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GENERALIZED LEVINSON'S INEQUALITY AND EXPONENTIAL CONVEXITY

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Abstract. We give a probabilistic version of Levinson's inequality under Mercer's assumption of equal variances for the family of 3-convex functions at a point. We also show that this is the largest family of continuous functions for which the inequality holds. New families of exponentially convex functions and related results are derived from the obtained inequality.

Keywords: Levinson's inequality, exponential convexity.

Mathematics Subject Classification: 26D15.

1. INTRODUCTION

In [5] Levinson proved the following inequality:

Theorem 1.1. *If $f:(0,2c)\to\mathbb{R}$ satisfies $f'''\geq 0$ and p_i, x_i, y_i, $i=1,2,\ldots,n$, are such that $p_i>0$, $\sum_{i=1}^n p_i=1$, $0\leqslant x_i\leqslant c$ and*

$$x_1+y_1=x_2+y_2=\ldots=x_n+y_n=2c, \tag{1.1}$$

then the inequality

$$\sum_{i=1}^n p_i f(x_i)-f(\overline{x})\leqslant\sum_{i=1}^n p_i f(y_i)-f(\overline{y}) \tag{1.2}$$

holds, where $\overline{x}=\sum_{i=1}^n p_i x_i$ and $\overline{y}=\sum_{i=1}^n p_i y_i$ denote the weighted arithmetic means.

Numerous papers have been devoted to generalizations and extensions of Levinson's result. Popoviciu showed in [10] that the assumptions on the differentiability of f can be weakened and for Theorem 1.1 to hold it is enough to assume that f is 3-convex. In [4] Bullen gave another proof of Popoviciu's result, as well as a converse of the inequality (rescaled to a general interval $[a,b]$). Bullen's result is the following.

Theorem 1.2.

(i) *If* $f : [a,b] \to \mathbb{R}$ *is 3-convex and* p_i, x_i, y_i, $i = 1, 2, \ldots, n$, *are such that* $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (1.1) *holds* (*for some* $c \in [a,b]$) *and*

$$\max(x_1, \ldots, x_n) \le \min(y_1, \ldots, y_n), \tag{1.3}$$

then (1.2) *holds.*

(ii) *If for a continuous function* f *inequality* (1.2) *holds for all* n, *all* $c \in [a,b]$, *all* $2n$ *distinct points satisfying* (1.1) *and* (1.3) *and all weights* $p_i > 0$ *such that* $\sum_{i=1}^{n} p_i = 1$, *then* f *is 3-convex.*

The aforementioned generalizations of Levinson's inequality assume that (1.1) holds, i.e. that the distribution of the points x_i is equal to the distribution of the points y_i reflected around the point $c \in [a,b]$. Mercer ([6]) made a significant improvement by replacing this condition of symmetric distribution with the weaker one that the variances of the two sequences are equal.

Theorem 1.3. *If* $f : [a,b] \to \mathbb{R}$ *satisfies* $f''' \ge 0$ *and* p_i, x_i, y_i, $i = 1, 2, \ldots, n$, *are such that* $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (1.3) *holds and*

$$\sum_{i=1}^{n} p_i (x_i - \overline{x})^2 = \sum_{i=1}^{n} p_i (y_i - \overline{y})^2, \tag{1.4}$$

then (1.2) *holds.*

In [11] Witkowski extended this result in several ways. Firstly, he showed that Levinson's inequality can be stated in a more general setting with random variables. Furthermore, he showed that it is enough to assume that f is 3-convex and that the assumption (1.4) of equality of the variances can be weakened to inequality in a certain direction. In the following, $\mathbb{E}(Z)$ and $\mathrm{Var}(Z)$ denote the expectation and variance, respectively, of a random variable Z.

