Opuscula Math. 35, no. 3 (2015), 411–419
http://dx.doi.org/10.7494/OpMath.2015.35.3.411 Opuscula Mathematica

HERMITE-HADAMARD TYPE INEQUALITIES FOR WRIGHT-CONVEX FUNCTIONS OF SEVERAL VARIABLES

Dorota Śliwińska and Szymon Wąsowicz

Communicated by Zbigniew Szkutnik

Abstract. We present Hermite-Hadamard type inequalities for Wright-convex, strongly convex and strongly Wright-convex functions of several variables defined on simplices.

Keywords: convex functions, Wright-convex functions, strongly Wright-convex functions, Hermite-Hadamard inequality.

Mathematics Subject Classification: 26B25, 26D15, 39B62, 65D32.

1. INTRODUCTION

One of the most classical inequalities in the theory of convex functions is the Hermite-Hadamard inequality. It states that if $f:[a,b]\to\mathbb{R}$ is convex then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\le\frac{f(a)+f(b)}{2}.$$

It plays an important role in convex analysis, in the literature one can find its various generalizations and applications. For example, an exhausting study of this inequality is given in the book [2].

Recall that a function $f:D\to\mathbb{R}$, where $D\subset\mathbb{R}^n$ is a convex set, is called *Wright-convex* (W-convex for short), if

$$f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big)\le f(x)+f(y)$$

for any $x,y\in D$ and $t\in[0,1]$. Trivially we can see that any convex function is necessarily W-convex and any W-convex function is Jensen-convex (i.e. it fulfills the above inequality with $t=\frac{1}{2}$). However, these inclusions are proper. It is evident,

© AGH University of Science and Technology Press, Krakow 2015

if one knows the famous Ng representation (cf. [8]). It states that any W-convex function defined on an open and convex set $D \subset \mathbb{R}^n$ is the sum of an additive function $a : \mathbb{R}^n \to \mathbb{R}$ and a convex function $g : D \to \mathbb{R}$. Therefore, if $f : \mathbb{R} \to \mathbb{R}$ is W-convex, then either f is continuous (and then convex), or the graph of f is a dense subset of a plane. Hence by putting $f(x) = |a(x)|$, where $a : \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, we obtain a Jensen-convex function, which is not W-convex. Of course, the function a is a W-convex function, which is not convex.

It was natural to generalize the Hermite-Hadamard inequality to functions of several variables. In the case of simplices, for the first time it was done by Neuman [7] (see also [1,3] and [13] for the functions defined on simplices and [2,9] for more general domains). Recently Olbryś [10] obtained the following inequality of Hermite-Hadamard type: if $f : \mathcal{I} \to \mathbb{R}$ (where $\mathcal{I} \subset \mathbb{R}$ is an open interval) is W-convex, then

$$2f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b \big(f(x) + f(a+b-x)\big)\,\mathrm{d}x \le f(a) + f(b) \tag{1.1}$$

for any $a, b \in \mathcal{I}$. Because the note [10] is actually unpublished, let us mention that (1.1) is an immediate consequence of the Hermite-Hadamard inequality. It is enough to apply it to the convex (due to Ng's representation of f) function $[a,b] \ni x \mapsto f(x) + f(a+b-x)$.

Motivated by this beautiful result of Olbryś we present in this paper its multivariate counterparts. We also give some related inequalities for strongly convex and strongly W-convex functions of several variables.

2. DEFINITIONS AND BASIC PROPERTIES

Let $v_0, \ldots, v_n \in \mathbb{R}^n$ be affinely independent and let $S = \operatorname{conv}\{v_0, \ldots, v_n\}$ be a simplex with vertices $v_0, \ldots, v_n$. Denote by $|S|$ its volume and by $\mathbf{b}$ its barycenter, i.e.

$$\mathbf{b} = \frac{1}{n+1}\sum_{i=0}^{n} v_i\,.$$

Any element $x \in S$ is uniquely represented by a convex combination of the vertices:

$$x = \sum_{i=0}^{n} t_i v_i\,,$$

where the coefficients $t_i \ge 0$, $i = 0, \ldots, n$, with $t_0 + \ldots + t_n = 1$, are called the *barycentric coordinates* of x. Moreover, any $x \in \mathbb{R}^n$ admits the above (unique) representation with real scalars summing up to 1.

