overset{\mathbf{def}}{=}(\zeta,1)$ and the perturbed interface $\Gamma^\zeta\overset{\mathbf{def}}{=}\{\zeta\}$. Clearly $\Omega=\Omega_1\cup\Gamma\cup\Omega_2=\Omega_1^\zeta\cup\Gamma^\zeta\cup\Omega_2^\zeta=(-1,1)$, see Figure 2. Notice that in the one dimensional case, the space V and its inner product given in Definition 2.3, reduce to

$$V=\left\{r\in H^1(-1,1):r(-1)=0\right\}, \tag{3.1a}$$

$$\langle\pi,\,\kappa\rangle_V=\int_{-1}^{1}\boldsymbol{\partial}\pi\,\boldsymbol{\partial}\kappa, \tag{3.1b}$$

where $\boldsymbol{\partial}$ indicates the weak derivative. Similarly, the problems (2.8a) and (2.8b) transform in

$$p\in V:\quad\int_{-1}^{0}\boldsymbol{\partial}p\,\boldsymbol{\partial}r+\frac{1}{\epsilon}\int_{0}^{1}\boldsymbol{\partial}p\,\boldsymbol{\partial}r=\int_{-1}^{1}F\,r+f\,(0)\,r\,(0)\quad\text{for all}\quad r\in V, \tag{3.2a}$$

$$q\in V:\quad\int_{-1}^{\zeta}\boldsymbol{\partial}q\,\boldsymbol{\partial}r+\frac{1}{\epsilon}\int_{\zeta}^{1}\boldsymbol{\partial}q\,\boldsymbol{\partial}r=\int_{-1}^{1}F\,r+f\,(\zeta)\,r\,(\zeta)\quad\text{for all}\quad r\in V. \tag{3.2b}$$

3.1. THE SUBSPACE H AND ITS ORTHOGONAL PROJECTION

In order to estimate the norms $\|p-q^\zeta\|_{1,\Omega}$, $\|p-q^\zeta\|_{0,\Omega}$, we need to project the solutions p and q^ζ into the adequate subspace, using the convenient geometry defined by the inner product (3.1b). For simplicity, from now on it will be assumed that $\zeta > 0$. Consider the subspaces

$$H \overset{\textbf{def}}{=} \{\kappa \in V : \boldsymbol{\partial}\kappa = 0 \text{ on } (0, \zeta)\}\,, \tag{3.3a}$$

$$H^{\perp} \overset{\textbf{def}}{=} \{\pi \in V : \langle \pi, \kappa \rangle_V = 0 \text{ for all } \kappa \in H\}\,. \tag{3.3b}$$

Next, we characterize the structure of $H^{\perp}$.

Lemma 3.1. *Let $H^{\perp}$ and H defined in* (3.3). *Then*

$$H^{\perp} = \{\pi \in V : \pi = 0 \textit{ on } (-1, 0),\ \boldsymbol{\partial}\pi = 0 \textit{ on } (\zeta, 1)\}\,. \tag{3.4}$$

Proof. It is direct to see that if $\pi \in V$ is such that $\pi = 0$ in $(-1, 0)$ and $\boldsymbol{\partial}\pi = 0$ in $(\zeta, 1)$ then $\pi \in H^{\perp}$. For the other inclusion take $\rho \in C_0^{\infty}(-1,\, 0)$ such that $\int_{-1}^{\mathbf{0}} \rho\, dx = 1$ and extend it by zero to the whole domain $(-1,\, 1)$. Choose any $\phi \in C_0^{\infty}(-1,\, 0)$, extend it by zero to $(-1,\, 1)$ and build the auxiliary function

$$\Phi(x) \overset{\textbf{def}}{=} \int_{-1}^{x} \phi(t)\, dt\, \mathbb{1}_{(-1,0)}(x) - \int_{-1}^{0} \phi(y)\, dy \int_{-1}^{x} \rho(t)\, dt\, \mathbb{1}_{(-1,0)}(x).$$

It is easy to see that $\Phi \in H$. Now take any $\pi \in H^{\perp}$, then

$$\begin{aligned} 0 = \langle \pi,\, \Phi \rangle_V &= \int_{-1}^{0} \boldsymbol{\partial}\pi\, \boldsymbol{\partial}\Phi = \int_{-1}^{0} \boldsymbol{\partial}\pi\,(x) \left[\phi\,(x) - \left(\int_{-1}^{0} \phi(y)\, dy \right) \rho(x) \right] dx \\ &= \int_{-1}^{0} \boldsymbol{\partial}\pi\,(x)\, \phi\,(x)\, dx - \int_{-1}^{0} \phi(y)\, dy \int_{-1}^{0} \boldsymbol{\partial}\pi\,(x)\, \rho(x)\, dx, \end{aligned}$$

i.e.,

$$\int_{-1}^{0} \boldsymbol{\partial}\pi\,(x)\, \phi\,(x)\, dx = \int_{-1}^{0} \phi(y)\, dy \int_{-1}^{0} \boldsymbol{\partial}\pi\,(x)\, \rho(x)\, dx$$

for all $\phi \in C_0^{\infty}(-1,\, 0)$. Therefore, we conclude $\boldsymbol{\partial}\pi$ must be constant in $(-1,\, 0)$. Using an analogous construction we also conclude $\boldsymbol{\partial}\pi$ must be constant in $(\zeta,\, 1)$. Now we prove that such constants must be zero. Consider any $\kappa \in H$, then we have

$$\begin{aligned} 0 = \langle \pi,\, \kappa \rangle_V &= \int_{-1}^{0} \boldsymbol{\partial}\pi\, \boldsymbol{\partial}\kappa + \int_{\zeta}^{1} \boldsymbol{\partial}\pi\, \boldsymbol{\partial}\kappa = \boldsymbol{\partial}\pi(-1) \int_{-1}^{0} \boldsymbol{\partial}\kappa + \boldsymbol{\partial}\pi(1) \int_{\zeta}^{1} \boldsymbol{\partial}\kappa \\ &= \boldsymbol{\partial}\pi(-1)\, \left(\kappa\,(0) - \kappa\,(-1)\right) + \boldsymbol{\partial}\pi(1)\, \left(\kappa\,(1) - \kappa\,(\zeta)\right). \end{aligned}$$

Since $\kappa \in H \subset V$, it holds $\kappa\,(-1) = 0$ and the above expression writes

$$\boldsymbol{\partial}\,\pi(-1)\, \kappa\,(0) + \boldsymbol{\partial}\,\pi(1)\, \left(\kappa\,(1) - \kappa\,(\zeta)\right) = 0. \tag{3.5}$$

Due to $\boldsymbol{\partial}\kappa = 0$ on $(0, \zeta)$, the function κ must be constant on this interval, therefore $\kappa(0) = \kappa(\zeta)$. Recalling that the above holds for any $\kappa \in H$, choose a test function such that $\kappa(0) = \kappa(\zeta) = 0$ and $\kappa(1) \neq 0$, then (3.5) reduces to $\boldsymbol{\partial}\pi(1)\,\kappa(1) = 0$ and we conclude $\boldsymbol{\partial}\pi(1) = 0$. Hence, (3.5) reduces to $\boldsymbol{\partial}\pi(-1)\,\kappa(0) = 0$. Since $\kappa \in H$ is arbitrary, we know that $\kappa(0)$ need not be zero for all $\kappa \in H$, then we conclude $\boldsymbol{\partial}\pi(-1) = 0$. Therefore, π must be constant on the intervals $(-1, 0)$ and $(\zeta, 1)$. Finally, the fact that $\pi \in H^{\perp} \subset V$ yields $\pi(-1) = 0$; this implies $\pi = 0$ on $(-1, 0)$ which completes the proof. □

Now we present the characterization of the orthogonal projections onto the subspaces H and $H^{\perp}$.

Theorem 3.2. *Let $H, H^{\perp}$ be the orthogonal complementary subspaces of V defined in* (3.3)*. Denote by P_H and $P_{H^{\perp}}$ the orthogonal projections onto H and $H^{\perp}$, respectively. Then, for any $r \in V$, it holds that*

$$P_H\, r(x) = r(x)\mathbb{1}_{[-1,\,0]}(x) + r(0)\,\mathbb{1}_{[0,\,\zeta]}(x) + \{r(x) - [r(\zeta) - r(0)]\}\,\mathbb{1}_{[\zeta,\,1]}(x), \tag{3.6a}$$

$$P_{H^{\perp}} r(x) = [r(x) - r(0)]\,\mathbb{1}_{[0,\,\zeta]}(x) + [r(\zeta) - r(0)]\,\mathbb{1}_{[\zeta,\,1]}(x), \tag{3.6b}$$

where $\mathbb{1}_A(\cdot)$ denotes the indicator function of the set A.