Theorem 1.4. *Let* I *be an open interval of* R (*bounded or unbounded*), $f : I \to R$ *be a* 3*-convex function and* $X, Y : (\Omega, \mu) \to I$ *be two random variables satisfying*

(i) $\mathbb{E}(X^2)$, $\mathbb{E}(Y^2)$, $\mathbb{E}(f(X))$, $\mathbb{E}(f(Y))$, $\mathbb{E}(f'(X))$, $\mathbb{E}(f'(Y))$, $\mathbb{E}(Xf'(X))$, $\mathbb{E}(Yf'(Y))$ *are finite,*

(ii) $\operatorname{ess\,sup} X \le \operatorname{ess\,inf} Y$,

(iii) $f''_+(\operatorname{ess\,sup} X) > 0$ *and* $\mathrm{Var}(X) \le \mathrm{Var}(Y)$, *or* $f''_-(\operatorname{ess\,inf} Y) < 0$ *and* $\mathrm{Var}(X) \ge \mathrm{Var}(Y)$, *or* $f''_+(\operatorname{ess\,sup} X) < 0 < f''_-(\operatorname{ess\,inf} Y)$.

Then

$$\mathbb{E}(f(X)) - f(\mathbb{E}(X)) \le \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)). \tag{1.5}$$

Notice that when $\mathrm{Var}(X) = \mathrm{Var}(Y)$, then assumption (iii) of Theorem 1.4 is automatically satisfied, and for discrete random variables assumption (i) is satisfied.

Therefore, for discrete random variables X taking values x_i with probabilities p_i and Y taking values y_j with probabilities q_j inequality (1.5) becomes

$$\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \le \sum_{j=1}^{m} q_j f(y_j) - f(\bar{y}), \tag{1.6}$$

and it holds for every 3-convex function f if $\max_i x_i \le \min_j y_j$, $p_i > 0$ and $q_j > 0$ are such that

$$\sum_{i=1}^{n} p_i = \sum_{j=1}^{m} q_j = 1 \quad \text{and} \quad \sum_{i=1}^{n} p_i (x - \bar{x})^2 = \sum_{j=1}^{m} q_j (y_j - \bar{y})^2, \tag{1.7}$$

where $\bar{x} = \sum_{i=1}^{n} p_i x_i$ and $\bar{y} = \sum_{j=1}^{m} q_j y_j$. This result for $n = m$ and $p_i = q_i$ was proven be Witkowski ([12]). Building on Witkowski's ideas ([12]), Baloch, Pečarić and Praljak ([3]) showed that, in this case, Levinson's inequality holds for a larger class of functions given in the following definition.

Definition 1.5. Let $f : I \to \mathbb{R}$ and $c \in I^\circ$, where I is an arbitrary interval I (open, closed or semi-open in either direction) in $\mathbb{R}$ and I° is its interior. We say that $f \in \mathcal{K}_1^c(I)$ (resp. $f \in \mathcal{K}_2^c(I)$) if there exists a constant A such that the function $F(x) = f(x) - \frac{A}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$.

A function $f \in \mathcal{K}_1^c(I)$ is said to be 3-convex at point c and $\mathcal{K}_1^c(I)$ generalizes 3-convex functions in the following sense: a function is 3-convex on I if and only if it is 3-convex at every $c \in I^\circ$. Baloch, Pečarić and Praljak ([3]) also proved the converse of Levinson's inequality for continuous functions.

Theorem 1.6.

(i) *Let $a < x_i \le c \le y_i < b$, $p_i > 0$ for $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$ and* (1.4) *holds. If $f \in \mathcal{K}_1^c((a,b))$, then inequality* (1.2) *holds and if $f \in \mathcal{K}_2^c((a,b))$, then* (1.2) *holds with the reverse sign of inequality.*

(ii) *Let $f : (a,b) \to \mathbb{R}$ be continuous and $c \in (a,b)$. If inequality* (1.2) (*resp. the reverse of* (1.2)) *holds for every $n \in \mathbb{N}$ and sequences p_i, x_i, y_i, $i = 1, \ldots, n$, such that $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $a < x_i \le c \le y_i < b$ and* (1.4) *holds, then $f \in \mathcal{K}_1^c((a,b))$ (resp. $f \in \mathcal{K}_2^c((a,b))$).*

In Section 2 of this paper we will prove the probabilistic version (1.5) of Levinson's inequality for the class of 3-convex functions at a point, which will generalize the results of Theorems 1.4 and 1.6 (i). We will also prove a converse stronger than Theorem 1.6 (ii). In addition to being more general, the proofs of the results from Section 2 will be significantly more elegant and intuitive than the rather technical and convoluted proofs from [3]. In Section 3 we will give mean value type results. In Section 4 we will give refinements of the results obtained in the second section by constructing certain exponentially convex functions and applying methods from [8]. The obtained results will generalize the results of Anwar and Pečarić given in [1] and [2].