Denote by C the set of all cyclic permutations of $\{0, \ldots, n\}$. Any $\sigma \in C$ generates an affine transformation $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\sigma\Big(\sum_{i=0}^{n} t_i v_i\Big) = \sum_{i=0}^{n} t_{\sigma(i)} v_i\,.$$

From now on we identify $\sigma \in C$ with the affine map σ given as above. For $\sigma \in C$ and for any function $f : S \to \mathbb{R}$ we define the function $f_\sigma : S \to S$ by

$$f_\sigma(x) = f\big(\sigma(x)\big).$$

Next we introduce the *symmetrization* F of a function f as follows:

$$F(x) = \sum_{\sigma \in C} f_\sigma(x)\,, \quad x \in S. \tag{2.1}$$

It is easy to observe that F is symmetric with respect to the barycenter, which means that $F\big(\sigma(x)\big) = F(x)$ for any $\sigma \in C$.

In our article we use the Hermite-Hadamard inequality on simplices, which was firstly given by Neuman [7], then reproved by Guessab and Schmeisser [3], Bessenyei [1] and the second-named author [13, Corollary 3]:

Theorem 2.1. *If $f : S \to \mathbb{R}$ is convex then*

$$f(\mathbf{b}) \le \frac{1}{|S|} \int\limits_S f(x)\,\mathrm{d}x \le \frac{1}{n+1} \sum_{i=0}^{n} f(v_i). \tag{2.2}$$

To prove the Hermite-Hadamard type inequality for W-convex functions, we require two lemmas. The first of them can be found in [12, Lemma 2.2].

Lemma 2.2. *If $g : S \to R$ is convex then so is g_σ.*

Lemma 2.3. *If $a : \mathbb{R}^n \to \mathbb{R}$ is additive, then its symmetrization $A = \sum_{\sigma\in C} a_\sigma$ is constant.*

Proof. Any $x \in \mathbb{R}^n$ may be written as $x = \sum_{i=0}^{n} t_i v_i$ with real scalars $t_0, \ldots, t_n$ (possibly not all positive) summing up to 1. Using additivity of a and the definition of σ we arrive at

$$\begin{aligned} A(x) &= \sum_{\sigma\in C} a_\sigma(x) = \sum_{\sigma\in C} a\big(\sigma(x)\big) = a\Big(\sum_{\sigma\in C} \sigma\Big(\sum_{i=0}^{n} t_i v_i\Big)\Big) \\ &= a\Big(\sum_{\sigma\in C}\sum_{i=0}^{n} t_{\sigma(i)} v_i\Big) = a\Big(\sum_{i=0}^{n}\sum_{\sigma\in C} t_{\sigma(i)} v_i\Big) = a\Big(\sum_{i=0}^{n}\Big(\sum_{\sigma\in C} t_{\sigma(i)}\Big) v_i\Big) = a\Big(\sum_{i=0}^{n} v_i\Big)\,, \end{aligned}$$

which is a constant. □

3. HERMITE-HADAMARD TYPE INEQUALITY FOR W-CONVEX FUNCTIONS

Theorem 3.1. *Let $D \subset \mathbb{R}^n$ be an open and convex set and $S \subset D$ be a simplex. If $f : D \to \mathbb{R}$ is W-convex, then its symmetrization F is convex on S.*

Proof. Because f is W-convex, then $f = a + g$ for some additive function $a : \mathbb{R}^n \to \mathbb{R}$ and a convex function $g : D \to \mathbb{R}$. The function A (symmetrization of a, cf. (2.1)) is constant by Lemma 2.3, while the function G (symmetrization of g on S), is convex by Lemma 2.2. Thus F is convex on S. □

Theorem 3.2. *Let $D \subset \mathbb{R}^n$ be an open and convex set and $S \subset D$ be a simplex with vertices $v_0, \ldots, v_n$ and barycenter $\mathbf{b}$. If $f : D \to \mathbb{R}$ is W-convex, then*

$$(n+1)f(\mathbf{b}) \le \frac{1}{|S|} \int\limits_S \Big(\sum_{\sigma \in C} f_\sigma(x) \Big) \,\mathrm{d}x \le \sum_{i=0}^{n} f(v_i).$$

Proof. Let $F = \sum_{\sigma \in C} f_\sigma$ be the symmetrization of f on S. By the previous theorem F is convex on S. By the Hermite-Hadamard inequality (cf. Theorem 2.1) we get

$$F(\mathbf{b}) \le \frac{1}{|S|} \int\limits_S F(x) \,\mathrm{d}x \le \frac{1}{n+1} \sum_{i=0}^{n} F(v_i)\,.$$