Proof. For any $r \in V$, it is direct to see that the function

$$x \mapsto r(x)\mathbb{1}_{[-1,\,0]}(x) + r(0)\,\mathbb{1}_{[0,\,\zeta]}(x) + \{r(x) - [r(\zeta) - r(0)]\}\,\mathbb{1}_{[\zeta,\,1]}(x)$$

is in H and that the map

$$x \mapsto [r(x) - r(0)]\;\mathbb{1}_{[0,\,\zeta]}(x) + [r(\zeta) - r(0)]\;\mathbb{1}_{[\zeta,\,1]}(x)$$

belongs to $H^{\perp}$. Also, their sum gives r. The result follows due to the characterization given in Lemma 3.1. □

Remark 3.3. In order to better understand the nature of the orthogonal decomposition we present Figure 2. An absolutely continuous function r (blue line) is decomposed in $P_H\, r$ (turquoise line) and $P_{H^{\perp}} r = (I - P_H) r$ (red line). Also, the domains Ω_1, Ω_2 and $\Omega_1^{\zeta}, \Omega_2^{\zeta}$ are depicted.

3.2. THE PROBLEMS RESTRICTED TO H

Test the problem (3.2a) with a function $\kappa \in H$, this gives

$$\int_{-1}^{0} \boldsymbol{\partial} p\,\boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial} p\,\boldsymbol{\partial}\kappa = \int_{-1}^{1} F\,\kappa + f(0)\,\kappa(0)\,.$$

Now decompose p in $P_H p$ and $P_{H^{\perp}} p$ using (3.6a) and (3.6b), we get

$$\int_{-1}^{0} \boldsymbol{\partial}\,(P_H\, p)\;\boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}\,(P_H\, p)\;\boldsymbol{\partial}\kappa = \int_{-1}^{0} F\,\kappa + \int_{\zeta}^{1} F\,\kappa + \left[\int_{0}^{\zeta} F + f(0)\right]\kappa(0)\,.$$

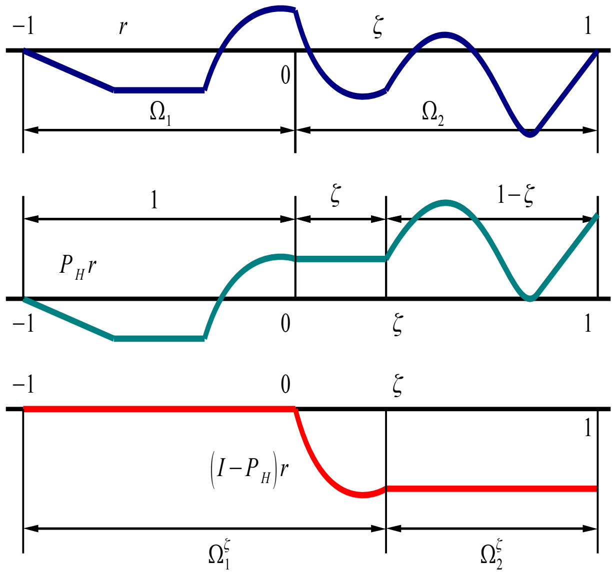

Fig. 2. Orthogonal decomposition

Here, the last equality used the fact that κ is constant in $(0,\zeta)$ for all $\kappa \in H$. We write the statement as

$$
\begin{aligned}
P_H p \in H : &\int_{-1}^{0} \boldsymbol{\partial}\,(P_H\, p)\ \boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}\,(P_H\, p)\ \boldsymbol{\partial}\kappa \\
&= \int_{-1}^{0} F\,\kappa + \int_{\zeta}^{1} F\,\kappa + \left[\int_{0}^{\zeta} F + f\,(0)\right]\kappa\,(0) \quad \text{for all} \quad \kappa \in H.
\end{aligned}
\tag{3.7}
$$

On the other hand, consider the problem

$$
\begin{aligned}
\sigma \in H : &\int_{-1}^{0} \boldsymbol{\partial}\sigma\ \boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}\sigma\ \boldsymbol{\partial}\kappa \\
&= \int_{-1}^{0} F\,\kappa + \int_{\zeta}^{1} F\,\kappa + \left[\int_{0}^{\zeta} F + f\,(0)\right]\kappa\,(0) \quad \text{for all} \quad \kappa \in H.
\end{aligned}
\tag{3.8}
$$

The bilinear the form $\mathcal{A}\,(\pi,\kappa) \overset{\mathbf{def}}{=} \int_{-1}^{0} \boldsymbol{\partial}\,\pi\,\boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}\pi\,\boldsymbol{\partial}\kappa$ is H-elliptic and continuous, therefore the problem (3.8) is well-posed and we conclude that $P_H p$ is the unique solution of (3.8). Repeating the same procedure on the perturbed problem (3.2b) we conclude $P_H q$ is the unique solution to the well-posed variational problem

$$
\begin{aligned}
P_H q \in H : \int_{-1}^{0} \boldsymbol{\partial}\, (P_H\, q)\ \boldsymbol{\partial}\kappa + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}\,(P_H\, q)\ \boldsymbol{\partial}\kappa \\
= \int_{-1}^{0} F\,\kappa\, dx + \int_{\zeta}^{1} F\,\kappa + \left[\int_{0}^{\zeta} F + f\,(\zeta)\right]\kappa\,(\zeta) \quad \text{for all} \quad \kappa \in H.
\end{aligned}
\tag{3.9}
$$

3.3. THE PROBLEMS RESTRICTED TO $H^{\perp}$

We repeat the same strategy of the previous section and get

$$
P_{H^{\perp}} p \in H^{\perp} : \frac{1}{\epsilon}\int_{0}^{\zeta} \boldsymbol{\partial}\, (P_{H^{\perp}}\, p)\ \boldsymbol{\partial}\kappa = \int_{0}^{\zeta} F\,\kappa + \int_{\zeta}^{1} F\,\kappa\,(\zeta) \quad \text{for all} \quad \kappa \in H^{\perp}. \tag{3.10}
$$

This is the weak solution of following strong problem

$$
\begin{aligned}
-\boldsymbol{\partial}\,\frac{1}{\epsilon}\,\boldsymbol{\partial} P_{H^{\perp}} p &= F \ \text{in} \ (0,\, \zeta)\,, \\
P_{H^{\perp}} p &= 0 \ \text{in} \ [-1,\, 0)\,, \\
P_{H^{\perp}} p &= \text{constant} \ \text{in} \ (\zeta,\, 1)\,, \\
\frac{1}{\epsilon}\,\boldsymbol{\partial} P_{H^{\perp}} p\left(\zeta^{-}\right) &= \int_{\zeta}^{1} F,
\end{aligned}
\tag{3.11}
$$

where $\boldsymbol{\partial} P_{H^{\perp}} p\,(\zeta^{-}) = \lim_{t\to\zeta^{-}} \boldsymbol{\partial} P_{H^{\perp}} p(t)$. In the same fashion

$$
P_{H^{\perp}} q \in H^{\perp} : \int_{0}^{\zeta} \boldsymbol{\partial}\, (P_{H^{\perp}} q)\ \boldsymbol{\partial}\kappa = \int_{0}^{\zeta} F\,\kappa + \left[\int_{\zeta}^{1} F + f\,(\zeta)\right]\kappa(\zeta) \quad \text{for all} \quad \kappa \in H^{\perp}, \tag{3.12}
$$

where the simplification on the term of the right hand side has been made, since $\kappa(x) = \kappa\,(\zeta)$ for $x \in (\zeta,\, 1)$. Thus $P_{H^{\perp}} q$ is the solution to the strong problem

$$
\begin{aligned}
-\boldsymbol{\partial}\,\boldsymbol{\partial} P_{H^{\perp}} q &= F \ \text{in} \ (0,\, \zeta)\,, \\
P_{H^{\perp}} q &= 0 \ \text{in} \ [-1,\, 0)\,, \\
P_{H^{\perp}} q &= \text{constant} \ \text{in} \ (\zeta,\, 1)\,, \\
\boldsymbol{\partial} P_{H^{\perp}} q\left(\zeta^{-}\right) &= \int_{\zeta}^{1} F + f\,(\zeta)\,.
\end{aligned}
\tag{3.13}
$$

3.4. ESTIMATES FOR THE H PROJECTIONS

Test (3.7) and (3.9) with $P_H p - P_H q$ and subtract the result to get

$$\int_{-1}^{0} \boldsymbol{\partial}(P_H p - P_H q)\, \boldsymbol{\partial}(P_H p - P_H q) + \frac{1}{\epsilon}\int_{\zeta}^{1} \boldsymbol{\partial}(P_H p - P_H\, q)\, \boldsymbol{\partial}(P_H p - P_H q)$$
$$= f\,(0)\,[P_H p(0) - P_H q(0)] - f\,(\zeta)\,[P_H p(\zeta) - P_H q(\zeta)]$$
$$= [f\,(0) - f\,(\zeta)]\,[P_H p(0) - P_H q(0)]\,.$$