2. MAIN RESULTS

Let us first recall the probabilistic version of Jensen's inequality (see, for example, [9, Theorem 2.3]).

Theorem 2.1. *Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ a convex function and $X : \Omega \to I$ a random variable such that $\mathbb{E}(X)$ and $\mathbb{E}(f(X))$ are finite. Then*

$$f(\mathbb{E}(X)) \leq \mathbb{E}(f(X)).$$

If f is concave, then the inequality is reversed.

The following theorem is our main result and it represents a probabilistic version of Levinson's inequality under the assumption of equal variances.

Theorem 2.2. *Let $X, Y : \Omega \to I$ be two random variables such that*

$$\mathrm{Var}(X) = \mathrm{Var}(Y) < \infty \tag{2.1}$$

and that there exists $c \in I^\circ$ such that

$$\text{ess sup}\, X \leq c \leq \text{ess inf}\, Y. \tag{2.2}$$

Then for every $f \in \mathcal{K}_1^c(I)$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite, inequality (1.5) *holds.*

Proof. Let $F(x) = f(x) - \frac{A}{2}x^2$, where A is the constant from Definition 1.5. Since $F : I \cap (-\infty, c] \to \mathbb{R}$ is concave, Jensen's inequality implies

$$\begin{aligned} 0 &\leq F(\mathbb{E}(X)) - \mathbb{E}(F(X)) \\ &= f(\mathbb{E}(X)) - \frac{A}{2}\mathbb{E}^2(X) - \mathbb{E}(f(X)) + \frac{A}{2}\mathbb{E}(X^2) \\ &= f(\mathbb{E}(X)) - \mathbb{E}(f(X)) + \frac{A}{2}\mathrm{Var}(X). \end{aligned} \tag{2.3}$$

Similarly, $F : I \cap [c, \infty) \to \mathbb{R}$ is convex, so

$$\begin{aligned} 0 &\leq \mathbb{E}(F(Y)) - F(\mathbb{E}(Y)) \\ &= \mathbb{E}(f(Y)) - \frac{A}{2}\mathbb{E}(Y^2) - f(\mathbb{E}(Y)) + \frac{A}{2}\mathbb{E}^2(Y) \\ &= \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)) - \frac{A}{2}\mathrm{Var}(Y). \end{aligned} \tag{2.4}$$

Adding up (2.3) and (2.4) we obtain

$$0 = \frac{A}{2}(\mathrm{Var}(Y) - \mathrm{Var}(X)) \leq \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)) - [\mathbb{E}(f(X)) - f(\mathbb{E}(X))],$$

which completes the proof. □

Remark 2.3. It is obvious from the proof that Levinson's inequality (1.5) holds if the equality (2.1) is replaced by the weaker condition

$$A(\mathrm{Var}(Y) - \mathrm{Var}(X)) \geq 0.$$

Since $f''_-(c) \leq A \leq f''_+(c)$ (see [3]), if, additionally, f is convex (resp. concave), this condition can be further weakened to $\mathrm{Var}(Y) - \mathrm{Var}(X) \geq 0$ (resp. ≤ 0).