Observe that $\sigma(\mathbf{b}) = \mathbf{b}$. Therefore

$$F(\mathbf{b}) = \sum_{\sigma \in C} f_\sigma(\mathbf{b}) = \sum_{\sigma \in C} f\left(\sigma(\mathbf{b})\right) = (n+1)f(\mathbf{b})\,, \tag{3.1}$$

whereas

$$\sum_{i=0}^{n} F(v_i) = \sum_{i=0}^{n} \sum_{\sigma \in C} f_\sigma(v_i) = \sum_{i=0}^{n} \big(f(v_0) + \ldots + f(v_n)\big) = (n+1) \sum_{i=0}^{n} f(v_i), \tag{3.2}$$

which completes the proof. □

Remark 3.3. For $n = 1$ and $S = [a, b]$ we obtain as a consequence inequality (1.1) orginally due to Olbryś [10].

Observe that in the Hermite-Hadamard inequality (2.2) one fact was essential: the integral mean value $\mathcal{T}[f] = \frac{1}{|S|} \int_S f(x) \,\mathrm{d}x$ is a positive linear operator. This motivated the second-named author to prove the operator version of the inequality (2.2) (cf. [13, Theorem 2]). Our next theorem offers an extension of Theorem 3.2 to positive linear operators going in this direction.

Theorem 3.4. *Let $D \subset \mathbb{R}^n$ be an open and convex set and $S \subset D$ be a simplex with vertices $v_0, \ldots, v_n$ and barycenter $\mathbf{b}$. Let $\mathcal{T}$ be a positive linear functional defined (at least) on a linear subspace of all functions mapping S into $\mathbb{R}$ generated by a cone of convex functions. Assume that*

$$\mathcal{T}(\pi_i) = \frac{1}{|S|} \int\limits_S \pi_i(x) \,\mathrm{d}x, \quad i = 1, \ldots, n\,,$$

where π_i is the projection onto the i–th axis and $\mathcal{T}(\mathbf{1}) = 1$. If $f : D \to \mathbb{R}$ is W-convex and F is the symmetrization of f on S, then

$$F(\mathbf{b}) \le \mathcal{T}[F] \le \frac{1}{n+1}\sum_{i=0}^{n} F(v_i)\,. \tag{3.3}$$

Proof. Take an arbitrary affine function $\varphi : \mathbb{R}^n \to \mathbb{R}$. It has a form

$$\varphi(x) = \sum_{i=0}^{n} \alpha_i x_i + \beta = \sum_{i=0}^{n} \alpha_i \pi_i(x) + \beta\,, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

for some scalars $\alpha_0, \ldots, \alpha_n, \beta$. The linearity yields

$$\begin{aligned}\mathcal{T}[\varphi] &= \mathcal{T}\Big[\sum_{i=0}^{n} \alpha_i \pi_i + \beta\Big] = \sum_{i=0}^{n} \alpha_i \mathcal{T}[\pi_i] + \beta \mathcal{T}[\mathbf{1}] = \sum_{i=0}^{n} \frac{\alpha_i}{|S|} \int\limits_S \pi_i(x)\,\mathrm{d}x + \beta \\ &= \frac{1}{|S|}\int\limits_S \Big(\sum_{i=0}^{n} \alpha_i \pi_i(x) + \beta\Big)\,\mathrm{d}x = \frac{1}{|S|}\int\limits_S \varphi(x)\,\mathrm{d}x\,.\end{aligned}$$

Therefore $\mathcal{T}$ meets the assumptions of [13, Theorem 2]. Hence, by convexity of F, the inequality (3.3) holds. □

Of course, taking in the above theorem $\mathcal{T}[f] = \frac{1}{|S|}\int_S f(x)\,\mathrm{d}x$, we obtain immediately Theorem 3.2.

4. HERMITE-HADAMARD TYPE INEQUALITY FOR STRONGLY CONVEX FUNCTIONS

From now on we consider the space $\mathbb{R}^n$ equipped with the Euclidean norm. Let $D \subset \mathbb{R}^n$ be a convex set and $c > 0$. The function $f : D \to \mathbb{R}$ is called *strongly convex with modulus c*, if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$. Strongly convex functions were introduced by Polyak [11] (see also [5] for some interesting remarks on this class of functions). Let us only mention that a strongly convex function is necessarily convex, but the converse does not hold (for instance, affine functions are not strongly convex).

Below we present the multivariate counterpart of a result due to Merentes and Nikodem [5].