The last equality holds true, since $\kappa\,(0) = \kappa\,(\zeta)$ for all $\kappa \in H$. Since $|r(x)| \leq \sqrt{2}\,\|\boldsymbol{\partial} r\|_{0,(-1,1)}$ for all $r \in V$, the expression above can be estimated by

$$\|\, P_H p - P_H\, q\|_V \leq C\,|f\,(0) - f(\zeta)|. \tag{3.14}$$

3.5. ESTIMATES FOR THE $H^{\perp}$ PROJECTIONS

Since (3.11) and (3.13) are both ordinary differential equations, the exact solutions can be found. These are given by

$$\begin{aligned} P_{H^\perp} p(x) = &-\epsilon \int_0^x\!\!\int_0^t F(s)\, ds\, dt\, \mathbb{1}_{[0,\,\zeta\,]}(x) \\ &+ \epsilon\, x \int_0^1 F\, \mathbb{1}_{[0,\,\zeta\,]}(x) - \epsilon\, \zeta \left[\int_0^\zeta\!\!\int_0^t F(s)\, ds\, dt - \int_0^1 F \right] \mathbb{1}_{[\zeta,\,1]}(x), \end{aligned} \tag{3.15}$$

$$\begin{aligned} P_{H^\perp} q(x) = &- \int_0^x\!\!\int_0^t F(s)\, ds\, dt\, \mathbb{1}_{[0,\,\zeta\,]}(x) + \left[\int_0^1 F + f(\zeta) \right] x\, \mathbb{1}_{[0,\,\zeta\,]}(x) \\ &- \left\{ \int_0^\zeta\!\!\int_0^t F(s)\, ds\, dt - \left[\int_0^1 F + f(\zeta) \right] \zeta \right\} \mathbb{1}_{[\zeta,\,1]}(x). \end{aligned} \tag{3.16}$$

We estimate the norm of the difference using the exact expressions (3.15) and (3.16). When computing the L^2-norm of the difference of derivatives we get

$$\begin{aligned} \|P_{H^\perp} p - P_{H^\perp} q\|_V &= \|\boldsymbol{\partial} P_{H^\perp} p - \boldsymbol{\partial} P_{H^\perp} q\|_{L^2(0,\zeta)} \\ &\leq (1-\epsilon) \left\| \int_0^{(\cdot)} F(t)\, dt\, \mathbb{1}_{[0,\,\zeta\,]}(\cdot) \right\|_{L^2(0,\,\zeta)} + \sqrt{\zeta} \left[(1-\epsilon) \int_0^1 F(t)\, dt + f(\zeta) \right] \\ &\leq \frac{\zeta(1-\epsilon)}{\sqrt{2}}\, \|F\|_{L^2(0,\,\zeta)} + \zeta \left| f(\zeta) + (1-\epsilon) \int_0^1 F \right|. \end{aligned}$$

Then we conclude that

$$\|P_{H^\perp} p - P_{H^\perp} q\|_V \leq C\,\zeta\,\{\|F\|_{L^2(-1,1)} + |f(\zeta)|\}, \tag{3.17}$$

where $C > 0$ is an adequate constant.

3.6. GLOBAL ESTIMATE OF THE PERTURBATION

For the global estimate of the difference recall $\|\mathbf{y}\|_2 \leq \|\mathbf{y}\|_1$ for all $\mathbf{y} \in \mathbb{R}^2$, and combine the estimates (3.14), (3.17) this gives

$$\begin{aligned} \|\, p - q\|_V &= \{\|P_H(p-q)\|_V^2 + \|P_{H^\perp}(p-q)\|_V^2\}^{\frac{1}{2}} \\ &\leq \|\, P_H p - P_H q\|_V + \|\, P_{H^\perp} p - P_{H^\perp} q\|_V \\ &\leq \sqrt{2}\,|f(0) - f(\zeta)| + C\,\zeta\,\{\|F\|_{L^2(-1,1)} + |f(\zeta)|\}. \end{aligned} \tag{3.18}$$

In order to have continuous dependence of the solutions with respect to perturbations of the interface, it is direct to see that the finiteness of $\|F\|_{L^2(-1,1)}$ is necessary, and that conditions on the forcing term f behavior need to be stated. In the last line of inequality (3.18), the third summand needs f to be bounded in a neighborhood $[0,\delta)$ while the first summand demands it to be right continuous in a neighborhood $[0,\delta)$ for some $\delta > 0$. Recalling $H_{\text{div}}(0,\delta) = H^1(\delta)$ in one dimension, the hypothesis that $f = \boldsymbol{\partial}\Phi$ for some $\Phi \in H_{\text{div}}(0,\delta)$ assumed in Section 2.3 is sufficient to satisfy these conditions, however, it is not necessary.

Finally, a repetition of the same procedure for perturbations to the left, i.e., when $\zeta < 0$ yields

$$\|\, p - q\|_V \leq \sqrt{2}\,|f(0) - f(\zeta)| + C\,|\zeta|\{\|F\|_{L^2(-1,1)} + |f(0)|\}. \tag{3.19}$$

The section is summarized in the following result.

Theorem 3.4. *Let $F \in L^2(-1,1)$ and $f \in C(-\delta,\delta)$ for some $\delta > 0$, then*

$$\|p - q^\zeta\|_V \xrightarrow[\zeta \to 0]{} 0, \tag{3.20}$$

i.e., the sequence of perturbed solutions $\{q^\zeta\}$ converge strongly to the original one.

Proof. The estimate (3.19) together with the hypotheses on the forcing terms yield the desired results. □

Remark 3.5. It is important to stress that the successful analysis in the one dimensional case, heavily relies on the dimension itself. The characterization of the right space H and its orthogonal projections can not be done in a multi dimensional setting, even in a very simple geometric domain such as the unit ball or the unit square. Also, the solutions provided by equations (3.15), (3.16) are possible only due to the one dimensional framework.

Remark 3.6. The estimates (3.18) and (3.19) heavily depend on the pointwise behavior of f, e.g. if $f(x) \equiv C\,x^s$ for $s > 0$ very small, the rate of convergence is very slow. They also reveal the nonlinear dependence of the solution q with respect to ζ.

4. A SIMPLE GEOMETRY IN MULTIPLE DIMENSIONAL SETTING

In this section we choose the simplest possible geometry in multiple dimensions, in order to illustrate the nonlinearities that the phenomenon of interface geometric perturbation involves. Let $\Gamma \subseteq \mathbb{R}^{N-1}$ be open connected, $\Omega_1 = \Gamma \times (0,1)$ and $\Omega_2 = \Gamma \times (-1,0)$, i.e., the domain on the right hand side of Figure 3. Also assume that the perturbation ζ is piecewise $C^1(\Gamma)$. We exploit the geometry defining the fractional bijective maps $\Lambda_i : \Omega_i^\zeta \to \Omega_i$ for $i = 1,2$ as follows:

$$\Lambda_i(\widetilde{\mathbf{X}}, X_N) \overset{\mathbf{def}}{=} \left(\widetilde{\mathbf{X}}, \frac{X_N - \zeta(\widetilde{\mathbf{X}})}{1 - (-1)^i \zeta(\widetilde{\mathbf{X}})}\right). \tag{4.1a}$$

Also define

$$\Lambda(\mathbf{X}) \overset{\mathbf{def}}{=} \sum_{i=1,2} \Lambda_i(\mathbf{X}) \mathbb{1}_{\Omega_i}(\mathbf{X}). \tag{4.1b}$$

Now set the new variables

$$z \overset{\mathbf{def}}{=} \Lambda(\widetilde{\mathbf{X}}, X_N) \cdot \widehat{\boldsymbol{e}}_N, \tag{4.2}$$

$$\widetilde{\mathbf{x}} \overset{\mathbf{def}}{=} \Lambda(\widetilde{\mathbf{X}}, X_N) - \Big(\Lambda(\widetilde{\mathbf{X}}, X_N) \cdot \widehat{\boldsymbol{e}}_N\Big)\, \widehat{\boldsymbol{e}}_N. \tag{4.3}$$