Corollary 2.4. *If $x_i \in I \cap (-\infty, c]$, $y_j \in I \cap [c, \infty)$, $p_i > 0$ and $q_j > 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$, are such that* (1.7) *holds, then for every $f \in \mathcal{K}_1^c(I)$ inequality* (1.6) *holds.*

Proof. Apply Theorem 2.2 to discrete random variables X taking values x_i with probabilities p_i, $i = 1, \ldots, n$, and Y taking values y_j with probabilities q_j, $j = 1, \ldots, m$. □

Our next goal is to prove the converse of Theorem 2.2, i.e. to show that inequality (1.5) characterizes the class $\mathcal{K}_1^c(I)$. In fact, we will show that it is enough to assume that inequality (1.5) holds for a very special type of random variable to insure that f belongs to $\mathcal{K}_1^c(I)$. If X is a random variable that takes values x_1 and x_2 with probabilities $\frac{1}{2}$ and Y is a random variable that takes values y_1 and y_2 with probabilities $\frac{1}{2}$, then (1.5) is equivalent to

$$\frac{f(x_1) + f(x_2)}{2} - f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(y_1) + f(y_2)}{2} - f\left(\frac{y_1 + y_2}{2}\right). \tag{2.5}$$

For a function g and points u and v, $u \neq v$, let us introduce the notation

$$|u, v|g = [u, v, \frac{u+v}{2}]g = \frac{\frac{g(u)+g(v)}{2} - g(\frac{u+v}{2})}{(\frac{v-u}{2})^2}.$$

Since $\mathrm{Var}(X) = \left(\frac{x_2 - x_1}{2}\right)^2$, one has

$$\mathbb{E}(f(X)) - f(\mathbb{E}(X)) = \mathrm{Var}(X)|x_1, x_2|f.$$

Therefore, if $\mathrm{Var}(X) = \mathrm{Var}(Y)$ (i.e. if $|x_2 - x_1| = |y_2 - y_1|$), then (2.5) is equivalent to

$$|x_1, x_2|f \leq |y_1, y_2|f. \tag{2.6}$$

We will use the following lemma to prove our main result.

Lemma 2.5. *Suppose that inequality* (2.6) *holds for all $x_1, x_2 \leq c \leq y_1, y_2$ such that $|y_2 - y_1| = \alpha|x_2 - x_1|$. Then* (2.6) *holds for all $x_1, x_2 \leq c \leq y_1, y_2$ and for all positive rational μ such that $|y_2 - y_1| = \mu\alpha|x_2 - x_1|$.*

Proof. Without loss of generality we can assume $y_1 < y_2$. Assume first that $\mu = k \in \mathbb{N}$ is natural and denote $d = \alpha|x_2 - x_1|$. Divide the interval $[y_1, y_2]$ into $2k$ segments of equal length. Denote by $a_0 = y_1, a_1, \ldots, a_k = y_2$ the even points of the division and by $b_0, \ldots, b_{k-1}$ the odd points (i.e. $a_i = y_1 + id$, $b_i = y_1 + (i + \frac{1}{2})d$). If $[u, v]$ is a segment of length dw centred at $(y_1 + y_2)/2$, then let us denote $S_w = \frac{f(u)+f(v)}{2} - f(\frac{y_1+y_2}{2})$.

The length of segments $[a_i, a_{i+1}]$ and $[b_i, b_{i+1}]$ equal d, so by the assumption of the lemma $|x_1, x_2|f \le |a_i, a_{i+1}|f$ and $|x_1, x_2|f \le |b_i, b_{i+1}|f$. Adding up these inequalities one can get

$$k|x_1, x_2|f \le \frac{4}{d^2}\sum_{i=0}^{k-1}\left(\frac{f(a_i)+f(a_{i+1})}{2} - f(b_i)\right),$$
$$(k-1)|x_1, x_2|f \le \frac{4}{d^2}\sum_{i=0}^{k-2}\left(\frac{f(b_i)+f(b_{i+1})}{2} - f(a_{i+1})\right).$$

Adding up and simplifying we get

$$(2k-1)|x_1, x_2|f \le \frac{4}{d^2}\left(\frac{f(a_0)+f(a_k)}{2} - \frac{f(b_0)+f(b_{k-1})}{2}\right) = \frac{4}{d^2}(S_k - S_{k-1}).$$