Theorem 4.1. *If $f : S \to \mathbb{R}$ is strongly convex with modulus c, then*

$$\begin{aligned} & f(\mathbf{b}) + c\left(\frac{1}{|S|}\int\limits_S \|x\|^2 \,\mathrm{d}x - \|\mathbf{b}\|^2\right) \\ & \le \frac{1}{|S|}\int\limits_S f(x)\,\mathrm{d}x \\ & \le \frac{1}{n+1}\sum_{i=0}^n f(v_i) + c\left(\frac{1}{|S|}\int\limits_S \|x\|^2\,\mathrm{d}x - \frac{1}{n+1}\sum_{i=0}^n \|v_i\|^2\right). \end{aligned}$$

Proof. We take a function $g : S \to \mathbb{R}$ of the form $g = f - c\|\cdot\|^2$. Since f is strongly convex with modulus c, then g is convex (for a quick reference see [4] or [5]). Therefore g satisfies the Hermite-Hadamard inequality (2.2):

$$g(\mathbf{b}) \le \frac{1}{|S|}\int\limits_S g(x)\,\mathrm{d}x \le \frac{1}{n+1}\sum_{i=0}^n g(v_i).$$

Thus

$$f(\mathbf{b}) - c\|\mathbf{b}\|^2 \le \frac{1}{|S|}\int\limits_S f(x)\,\mathrm{d}x - \frac{c}{|S|}\int\limits_S \|x\|^2\,\mathrm{d}x \le \frac{1}{n+1}\sum_{i=0}^n f(v_i) - \frac{c}{n+1}\sum_{i=0}^n \|v_i\|^2$$

and our result follows by adding the term $\frac{c}{|S|}\int_S \|x\|^2\,\mathrm{d}x$ to both sides of the above inequality. □

Denote by S_1 the unit simplex in $\mathbb{R}^n$, i.e. the simplex with vertices $e_0 = (0,0,\dots,0)$, $e_1 = (1,0,\dots,0)$, $\dots$, $e_n = (0,\dots,0,1)$. Then

$$\mathbf{b} = \frac{1}{n+1}\sum_{i=0}^n e_i\,.$$

It is well-known that $|S| = \frac{1}{n!}$. The second-named author noticed in [13] (proof of Corollary 8) that

$$\int\limits_{S_1} \pi_i^2(x)\,\mathrm{d}x = \frac{2}{(n+2)!}\,, \quad i = 1,\dots,n\,. \tag{4.1}$$

For strongly convex functions defined on the unit simplex S_1, Theorem 4.1 together with (4.1) gives us the following corollary.

Corollary 4.2. *Suppose that $f : S_1 \to \mathbb{R}$ is strongly convex with modulus c. Then*

$$f(\mathbf{b}) + \frac{cn^2}{(n+1)^2(n+2)} \le n!\int\limits_{S_1} f(x)\,\mathrm{d}x \le \frac{1}{n+1}\sum_{i=0}^n f(e_i) - \frac{cn^2}{(n+1)(n+2)}.$$

Remark 4.3. For $n = 1$ we obtain the inequality

$$f\left(\frac{1}{2}\right) + \frac{c}{12} \leq \int_0^1 f(x)\,\mathrm{d}x \leq \frac{f(0) + f(1)}{2} - \frac{c}{6}\,,$$

which corresponds to the result of Merentes and Nikodem presented in [5, Theorem 6] (the full version of their inequality could be derived from Theorem 4.1 by setting $n = 1$ and an arbitrary compact interval $[a, b]$ in the role of S).

5. HERMITE-HADAMARD TYPE INEQUALITY FOR STRONGLY W-CONVEX FUNCTIONS

Let $D \subset \mathbb{R}^n$ be a convex set and $c > 0$. The function $f : D \to \mathbb{R}$ is called *strongly W-convex with modulus* c, if

$$f\big(tx + (1-t)y\big) + f\big((1-t)x + ty\big) \leq f(x) + f(y) - 2ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$. Such functions were introduced by Merentes, Nikodem and Rivas in [6].