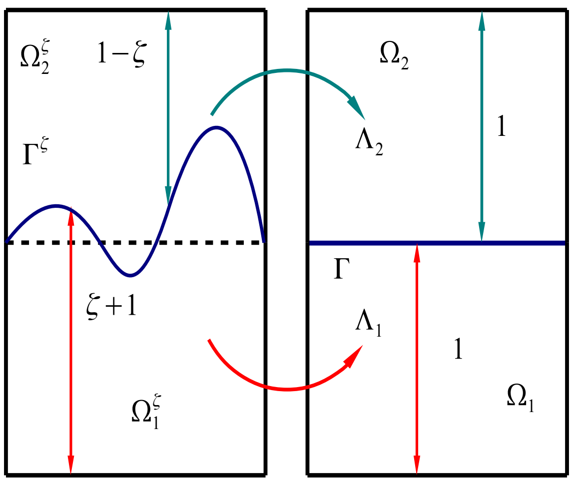

Fig. 3. Flattening of the interface

4.1. CHANGES ON GRADIENT STRUCTURE

For the maps Λ_i denote by Λ_i' its derivative or Jacobian matrix, then for $i = 1, 2$ holds

$$\Lambda_i' = \begin{bmatrix} I & \mathbf{0} \\ (1-(-1)^i z)\widetilde{\boldsymbol{\nabla}}^T\zeta & 1-(-1)^i\zeta \end{bmatrix}. \tag{4.4a}$$

Since $\|\zeta\|_{C(\Gamma)} < 1$ for functions belonging to $\mathscr{T}(\Gamma, \Omega)$ (defined in equation (2.1)), the absolute value of the determinant of the Jacobian matrix is given by

$$|\det(\Lambda_i')| = \left|1-(-1)^i\zeta(\widetilde{\mathbf{x}})\right| = 1-(-1)^i\zeta(\widetilde{\mathbf{x}}). \tag{4.4b}$$

For a scalar function we observe that whenever $\mathbf{X} \in \Omega_i^{\zeta}$ the gradient has the following structure:

$$\begin{Bmatrix} \widetilde{\boldsymbol{\nabla}}_X \\ \dfrac{\partial}{\partial X_N} \end{Bmatrix} = \begin{bmatrix} I & -(-1)^i\dfrac{z-(-1)^i}{1-(-1)^i\zeta}\,\widetilde{\boldsymbol{\nabla}}\zeta \\ \mathbf{0}^T & \dfrac{1}{1-(-1)^i\zeta} \end{bmatrix} \begin{Bmatrix} \widetilde{\boldsymbol{\nabla}}_x \\ \dfrac{\partial}{\partial z} \end{Bmatrix}, \tag{4.5a}$$

in matrix notation

$$\boldsymbol{\nabla}_X = A_i^{\zeta}\,\boldsymbol{\nabla}_x \quad \text{for all } X \in \Omega_i^{\zeta} \text{ and } i = 1, 2. \tag{4.5b}$$

4.2. FRACTIONAL MAPPING AND THE $H^1(\Omega)$ SPACE

From now on, we endow the set $\mathscr{T}(\Omega, \Gamma)$ with the norm $W^{1,\infty}(\Gamma)$, i.e., the sum of the essential suprema for the function and its gradient. Define the following change of variable.

Definition 4.1. For each element $r \in H^1(\Omega)$, we define the *fractional mapping* operator by

$$T\,r \overset{\mathbf{def}}{=} \sum_{i=1,2} \left(r \circ \Lambda_i^{-1}\right) \mathbb{1}_{\Omega_i}. \tag{4.6}$$

Lemma 4.2. *Let $r \in H^1(\Omega)$. Then for each $1 \leq \ell \leq N$ holds*

$$\frac{\partial}{\partial x_\ell} Tr = \sum_{i=1,2} \frac{\partial}{\partial x_\ell}\left(r \circ \Lambda_i^{-1}\right) \mathbb{1}_{\Omega_i}, \tag{4.7}$$

i.e., *the weak derivative does not have pulses/jumps on lower dimensional manifolds.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} \left\langle \frac{\partial}{\partial x_\ell} Tr, \varphi \right\rangle_{D'(\Omega), D(\Omega)} &= -\int_\Omega Tr\,\frac{\partial \varphi}{\partial x_\ell} = -\sum_{i=1,2}\int_{\Omega_i} r \circ \Lambda_i^{-1}\,\frac{\partial\varphi}{\partial x_\ell} \\ &= \sum_{i=1,2}\int_{\Omega_i} \frac{\partial}{\partial x_\ell}\left(r \circ \Lambda_i^{-1}\right)\varphi - \sum_{i=1,2}\int_{\partial\Omega_i}\left(r\circ\Lambda_i^{-1}\right)\varphi\,(\widehat{\boldsymbol{\nu}}_{\boldsymbol{i}}\cdot\widehat{\mathbf{e}}_\ell)\,dS. \end{aligned} \tag{4.8}$$

We focus on the last two summands. Since $\varphi = 0$ on $\partial\Omega$, this implies

$$
\begin{aligned}
&- \sum_{i=1,2} \int_{\partial\Omega_i} \left(r \circ \Lambda_i^{-1}\right) \varphi\, (\widehat{\boldsymbol{\nu}}_{\boldsymbol{i}} \cdot \widehat{\mathbf{e}}_\ell)\, dS \\
&= - \int_\Gamma \left\{ (r \circ \Lambda_1^{-1}) \varphi\, (\widehat{\boldsymbol{\nu}}_{\boldsymbol{1}} \cdot \widehat{\mathbf{e}}_\ell) + \left(r \circ \Lambda_2^{-1}\right) \varphi\, (\widehat{\boldsymbol{\nu}}_{\boldsymbol{2}} \cdot \widehat{\mathbf{e}}_\ell) \right\} dS = 0.
\end{aligned}
$$

The last equality holds since $\widehat{\boldsymbol{\nu}}_{\boldsymbol{1}} = -\widehat{\boldsymbol{\nu}}_{\boldsymbol{2}}$ and $\Lambda_1^{-1} = \Lambda_2^{-1}$ on Γ. Combining this fact with (4.8), we conclude (4.7). □

Theorem 4.3. *The map T is an isomorphism from $H^1(\Omega)$ onto itself.*

Proof. Since the application Λ is a bijection from Ω into itself the map T is clearly bijective and linear. For the calculation of the norms we use the change of variables theorem

$$
\int_{\Omega_i} |r \circ \Lambda_i^{-1}|^2 = \int_{\Omega_i^\zeta} |r|^2\, |\det(\Lambda_i')| \leq 2 \int_{\Omega_i^\zeta} |r|^2, \quad i = 1, 2,
$$

where the last inequality holds due to (4.4b). Equivalently, $\|Tr\|_{0,\Omega_i}^2 \leq 2\, \|r\|_{0,\Omega_i^\zeta}^2$, i.e., T is a bounded operator in $L^2(\Omega)$. For the derivative first consider $u \in C^1(\Omega)$ and take $\ell \in \{1, \ldots, N-1\}$. For the vector function $\Lambda_i^{-1} : \Omega_i \to \Omega_i^\zeta$ denote by $\Lambda_{i,k}^{-1}$ its k-th component function, thus

$$
\begin{aligned}
\int_{\Omega_i} \left| \frac{\partial}{\partial x_\ell} (u \circ \Lambda_i^{-1}) \right|^2 &= \int_{\Omega_i^\zeta} \left| \sum_{k=1}^N \frac{\partial u}{\partial x_k} \frac{\partial \Lambda_{i,k}^{-1}}{\partial x_\ell} \right|^2 |\det(\Lambda_i')| \\
&= \int_{\Omega_i^\zeta} \left| \frac{\partial u}{\partial x_\ell} + \frac{\partial u}{\partial z} \left[1 + (-1)^i z\right] \frac{\partial \zeta}{\partial x_\ell} \right|^2 |\det(\Lambda_i')| \\
&\leq 2 \int_{\Omega_i^\zeta} \left| \frac{\partial u}{\partial x_\ell} \right|^2 + 4 \int_{\Omega_i^\zeta} \left| \frac{\partial u}{\partial z} \right|^2 \left| \frac{\partial \zeta}{\partial x_\ell} \right|^2 \\
&\leq \max\{2,\, 4\, \|\zeta\|_{W^{1,\infty}(\Gamma)}^2\} \int_{\Omega_i^\zeta} |\boldsymbol{\nabla} u|^2 .
\end{aligned}
$$

For the derivative with respect to z we get

$$
\begin{aligned}
\int_{\Omega_i} \left| \frac{\partial}{\partial z} (u \circ \Lambda_i^{-1}) \right|^2 &= \int_{\Omega_i^\zeta} \left| \sum_{k=1}^N \frac{\partial u}{\partial x_k} \frac{\partial \Lambda_{i,k}^{-1}}{\partial z} \right|^2 |\det(\Lambda_i')| \\
&= \int_{\Omega_i^\zeta} \left| \frac{\partial u}{\partial z} \left[1 + (-1)^i \zeta\right] \right|^2 |\det(\Lambda_i')|\, 2 \int_{\Omega_i^\zeta} |\boldsymbol{\nabla} u|^2 .
\end{aligned}
$$