Repeating the same reasoning with segments $[b_0, b_{k-1}]$ (of length $b_{k-1} - b_0 = (k-1)d$) and $[a_1, a_{k-1}]$ (of length $a_{k-1} - a_1 = (k-2)d$), and so on we get

$$(1 + 3 + \ldots + 2k - 1)|x_1, x_2|f \le \frac{4}{d^2}S_k$$

which is equivalent to

$$|x_1, x_2|f \le \frac{1}{(\frac{kd}{2})^2}\left(\frac{f(y_1)+f(y_2)}{2} - f\left(\frac{y_1+y_2}{2}\right)\right) = |y_1, y_2|f.$$

Now let $\mu = k/m$. If $[u, v] \subset [x_1, x_2]$ is a subsegment of length $|x_2 - x_1|/m$, then $|y_2 - y_1| = k\alpha|v - u|$, so the previous part yields $|u, v|f \le |y_1, y_2|f$. Divide the segment $[x_1, x_2]$ into $2m$ equal parts and proceed as above to obtain the result. □

Now we are ready to state and prove the main result.

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be a continuous function. Suppose that for fixed $c \in I^\circ$ and all $x_1, x_2 \in I \cap (-\infty, c]$, $y_1, y_2 \in I \cap [c, \infty)$ such that $|x_2 - x_1| = |y_2 - y_1|$ inequality* (2.5) *holds. Then $f \in \mathcal{K}_1^c(I)$.*

Proof. Clearly, (2.5) is equivalent to (2.6). Choose arbitrary $x_1, x_2 \in I \cap (-\infty, c]$, $y_1, y_2 \in I \cap [c, \infty)$ and choose rational sequences such that

$$x_i^{(1)} \to x_1, x_i^{(2)} \to x_2, y_i^{(1)} \to y_1, y_i^{(2)} \to y_2.$$

By Lemma 2.5, for every $i \in \mathbb{N}$, we have $|x_i^{(1)}, x_i^{(2)}|f \le |y_i^{(1)}, y_i^{(2)}|f$, and continuity of f implies $|x_1, x_2|f \le |y_1, y_2|f$.

This means there exists a number A such that

$$\sup_{x_1, x_2 \in I \cap (-\infty, c]} |x_1, x_2|f \le \frac{A}{2} \le \inf_{y_1, y_2 \in I \cap [c, \infty)} |y_1, y_2|f.$$

Notice that $F(x) = f(x) - Ax^2/2$ satisfies $|u, v|F = |u, v|f - A/2$. Therefore,

$$\sup_{x_1,x_2 \in I\cap(-\infty,c]} |x_1, x_2|F \leq 0 \leq \inf_{y_1,y_2\in I\cap[c,\infty)} |y_1, y_2|F$$

which implies that F is Jensen concave on $I\cap(-\infty, c]$ and Jensen convex on $I\cap[c, \infty)$. Since the case of continuous function Jensen convexity (Jensen concavity) implies convexity (concavity) (see [9]), this finishes the proof. □

3. MEAN VALUE THEOREMS

Notice that Levinson's inequality (1.5) is linear in f. This motivates us to define the following linear functional: for fixed random variables $X, Y : \Omega \to I$ and $c \in I^\circ$ such that (2.1) and (2.2) hold, we define

$$\Lambda(f) = \mathbb{E}(f(Y)) - f(\mathbb{E}(Y)) - \mathbb{E}(f(X)) + f(\mathbb{E}(X)) \tag{3.1}$$

for functions $f : I \to \mathbb{R}$ such that $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite. Notice that Theorem 2.2 guarantees that $\Lambda(f) \geq 0$ for $f \in \mathcal{K}_1^c(I)$.

We will give two mean value results.

Theorem 3.1. *Let $-\infty < a < c < b < \infty$, $I = [a, b]$, $X, Y : \Omega \to I$ be two random variables such that* (2.1) *and* (2.2) *hold, and let Λ be given by* (3.1). *Then for $f \in C^3([a, b])$ there exists $\xi \in [a, b]$ such that*

$$\Lambda(f) = \frac{f'''(\xi)}{6} \left[\mathbb{E}(Y^3 - X^3) - \mathbb{E}^3(Y) + \mathbb{E}^3(X)\right]. \tag{3.2}$$