Theorem 5.1. *Let $D \subset \mathbb{R}^n$ be an open and convex set and let $S \subset D$ be a simplex with vertices $v_0, \ldots, v_n$ and barycenter $\mathbf{b}$. If $f : D \to \mathbb{R}$ is strongly W-convex with modulus c, then*

$$\begin{aligned}
&f(\mathbf{b}) + c\bigg(\frac{1}{|S|}\int_S \|x\|^2\,\mathrm{d}x - \|\mathbf{b}\|^2\bigg)\\
&\leq \frac{1}{(n+1)|S|}\int_S \bigg(\sum_{\sigma\in C} f_\sigma(x)\,\mathrm{d}x\bigg)\\
&\leq \frac{1}{n+1}\sum_{i=0}^{n} f(v_i) + c\bigg(\frac{1}{|S|}\int_S \|x\|^2\,\mathrm{d}x - \frac{1}{n+1}\sum_{i=0}^{n}\|v_i\|^2\bigg).
\end{aligned}$$

Proof. Because f is strongly W-convex with modulus c, the function $f - c\|\cdot\|^2$ is W-convex (cf. [6, Theorem 4]). Then the desired inequality follows by Theorem 3.2. To perform the easy computation let us only observe that by [12, Lemma 2.1] we have $\int_S \|x\|^2\,\mathrm{d}x = \int_S \|\sigma(x)\|^2\,\mathrm{d}x$ for any $\sigma \in C$. We omit further details. □

For strongly W-convex functions on the unit simplex S_1, Theorem 5.1 together with (4.1) gives us the following corollary.

Corollary 5.2. *Let $D \subset \mathbb{R}^n$ be an open and convex such that $S_1 \subset D$. If $f : D \to \mathbb{R}$ is strongly W-convex with modulus c, then*

$$f(\mathbf{b}) + \frac{cn^2}{(n+1)^2(n+2)} \le \frac{n!}{(n+1)} \int\limits_{S_1} \left(\sum_{\sigma \in C} f_\sigma(x)\,\mathrm{d}x \right) \le \frac{1}{n+1} \sum_{i=0}^{n} f(e_i) - \frac{cn^2}{(n+1)(n+2)}.$$

Remark 5.3. For $n = 1$ we obtain the inequality

$$f\left(\frac{1}{2}\right) + \frac{c}{12} \le \frac{1}{2} \int\limits_0^1 \big(f(x) + f(1-x)\big)\,\mathrm{d}x \le \frac{f(0)+f(1)}{2} - \frac{c}{6}$$

(compare with Remark 4.3). The similar inequality for an arbitrary compact interval $[a, b]$ could be derived directly from Theorem 5.1.

REFERENCES

[1] M. Bessenyei, *The Hermite–Hadamard inequality on simplices*, Amer. Math. Monthly **115** (2008), 339–345.

[2] S.S. Dragomir, C.E.M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2002.

[3] A. Guessab, G. Schmeisser, *Convexity results and sharp error estimates in approximate multivariate integration*, Math. Comp. **73** (2004), 1365–1384.

[4] J.B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of convex analysis*, Springer–Verlag, Berlin Heidelberg, 2001.

[5] N. Merentes, K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math. **80** (2010), 193–199.

[6] N. Merentes, K. Nikodem, S. Rivas, *Remarks on strongly Wright-convex functions*, Ann. Polon. Math. **102** (2011), 271–278.

[7] E. Neuman, *Inequalities involving multivariate convex functions II*, Proc. Amer. Math. Soc. **109** (1990), 965–974.

[8] C.T. Ng. *Functions generating Schur-convex sums*, [in:] *General inequalities, 5 (Oberwolfach, 1986)*, vol. 80 of Internat. Schriftenreihe Numer. Math., pp. 433–438. Birkhäuser, Basel, 1987.

[9] C.P. Niculescu, L.E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, Springer, New York 2006.

[10] A. Olbryś, *On some inequalities equivalent to the Wright convexity*, submitted.

[11] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. **7** (1966), 72–75.

[12] Sz. Wąsowicz, A. Witkowski, *On some inequality of Hermite-Hadamard type*, Opuscula Math. **32** (2012), 591–600.

[13] Sz. Wąsowicz, *Hermite–Hadamard-type inequalities in the approximate integration*, Math. Inequal. Appl. **11** (2008), 693–700.

Dorota Śliwińska
dsliwinska@ath.bielsko.pl

University of Bielsko-Biała
Department of Mathematics and Computer Science
Willowa 2, 43–309 Bielsko-Biała, Poland

Szymon Wąsowicz
swasowicz@ath.eu

University of Bielsko-Biała
Department of Mathematics and Computer Science
Willowa 2, 43–309 Bielsko-Biała, Poland

Received: December 19, 2013.
Revised: September 29, 2014.
Accepted: September 29, 2014.