Define

$$C_\zeta \overset{\textbf{def}}{=} \sqrt{\max\{2,\, 4\,\|\zeta\|^2_{W^{1,\infty}(\Gamma)}\}}. \tag{4.9}$$

Combining both previous inequalities, it follows that

$$\|\boldsymbol{\nabla}(Tu)\|^2_{0,\Omega_i} \leq C_\zeta^2\,\|\boldsymbol{\nabla} u\|^2_{0,\Omega_i^\zeta} \quad \text{for all} \quad u \in C^1(\Omega),$$

for $i = 1, 2$. Therefore,

$$\|\boldsymbol{\nabla}(Tu)\|_{0,\Omega} \leq C_\zeta\,\|\boldsymbol{\nabla} u\|_{0,\Omega} \quad \text{for all} \quad u \in C^1(\Omega).$$

The inequality above extends to the whole space $H^1(\Omega)$ by density of $C^1(\Omega)$ in $H^1(\Omega)$. Finally, combining the first and second parts we have

$$\|\,T\,r\,\|_{1,\Omega} \leq C_\zeta\,\|\,r\,\|_{1,\Omega} \quad \text{for all} \quad r \in H^1(\Omega), \tag{4.10}$$

i.e., T is a bounded operator on $H^1(\Omega)$. □

Corollary 4.4. *The map T is an isomorphism from V onto itself.*

Proof. Observe that $\Lambda_1^{-1}|_{\partial\Omega_1-\Gamma} = I\,|_{\partial\Omega_1-\Gamma}$, then $Tr = 0$ on $\partial\Omega_1 - \Gamma$, i.e., T is a bijection from V into itself and due to previous theorem the result follows. □

4.3. THE FRACTIONAL MAPPING OPERATOR ON $\mathscr{T}(\Omega,\Gamma)$

Consider the application $T : \mathscr{T}(\Gamma,\Omega) \to \mathcal{L}(H^1(\Omega))$, where $\zeta \mapsto T(\zeta)$ is defined by equation (4.6). Since $\mathscr{T}(\Gamma,\Omega)$ is not a linear space, only a convex set, T can not be linear; however T does not respect convex combinations either. Therefore, the nonlinearity of T does not lie only on its domain of definition but also on its algebraic structure. Clearly, $T(0) = I$. We will show that T is continuous at 0 in the pointwise topology.

Lemma 4.5. *Let $\{\zeta_n\} \subseteq \mathscr{T}(\Gamma,\Omega)$ be bounded in $W^{1,\infty}(\Gamma)$ and such that $\|\zeta_n\|_{C(\Gamma)} \to 0$, then*

$$\|T(\zeta_n)u - u\|_{H^1(\Omega)} \to 0 \quad \textit{for all} \quad u \in C^1(\Omega).$$

Proof. First notice that for $i = 1, 2$

$$\Lambda_i^{-1}(\zeta_n)(\widetilde{\mathbf{X}}, X_N) = (\widetilde{\mathbf{x}}, z(1-(-1)^i)\zeta_n(\widetilde{\mathbf{x}}) + \zeta_n(\widetilde{\mathbf{x}})).$$

Then, the uniform convergence $\|\zeta_n\|_{C(\Gamma)} \to 0$ implies

$$\|\Lambda_1^{-1}(\zeta_n)\mathbb{1}_{\Omega_1} + I\mathbb{1}_\Gamma + \Lambda_2^{-1}(\zeta_n)\mathbb{1}_{\Omega_2} - I\|_{C(\Omega)} \xrightarrow[n\to\infty]{} 0. \tag{4.11}$$

Here $I\,\mathbb{1}_\Gamma$ has to be introduced for the convergence in the space of continuous functions. Also recall the fact that $\Lambda_1^{-1}(\zeta)|_\Gamma = \Lambda_2^{-1}(\zeta)|_\Gamma = I|_\Gamma$ for all $\zeta \in \mathscr{T}(\Gamma,\Omega)$. Take $u \in C^1(\Omega)$ and $\mathbf{x} \in \Omega$ fixed then, due to the mean value theorem in multiple dimensions, we have

$$|T(\zeta_n)u(\mathbf{x}) - u(\mathbf{x})| = |u \circ \Lambda_i^{-1}(\zeta_n)(\mathbf{x}) - u(\mathbf{x})| \leq |\boldsymbol{\nabla} u(\boldsymbol{\xi})|\,|\Lambda_i^{-1}(\zeta_n)(\mathbf{x}) - \mathbf{x}|,$$

where $\boldsymbol{\xi}$ lies in the segment uniting $\mathbf{x}$ and $\Lambda_i^{-1}(\zeta_n)(\mathbf{x})$. Thus

$$\int_{\Omega_i} |T(\zeta_n)u - u|^2 \leq \| u \|_{H^1(\Omega)}^2 \, \|\Lambda_i^{-1}(\zeta_n) - I\|_{C(\Omega_i)}^2 \quad \text{for } i = 1, 2. \tag{4.12}$$

Due to (4.11) we conclude

$$\|T(\zeta_n)u - u\|_{0,\Omega} \to 0. \tag{4.13}$$

For the $H^1(\Omega)$-convergence, first notice that due to the inequality (4.10) and definition (4.9) it follows that

$$\|T(\zeta_n)u\|_{1,\Omega} \leq \sup_{n \in \mathbb{N}} \sqrt{\max\{2, 4\|\zeta_n\|_{W^{1,\infty}(\Omega)}^2\}} \, \|u\|_{1,\Omega}, \tag{4.14}$$

where the supremum is finite because of the boundedness of $\{\zeta_n\}$ in $W^{1,\infty}(\Gamma)$. Then the sequence $\{T(\zeta_n)u\}$ has a weakly convergent subsequence in $H^1(\Omega)$ and due to (4.13) the weak limit must be u. Moreover, the Rellich-Kondrachov compactness theorem implies that the whole sequence converges weakly to the same limit u.

Next we prove that the $H^1(\Omega)$-norms converge. The strong convergence in $L^2(\Omega)$ is given by the Rellich-Kondrachov theorem, therefore we focus only on the derivatives. For any $1 \leq \ell \leq N-1$, we have

$$\begin{aligned}
\int_{\Omega_i} \left| \frac{\partial}{\partial x_\ell} \left[u \circ \Lambda_i^{-1}(\zeta_n) \right] \right|^2 &= \int_{\Omega_i^{\zeta_n}} |\partial_\ell u + \partial_z u (1 + (-1)^i z) \partial_\ell \zeta_n|^2 |\det \Lambda_i'(\zeta_n)| \\
&= \int_{\Omega_i^{\zeta_n}} |\partial_\ell u + \partial_z u (1 + (-1)^i z) \partial_\ell \zeta_n|^2 |\det \Lambda_i'(\zeta_n)| \\
&= \int_{\Omega_i^{\zeta_n}} |\partial_\ell u + \partial_z u (1 + (-1)^i z) \partial_\ell \zeta_n|^2 (1 - (-1)^i \zeta_n).
\end{aligned}$$

On one hand, it is clear that the integrand converges to $\left| \frac{\partial u}{\partial x_\ell} \right|^2 \mathbb{1}_{\Omega_i}$, on the other hand, we have the estimate

$$\begin{aligned}
&\left| \partial_\ell u + \partial_z u (1 + (-1)^i z) \partial_\ell \zeta_n|^2 (1 - (-1)^i \zeta_n) \mathbb{1}_{\Omega_i^{\zeta_n}} \right| \\
&\quad \leq 2 \{ |\partial_\ell u|^2 + |1 + (-1)^i z| |\partial_\ell \zeta_n|^2 |\partial_z u|^2 \} \, 2 \\
&\quad \leq 4 \{ |\partial_\ell u|^2 + 2 \|\zeta_n\|_{W^{1,\infty}(\Gamma)}^2 |\partial_z u|^2 \} \\
&\quad \leq 4 \max\{1 + 2 \sup_{n \in \mathbb{N}} \|\zeta_n\|_{W^{1,\infty}(\Gamma)}^2\} \, |\boldsymbol{\nabla} u|^2 \in L^1(\Omega).
\end{aligned}$$

Thus, Lebesgue's dominated convergence theorem yields

$$\left\| \frac{\partial}{\partial x_\ell} (T(\zeta_n)u) \right\|_{0,\Omega_i}^2 \xrightarrow[n \to \infty]{} \left\| \frac{\partial u}{\partial x_\ell} \right\|_{0,\Omega_i}^2, \quad 1 \leq \ell \leq N-1, \quad i = 1, 2. \tag{4.15}$$