Proof. Since f is bounded, $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite and $\Lambda(f)$ is well defined. Furthermore, since $f \in C^3([a, b])$, there exist $m = \min_{x\in[a,b]} f'''(x)$ and $M = \max_{x\in[a,b]} f'''(x)$. The functions

$$f_1(x) = f(x) - \frac{m}{6}x^3,$$
$$f_2(x) = \frac{M}{6}x^3 - f(x)$$

are 3-convex since $f_i'''(x) \geq 0$, $i = 1, 2$. Hence, by Theorem 2.2, we have $\Lambda(f_i) \geq 0$, $i = 1, 2$, and we get

$$\frac{m}{6}\Lambda(id^3) \leq \Lambda(f) \leq \frac{M}{6}\Lambda(id^3), \tag{3.3}$$

where $id(x) = x$. Since id^3 is 3-convex, by Theorem 2.2, we have

$$0 \leq \Lambda(id^3) = \mathbb{E}(Y^3 - X^3) - \mathbb{E}^3(Y) + \mathbb{E}^3(X).$$

If $\Lambda(id^3) = 0$, then (3.3) implies $\Lambda(f) = 0$ and (3.2) holds for every $\xi \in [a, b]$. Otherwise, dividing (3.3) by $0 < \Lambda(id^3)/6$ we get

$$m \leq \frac{6\Lambda(f)}{\Lambda(id^3)} \leq M,$$

so continuity of f''' insures existence of $\xi \in [a, b]$ satisfying (3.2). □

Theorem 3.2. *Let I, c, X, Y and Λ be as in Theorem 3.1 and let $f, g \in C^3([a,b])$. If $\Lambda(g) \neq 0$, then there exists $\xi \in [a,b]$ such that either*

$$\frac{\Lambda(f)}{\Lambda(g)} = \frac{f'''(\xi)}{g'''(\xi)},$$

or $f'''(\xi) = g'''(\xi) = 0$.

Proof. Define $h \in C^3([a,b])$ by $h(x) = \alpha f(x) - \beta g(x)$, where $\alpha = \Lambda(g)$, $\beta = \Lambda(f)$. Due to the linearity of Λ we have $\Lambda(h) = 0$. Now, by Theorem 3.1, there exist $\xi, \xi_1 \in [a,b]$ such that

$$\begin{aligned} 0 = \Lambda(h) &= \frac{h'''(\xi)}{6}\Lambda(id^3), \\ 0 \neq \Lambda(g) &= \frac{g'''(\xi_1)}{6}\Lambda(id^3), \end{aligned}$$

where $id(x) = x$. Therefore, $\Lambda(id^3) \neq 0$ and

$$0 = h'''(\xi) = \alpha f'''(\xi) - \beta g'''(\xi),$$

which gives the claim of the theorem. □

Remark 3.3. Theorems 3.1 and 3.2 are generalizations of mean value results from [1]. Indeed, let $I = [0, 2a]$, $c = a$ be the midpoint of the segment and X be the discrete random variable taking values $x_i \in [0, c]$ with probabilities p_i, $i = 1, \dots, n$. The random variables $Y_1 = 2a - X$ and $Y_2 = X + a$ satisfy $\mathrm{Var}(Y_1) = \mathrm{Var}(Y_2) = \mathrm{Var}(X)$. The results from [1] can be recovered by applying Theorems 3.1 and 3.2 with the pair of random variables X and Y_1 or X and Y_2.

4. EXPONENTIAL CONVEXITY

We will first give some basic definitions and results on exponential convexity that we will use in this section.

Definition 4.1. A function $g : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j g\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \dots, n$.

A function $g : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 4.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \leq n$, $k \in \mathbb{N}$.

Definition 4.3. A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense on I for every $n \in \mathbb{N}$. A function $g : I \to \mathbb{R}$ is exponentially convex on I if it is exponentially convex in the Jensen sense and continuous on I.