For the derivative with respect to z we get

$$\int\limits_{\Omega_i} \left| \frac{\partial}{\partial z} \left[u \circ \Lambda_i^{-1}(\zeta_n) \right] \right|^2 = \int\limits_{\Omega_i^{\zeta_n}} \left| \frac{\partial u}{\partial z} \left[1 + (-1)^i \zeta_n \right] \right|^2 \left| \det \left(\Lambda_i'(\zeta_n) \right) \right|$$

$$= \int\limits_{\Omega} \left| \frac{\partial u}{\partial z} \right|^2 \left[1 + (-1)^i \zeta_n \right]^3 \mathbb{1}_{\Omega_i^{\zeta_n}} .$$

Again, the integrand converges to $|\frac{\partial u}{\partial z}|^2 \, \mathbb{1}_{\Omega_i}$ and it is bounded by $2\,|\frac{\partial u}{\partial z}|^2$ which is an element of $L^1(\Omega)$. Hence, the Lebesgue dominated convergence theorem yields

$$\left\| \frac{\partial}{\partial z} \left(T(\zeta_n) u \right) \right\|_{0,\,\Omega_i}^2 \xrightarrow[n \to \infty]{} \left\| \frac{\partial\, u}{\partial z} \right\|_{0,\,\Omega_i}^2 \quad \text{for } i = 1, 2. \tag{4.16}$$

The equations (4.15) and (4.16) give the convergence of the $L^2(\Omega)$-norms of the gradients $\|\boldsymbol{\nabla} T(\zeta_n) u\|_{0,\Omega} \to \|\boldsymbol{\nabla} u\|_{0,\Omega}$ and then $\|T(\zeta_n)u\|_{1,\Omega} \to \|u\|_{1,\Omega}$. This fact together with the weak convergence in $H^1(\Omega)$ imply $\|T(\zeta_n)u - u\|_{1,\Omega} \to 0$. □

Theorem 4.6. *Let $\{\zeta_n\} \subseteq \mathscr{T}(\Gamma, \Omega)$ be bounded in $W^{1,\infty}(\Gamma)$ such that $\|\zeta_n\|_{C(\Gamma)} \to 0$. Then*

$$\|T(\zeta_n)\, r - r\|_{H^1(\Omega)} \to 0 \quad \textit{for all} \quad r \in H^1(\Omega).$$

Hence, $T(\zeta_n)$ converges to the identity I in the strong operator topology.

Proof. We use the standard density argument. Let $r \in H^1(\Omega)$, take $\{u_j\} \subseteq C^1(\Omega)$ such that $\|u_j - r\|_{1,\Omega} \to 0$. Recall Definition 4.9 and inequality (4.10), then

$$\begin{aligned}
\|\, T(\zeta_n)\, r - r\|_{H^1(\Omega)} &\le \|\, T(\zeta_n)\, r - T(\zeta_n) u_j\|_{H^1(\Omega)} \\
&\quad + \|\, T(\zeta_n) u_j - u_j\|_{H^1(\Omega)} + \|\, u_j - r\|_{H^1(\Omega)} \\
&\le (1 + \|\, T(\zeta_n)\|) \|u_j - r\|_{H^1(\Omega)} + \|\, T(\zeta_n) u_j - u_j\|_{H^1(\Omega)} \\
&\le (1 + \sup_{k \in \mathbb{N}} \sqrt{\max\{2,\, 4\, \|\zeta_k\|_{W^{1,\infty}(\Gamma)}\}})\, \|\, u_j - r\|_{H^1(\Omega)} \\
&\quad + \|\, T(\zeta_n) u_j - u_j\|_{H^1(\Omega)} .
\end{aligned}$$

Fix $j \in \mathbb{N}$ such that the first summand of the right hand side is less than $\frac{\epsilon}{2}$. Due to Lemma 4.5 there exists $N \in \mathbb{N}$ such that $n \ge N$, implies that the second summand on the right hand side of the expression above is less than $\frac{\epsilon}{2}$. This completes the proof. □

Corollary 4.7. *Let $\{\zeta_n\} \subseteq \mathscr{T}(\Gamma, \Omega)$ be bounded in $W^{1,\infty}(\Gamma)$ and such that $\|\zeta_n\|_{C(\Gamma)} \to \mathbf{0}$, then*

$$\left\|\, T^{-1}(\zeta_n)\, r - r \,\right\|_{H^1(\Omega)} \xrightarrow[n \to \infty]{} 0,$$

i.e., the sequence of inverse operators converge.

Proof. Repeating the techniques exposed in Lemma 4.5 and in Theorem 4.6, applied to the fractional bijective maps $\{\Lambda_i(\zeta_n)\}$ defined in (4.1a), the result follows. □

Remark 4.8. Using Theorem 4.6 it can be proved that $\| T(\zeta_n) - \jmath \|_{\mathcal{L}(C^1(\Omega),\, L^2(\Omega))} \to 0$ as $\|\zeta_n\|_{C(\Gamma)} \to 0$ if $\{\zeta_n\}$ is bounded in $W^{1,\infty}(\Gamma)$. Here $\jmath : C^1(\Omega) \hookrightarrow L^2(\Omega)$ is the embedding operator $\jmath(\varphi) \stackrel{\mathbf{def}}{=} \varphi$ and $T(\zeta_n)$ is regarded as an operator from $C^1(\Omega)$ to $L^2(\Omega)$. Also, using the same technique in obtaining the estimate (4.12) we can show $\| T(\zeta_n) - \jmath \|_{\mathcal{L}(C^2(\Omega),\, H^1(\Omega))} \to 0$ with $\jmath : C^2(\Omega) \hookrightarrow H^1(\Omega)$, i.e., in order to get convergence in the norm of the operators, higher degrees of regularity are needed.

4.4. THE FLATTENED PROBLEM

Consider the problem (2.8b) subject to the changes of variable described above. Introducing (4.4b) and (4.5) in each summand of the left hand side in (2.8b), we have

$$\int_{\Omega_i^\zeta} \boldsymbol{\nabla}_{\!X}\, q \cdot \boldsymbol{\nabla}_{\!X}\, r = \int_{\Omega_i} \boldsymbol{\nabla}_{\!X}\, Tq \cdot \boldsymbol{\nabla}_{\!X} Tr \;|\det \Lambda_i'| \; = \int_{\Omega_i} (A_i^\zeta)^T A_i^\zeta\, \boldsymbol{\nabla}_{\!x} Tq \cdot \boldsymbol{\nabla}_{\!x} Tr\, (1-(-1)^i\zeta),$$

where

$$(A_i^\zeta)^T A_i^\zeta = \begin{bmatrix} (1-(-1)^i\zeta) I & (-1)^i (z-(-1)^i)\widetilde{\boldsymbol{\nabla}}\zeta \\ (-1)^i (z-(-1)^i)\widetilde{\boldsymbol{\nabla}}^T\zeta & \dfrac{|(z-(-1)^i)\boldsymbol{\nabla}\zeta|^2+1}{1-(-1)^i\zeta} \end{bmatrix}. \tag{4.17}$$

Due to Corollary 4.4, the quantifiers $\forall\, Tr \in V$ and $\forall\, r \in V$ are equivalent, therefore we conclude that the solution q of problem (2.8b), satisfies the variational statement

$$\begin{aligned} q \in V : \sum_{i=1,2} \int_{\Omega_i} \frac{1-(-1)^i\,\zeta}{\epsilon^{\,i-1}}\, (A_i^\zeta)^T A_i^\zeta\, \boldsymbol{\nabla} Tq \cdot \boldsymbol{\nabla} r \\ = \int_\Gamma \frac{1}{|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|}\, (Tf)\; r\, dS + \sum_{i=1,2} \int_{\Omega_i} (1-(-1)^i\,\zeta\,)\,(TF)\, r \quad \text{for all} \quad r \in V. \end{aligned} \tag{4.18}$$

In the first summand of the left hand side, the notation Tf stands for $f \circ \Lambda_1^{-1}$ or $f \circ \Lambda_2^{-1}$ indistinctively since $\Lambda_1^{-1} = \Lambda_2^{-1}$ on Γ. The surface integral summand implicitly uses the fact, that the upwards unitary vector normal to the surface $\Gamma^{\,\zeta}$ is given by

$$\widehat{\boldsymbol{n}} = \frac{(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)}{|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|}.$$

Declaring $\varrho \stackrel{\mathbf{def}}{=} Tq$, the problem (4.18) is equivalent to

$$\begin{aligned} \varrho \in V : \sum_{i=1,2} \int_{\Omega_i} \frac{1-(-1)^i\,\zeta}{\epsilon^{\,i-1}}\, (A_i^\zeta)^T A_i^\zeta\, \boldsymbol{\nabla}\varrho \cdot \boldsymbol{\nabla} r \\ = \int_\Gamma \frac{1}{|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|}\, (Tf)\; r\, dS + \sum_{i=1,2} \int_{\Omega_i} (1-(-1)^i\,\zeta\,)\,(TF)\, r \quad \text{for all} \quad r \in V. \end{aligned} \tag{4.19}$$

Next, we focus on some properties of the involved matrices.

Lemma 4.9. *For $i = 1, 2$ the inverse matrix of A_i^ζ is given by*

$$(A_i^\zeta)^{-1} = \begin{bmatrix} I & (1 - (-1)^i z)\widetilde{\boldsymbol{\nabla}}\zeta \\ \mathbf{0}^T & 1 - (-1)^i\zeta \end{bmatrix}. \tag{4.20}$$

The proof of Lemma 4.9 follows by a direct calculation.

Corollary 4.10. *Let $(\widetilde{\mathbf{x}}, z) \in \Omega_i^\zeta$ be fixed, with $i \in \{1, 2\}$. Then, the linear operator $\boldsymbol{\xi} \mapsto (A_i^\zeta(\mathbf{x}))^{-1}\boldsymbol{\xi}$ from $\mathbb{R}^N$ into itself, endowed with the canonical inner product, satisfies*

$$\|(A_i^\zeta(\mathbf{x}))^{-1}\|_{\mathcal{L}(\mathbb{R}^N)} \leq \sqrt{N + 3 + 4\,\|\zeta\|^2_{W^{1,\infty}(\Gamma)}} \text{ for all } \mathbf{x} \in \Omega_i^\zeta \text{ and } i \in \{1, 2\}. \tag{4.21}$$

Proof. We compute the Frobenius norm of the operator adding the squared inner product norms $|\cdot|$ of each column vector in (4.20). This gives

$$\|(A_i^\zeta(\mathbf{x}))^{-1}\|^2_{\mathcal{L}(\mathbb{R}^N)} \leq (N - 1) + (1 - (-1)^i z)^2 |\boldsymbol{\nabla}\zeta(\widetilde{\mathbf{x}})\,|^2 + (1 - (-1)^i\zeta)^2.$$

Since $z, \zeta(\widetilde{\mathbf{x}}) \in [-1, 1]$ for all $\mathbf{x} = (\widetilde{\mathbf{x}}, z) \in \Omega$ we estimate $|1 \pm z| \leq 2$ and $|1 \pm \zeta| \leq 2$. The gradient of ζ is estimated by the $W^{1,\infty}(\Gamma)$-norm and the result follows. □

Proposition 4.11. *The matrices $(1-(-1)^i\zeta)(A_i^\zeta)^T A_i^\zeta$ are uniformly coercive in $\mathbb{R}^N$ for $i = 1, 2$, i.e., there exists $e(\zeta) > 0$ such that*

$$e(\zeta)\,|\boldsymbol{\xi}|^2 \leq (1 - (-1)^i\zeta)(A_i^\zeta)^T A_i^\zeta\, \boldsymbol{\xi} \cdot \boldsymbol{\xi} \tag{4.22}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and each $(\widetilde{\mathbf{x}}, z) \in \Omega$.

Proof. Let $\mathbf{x} \in \Omega_i$ be fixed and $\boldsymbol{\xi} \in \mathbb{R}^N$ unitary then

$$\begin{aligned}(1 - (-1)^i\zeta)(A_i^\zeta)^T A_i^\zeta\, \boldsymbol{\xi} \cdot \boldsymbol{\xi} &= (1 - (-1)^i\zeta) A_i^\zeta\, \boldsymbol{\xi} \cdot A_i^\zeta\, \boldsymbol{\xi} \\ &\geq (1 - \|\zeta\|_{C(\Gamma)}) \left|A_i^\zeta\, \boldsymbol{\xi}\right|^2 \geq (1 - \|\zeta\|_{C(\Gamma)}) \min_{|\boldsymbol{\eta}|=1} \left|A_i^\zeta\, \boldsymbol{\eta}\right|^2 \\ &= \frac{1 - \|\zeta\|_{C(\Gamma)}}{\|(A_i^\zeta)^{-1}\|^2_{\mathcal{L}(\mathbb{R}^N)}} \geq \frac{1 - \|\zeta\|_{C(\Gamma)}}{N + 3 + 4\,\|\zeta\|^2_{W^{1,\infty}(\Gamma)}},\end{aligned}$$

where, the last inequality comes from Corollary 4.10. Defining

$$e(\zeta) \overset{\mathbf{def}}{=} \frac{1 - \|\zeta\|_{C(\Gamma)}}{N + 3 + 4\,\|\zeta\|^2_{W^{1,\infty}(\Gamma)}}$$

the statement (4.22) follows. □

Corollary 4.12. *The form* $[\,\cdot,\,\cdot\,] : V \times V \to \mathbb{R}$

$$[\kappa, \pi] \overset{\text{def}}{=} \sum_{i=1,2} \int_{\Omega_i} \frac{1-(-1)^i\zeta}{\epsilon^{\,i-1}} \, (A_i^\zeta)^T A_i^\zeta \, \boldsymbol{\nabla}\kappa \cdot \boldsymbol{\nabla}\pi, \tag{4.23}$$

defines an inner product on V *which induces the same topology as the induced by the standard inner product on* $H^1(\Omega)$*.*

Proof. The form (4.23) is well-defined since the application

$$(\widetilde{\mathbf{x}}, z) \in \Omega_i \mapsto (1-(-1)^i\zeta(\widetilde{\mathbf{x}}))(A_i^\zeta(\widetilde{\mathbf{x}}, z))^T A_i^\zeta(\widetilde{\mathbf{x}}, z),$$

for $i = 1, 2$ is in $L^\infty(\Omega_i, \mathbb{R}^{N\times N})$. Clearly the form is bilinear and symmetric. For the continuity we have

$$[\kappa, \pi] \le \tfrac{2}{\epsilon} \, \|A_i^\zeta\|_\infty^2 \, \|\boldsymbol{\nabla}\kappa\|_{0,\,\Omega} \, \|\boldsymbol{\nabla}\pi\|_{0,\,\Omega} \le \tfrac{2}{\epsilon} \, \|A_i^\zeta\|_\infty^2 \, \|\kappa\,\|_{1,\,\Omega} \, \|\pi\|_{1,\,\Omega}$$

for all $\kappa, \pi \in V$. In particular $[\kappa, \kappa] \le \frac{2}{\epsilon} \, \|A_i^\zeta\|_\infty^2 \, \|\kappa\,\|_{1,\,\Omega}^2$. The homogeneous condition of the induced norm comes from the uniform coercivity of the matrices shown in Proposition 4.11. Hence

$$\frac{e(\zeta)}{1+C_\Omega} \, \|\kappa\|_{1,\Omega}^2 \le e(\zeta) \, \|\boldsymbol{\nabla}\kappa\|_{0,\,\Omega}^2 \le [\kappa, \kappa] \quad \text{for all} \quad \kappa \in V,$$

where C_Ω is the Poincaré constant associated to the domain Ω valid for all elements of V, given the boundary conditions. Therefore, the induced norms are equivalent. □

Theorem 4.13. *The problem* (4.19) *is well-posed.*

Proof. The equivalence of the norm induced by the inner product $[\,\cdot, \cdot\,]$ to the standard $H^1(\Omega)$-norm on V is shown in Corollary 4.12, therefore the well-posedness of problem (4.19) follows from Lax-Milgram's lemma. □

4.5. GEOMETRIC PERTURBATION AND INNER PRODUCTS

This section is aimed at analyzing the highly nonlinear impact of the geometry in terms of the inner product. Consider the bounds

$$\|p - q^\zeta\|_V \le \|p - T^{-1}(\zeta)p\,\|_V + \|T^{-1}(\zeta)p - q^\zeta\|_V. \tag{4.24}$$

The first summand converges due to Theorem 4.6. For the convergence of the second summand, due to Corollary 4.7 it is equivalent to prove

$$\|p - T(\zeta)q^\zeta\|_V = \|p - \varrho^\zeta\|_V \to 0 \text{ as } \|\zeta\|_{C(\Gamma)} \to 0 \text{ with } \|\zeta\|_{W^{1,\infty}(\Gamma)} \text{ bounded.} \tag{4.25}$$

We are to estimate the norm above by comparing the problems (2.8a) and (4.19). In problem (4.19) the effect of the geometry on the inner product structure is fully contained in the matrices

$$\{(1-(-1)^i\zeta)(A_i^\zeta)^T A_i^\zeta : \mathbf{x} \in \Omega_i\}, \quad i = 1, 2. \tag{4.26}$$

Notice that this is a family of symmetric and therefore diagonalizable matrices. However, they depend on the point $\mathbf{x} \in \Omega$ and, in general, they do not commute for $\mathbf{x}, \mathbf{x}' \in \Omega$ different. Therefore, it can not be assured that the family (4.26) is simultaneously diagonalizable. Observe that the matrices (4.5a) have entries multiplied by the factors $\frac{1}{1-(-1)^i\zeta(\widetilde{\mathbf{x}})}$, for $i = 1, 2$ respectively. This implies that the maps induced by the matrices are not linear with respect to the perturbation ζ. The following hypothetic assumptions illustrate the nonlinearity of the dependence.