Remark 4.4. A function $g : I \to \mathbb{R}$ is log-convex in the Jensen sense, i.e.

$$g\left(\frac{x_1 + x_2}{2}\right)^2 \leq g(x_1)g(x_2), \quad \text{for all } x_1, x_2 \in I, \tag{4.1}$$

if and only if

$$\xi_1^2 g(x_1) + 2\xi_1\xi_2 g\left(\frac{x_1 + x_2}{2}\right) + \xi_2^2 g(x_2) \geq 0$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}$ and $x_1, x_2 \in I$, i.e., if and only if g is 2-exponentially convex in the Jensen sense. If $g(x_1) = 0$ for some x_1 and $[a, b] \subset I$ is an arbitrary interval containing x_1, then it follows from (4.1) and non-negativity of g (see Remark 4.2) that g vanishes on $[a_1, b_1]$, where $a_1 = (a + x_1)/2$ and $b_1 = (x_1 + b)/2$. Applying the same reasoning to intervals $[a, a_1]$ and $[b_1, b]$ we obtain sequences $a_n \searrow a$ and $b_n \nearrow b$ with g vanishing on $[a_n, b_n]$. Thus g is zero on (a, b) and a function that is 2-exponentially convex in the Jensen sense is either identically equal to zero or it is strictly positive and log-convex in the Jensen sense.

The following lemma is equivalent to the definition of convex functions (see [9, p. 2]).

Lemma 4.5. *A function $g : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)g(x_1) + (x_1 - x_3)g(x_2) + (x_2 - x_1)g(x_3) \geq 0$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

We will also need the following result (see [9, p. 2]).

Lemma 4.6. *If g is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$ and $y_1 \neq y_2$, then the following inequality holds*

$$\frac{g(x_2) - g(x_1)}{x_2 - x_1} \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1}. \tag{4.2}$$

If the function g is concave then the sign of the above inequality is reversed.

The following results will enable us to construct exponentially convex functions.

Theorem 4.7. *Let $X, Y : \Omega \to I$ be two random variables and $c \in I^\circ$ such that (2.1) and (2.2) hold and let Λ be given by (3.1). Furthermore, let $\Upsilon = \{f_t : I \to \mathbb{R} \mid t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that, for every $t \in J$, $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite and for every four mutually different points $u_0, u_1, u_2, u_3 \in I$ the mapping $t \mapsto [u_0, u_1, u_2, u_3]f_t$ is n-exponentially convex. Then the mapping $t \mapsto \Lambda(f_t)$ is n-exponentially convex in the Jensen sense on J. If the mapping $t \mapsto \Lambda(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$ and $t_i \in J$, $i = 1, \dots, n$, we define the function

$$f(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(x).$$

Due to linearity of the divided differences and the assumption that the function $t \mapsto [u_0, u_1, u_2, u_3]f_t$ is n-exponentially convex in the Jensen sense we have

$$[u_0, u_1, u_2, u_3]f = \sum_{i,j=1}^{n} \xi_i \xi_j [u_0, u_1, u_2, u_3] f_{\frac{t_i+t_j}{2}} \geq 0.$$

This implies that f is 3-convex, so $f \in \mathcal{K}_1^c(I)$. Due to linearity of the expectation, $\mathbb{E}(f(X))$ and $\mathbb{E}(f(Y))$ are finite, so by Theorem 2.2

$$0 \leq \Lambda(f) = \sum_{i,j=1}^{n} \xi_i \xi_j \Lambda(f_{\frac{t_i+t_j}{2}}).$$

Therefore, the mapping $t \mapsto \Lambda(f_t)$ is n-exponentially convex. If it is also continuous, it is n-exponentially convex by definition. □

If the assumptions of Theorem 4.7 hold for all $n \in \mathbb{N}$, then we immediately get the following corollary.