(i) Suppose that ζ is a piecewise linear affine function. Although $\widetilde{\boldsymbol{\nabla}}\zeta$ is piecewise constant, the map $\mathbf{x} \mapsto (1-(-1)^i\zeta)(A_i^\zeta)^T A_i^\zeta$ is not piecewise constant.

(ii) Assume that the family of matrices (4.26) is diagonal. Testing the problems (2.8a) and (4.19) with $p - \varrho$ and subtracting them, yields

$$\begin{aligned} &\sum_{i=1,2} \int_{\Omega_i} \frac{\boldsymbol{\nabla} p - \boldsymbol{\nabla}\,\varrho}{\epsilon^{i-1}} \cdot \boldsymbol{\nabla} p - \begin{bmatrix} \mu_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \mu_N \end{bmatrix} \boldsymbol{\nabla}\,\varrho \\ &= \int_\Omega F(p-\varrho) - \sum_{i=1,2} \int_{\Omega_i} (1-(-1)^i\,\zeta)(TF)(p-\varrho)\,dS \\ &\quad + \int_\Gamma f(p-\varrho)\,dS - \int_\Gamma \frac{1}{|(-\widetilde{\boldsymbol{\nabla}}\zeta,\,1)|}\,(Tf)\,(p-\varrho)\,dS, \end{aligned} \tag{4.27}$$

where $\mu_1, \dots, \mu_N$ are the eigenvalues. Nevertheless $\mu_j = \mu_j(\mathbf{x})$ for $1 \le j \le N$, i.e., they depend on the position $\mathbf{x}$ within the domain Ω.

We close this section reviewing the simplest possible scenario.

4.6. FRACTIONAL MAPPING OF DOMAINS IN THE ONE DIMENSIONAL CASE

In this section we analyze the fractional mapping technique in the one dimensional setting. For this case the problem (4.19) reduces to

$$\begin{aligned} &\frac{1}{1+\zeta} \int_{-1}^{0} \boldsymbol{\partial}\varrho \cdot \boldsymbol{\partial} r + \frac{1}{\epsilon}\,\frac{1}{1-\zeta} \int_{0}^{1} \boldsymbol{\partial}\varrho \cdot \boldsymbol{\partial} r \\ &= f(\zeta)\,r(0) + \sum_{i=1,2} (1-(-1)^i\zeta) \int_{\frac{(-1)^i-1}{2}}^{\frac{(-1)^i+1}{2}} r(x)\,F\left(\frac{x-\zeta}{1-(-1)^i\zeta}\right)\,dx. \end{aligned} \tag{4.28}$$

In order to get a-priori estimates the left hand side of the problem (4.28) suggests testing equations (3.2a) and (4.28) with the following function:

$$\Big(p - \frac{1}{1+\zeta}\,\varrho\Big)\mathbb{1}_{(-1,\,0]} + \Big(p - \frac{1}{1-\zeta}\,\varrho\Big)\mathbb{1}_{[0,\,1)} = p - \Big\{\frac{1}{1+\zeta}\,\varrho\,\mathbb{1}_{(-1,\,0]} + \frac{1}{1-\zeta}\,\varrho\,\mathbb{1}_{[0,\,1)}\Big\}. \tag{4.29}$$

However, the test function presented above (as the second summand in the right hand side illustrates) is not eligible, because it does not belong to V unless $\zeta = 0$ or $\varrho(0) = 0$. The first condition removes the perturbation leaving the original problem (3.2a) and the second can not be assured. Any other attempt to estimate $\|p - \varrho\|_V$ demands test functions equivalent to (4.29), because of the coefficients disagreement in problems (3.2a) and (4.28). Of course, such piecewise functions are not in the test space V due to the trace continuity requirements.

In the one dimensional case, the fractional mapping technique defines an inner product much easier to understand than the one corresponding to the multidimensional case (4.19). It is clear that the dependence of the inner product with respect to the perturbation, is piecewise linear fractional as the map $\Lambda : (0, 1) \to (0, 1)$ itself. Additionally, direct calculations can be done to find explicitly, the dependence of the eigenvalues associated to the problem (3.2b). This dependence also turns out to be piecewise fractional on ζ, closely related to Λ. For a deep discussion on boundary perturbation of the Laplace eigenvalues see [8, 9].

Finally, the question of strong convergence (4.25) can be solved using the Rellich-Kondrachov theorem. However, this is not a constructive result and it does not yield explicit estimates such as inequality (3.19) in Section 3.6.

5. CONCLUDING REMARKS AND DISCUSSION

The present work yields several conclusions and open questions.

(i) In Section 2.3, extra hypothesis of regularity on the forcing terms involved, were introduced in order to conclude weak convergence in a first step, and strong convergence in a second one. Also, for the geometric perturbations $\zeta \in \mathscr{T}(\Omega, \Gamma)$, it is not enough to have convergence in $C(\Gamma)$, there is also need for boundedness in $W^{1,\infty}(\Gamma)$.

(ii) The conditions of convergence for the interface are acceptable in the context of saturated porous media fluid flow. Moreover, for the modeling of saturated fluid flow through deformable porous media in the *elastic regime*, these conditions are natural because, for high gradients of deformation the *elasto-plastic* and *plastic* regimes start taking place, see [17].

(iii) Mapping the perturbed domains with fractional applications as in Section 4, decomposes the nonlinearity of the question in two parts: the fractional mapping operator $T : \mathscr{T}(\Omega, \Gamma) \to \mathcal{L}(H^1(\Omega))$ which is clearly nonlinear, and the effect of the geometric perturbation on the inner product that the problem (2.8b) defines on V. The latter is reflected in the matrices (4.17) of the problem (4.19) above.

(iv) Although the operators $T(\zeta)$ converge in the strong operator topology as the perturbations $\zeta \in \mathscr{T}(\Omega, \Gamma)$ converge, according to the hypothesis of Theorem 4.6, it is an abstract statement. There are no explicit estimates depending on $\|\zeta\|_{W^{1,\infty}(\Gamma)}$ or $\|\zeta\|_{C(\Gamma)}$, analogous to inequality (3.19) presented in Section 3.6.

(v) The inner product, that a geometric perturbation implicitly defines on the function space V, is the most important nonlinearity of the problem. It introduces a notion of orthogonality in the space which is hardly comparable with the standard one, beyond the topological equivalence of the induced norms.

(vi) The fractional mapping technique is a much simpler approach than the local charts strategy used in trace theorems. Its main contribution to this work, consists in exposing the challenges of the convergence rate question, as well as the non-linearities involved, in a much neater way than the local charts approach.

(vii) The strong convergence statements attained in Section 2.4, realizing problem (2.8a) as the strong limit of the family of problems (2.8b), suggest numerical experimentation and a-posteriori estimates as the most feasible approach to gain insight into the convergence rate question as well as the dependence with respect to the norms $W^{1,\infty}(\Gamma)$ and/or $C(\Gamma)$ of the perturbation ζ.

(viii) The solution of the one dimensional case presented in Section 3, using orthogonal decompositions on carefully chosen subspaces, illustrates the complexity of the convergence rate problem. This fact becomes even more dramatic due to the strong dependence on the one dimensional setting.

(ix) In both settings, multi and one dimensional, the necessity of testing the variational statements with functions of the structure (4.29) (i.e., discontinuous across the interfaces), to obtain explicit a-priori estimates of the difference $p - q$ is self-evident. Such functions break the linear structure of the domain V and obey to the disagreement of scaling coefficients in problems (2.8a), (2.8b).

(x) Both of the classical mixed variational formulations: $\mathbf{L}^2$-H^1 and $\mathbf{H}_{\text{div}}$-L^2 demand coupling conditions on the function spaces, for the trace on the interfaces Γ, Γ^{ζ}. These conditions do not allow testing with discontinuous functions such as (4.29). However, a mixed formulation setting $\mathbf{L}^2$-H^1 in one region, namely Ω_1 and $\mathbf{H}_{\text{div}}$-L^2 in the other region, as the one introduced in [16], does not require continuity of the test functions across the interfaces. Hence, this is the formulation where convergence rate estimates (implicit or explicit, given the nonlinearity of the problem) can most likely be attained.

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