Corollary 4.8. *Let X, Y, c and Λ be as in Theorem 4.7. Furthermore, let $\Upsilon = \{f_t : I \to \mathbb{R} \mid t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that, for every $t \in J$, $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite and for every four mutually different points $u_0, u_1, u_2, u_3 \in I$ the mapping $t \mapsto [u_0, u_1, u_2, u_3]f_t$ is exponentially convex. Then the mapping $t \mapsto \Lambda(f_t)$ is exponentially convex in the Jensen sense on J. If the mapping $t \mapsto \Lambda(f_t)$ is continuous on J, then it is exponentially convex on J.*

Corollary 4.9. *Let X, Y, c and Λ be as in Theorem 4.7. Furthermore, let $\Upsilon = \{f_t : I \to \mathbb{R} \mid t \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions such that, for every $t \in J$, $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite and for every four mutually different points $u_0, u_1, u_2, u_3 \in I$ the mapping $t \mapsto [u_0, u_1, u_2, u_3]f_t$ is 2-exponentially convex in the Jensen sense. Then the following statements hold:*

(i) *if the mapping $t \mapsto \Lambda(f_t)$ is continuous on J, then for $r, s, t \in J$ such that $r < s < t$, we have*

$$\Lambda(f_s)^{t-r} \leq \Lambda(f_r)^{t-s} \Lambda(f_t)^{s-r}, \tag{4.3}$$

(ii) *if the mapping $t \mapsto \Lambda(f_t)$ is strictly positive and differentiable on J, then for all $s, t, u, v \in J$ such that $s \leq u$ and $t \leq v$ we have*

$$\mu_{s,t}(\Upsilon) \leq \mu_{u,v}(\Upsilon),$$

where

$$\mu_{s,t}(\Upsilon) = \begin{cases} \left(\frac{\Lambda(f_s)}{\Lambda(f_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}(\Lambda(f_s))}{\Lambda(f_s)} \right), & s = t. \end{cases} \tag{4.4}$$

Proof. (i) By Theorem 4.7, the mapping $t \mapsto \Lambda(f_t)$ is 2-exponentially convex. Hence, by Remark 4.4, this mapping is either identically equal to zero, in which case inequality (4.3) holds trivially with zeros on both sides, or it is strictly positive and log-convex. Therefore, for $r, s, t \in J$ such that $r < s < t$ Lemma 4.5 with $g(t) = \log \Lambda(f_t)$ gives

$$(t-s)\log \Lambda(f_r) + (r-t)\log \Lambda(f_s) + (s-r)\log \Lambda(f_t) \geq 0.$$

This is equivalent to inequality (4.3).

(ii) By (i), the mapping $t \mapsto \Lambda(f_t)$ is log-convex on J, which means that the function $t \mapsto \log \Lambda(f_t)$ is convex on J. Hence, by using Lemma 4.6 with $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, we obtain

$$\frac{\log \Lambda(f_s) - \log \Lambda(f_t)}{s-t} \leq \frac{\log \Lambda(f_u) - \log \Lambda(f_v)}{u-v},$$

that is,

$$\mu_{s,t}(\Upsilon) \leq \mu_{u,v}(\Upsilon).$$

Finally, the limiting cases $s = t$ are $u = v$ are obtained by applying the standard continuity argument. $\square$

Consider now the family of functions

$$\Upsilon_1 = \{f_t : I \to \mathbb{R} \mid t \in \mathbb{R}\}, \qquad I \subset (0,\infty),$$

defined by

$$f_t(x) = \begin{cases} \frac{x^t - \frac{t(t-1)}{2}x^2 + t(t-2)x - \frac{(t-1)(t-2)}{2}}{t(t-1)(t-2)}, & t \neq 0,1,2, \\ \frac{1}{2}\ln x, & t = 0, \\ -x\ln x, & t = 1, \\ \frac{1}{2}x^2 \ln x, & t = 2. \end{cases} \tag{4.5}$$

The functions f_t are 3-convex since $f_t'''(x) = x^{t-3} \geq 0$. Moreover, the function

$$f(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(x)$$

satisfies

$$f'''(x) = \sum_{i,j=1}^{n} \xi_i \xi_j f'''_{\frac{t_i+t_j}{2}}(x) = \left(\sum_{i=1}^{n} \xi_i e^{\frac{t_i-3}{2}\ln x}\right)^2 \geq 0,$$

so f is 3-convex. Therefore

$$0 \leq [u_0,u_1,u_2,u_3]f = \sum_{i,j=1}^{n} \xi_i \xi_j [u_0,u_1,u_2,u_3] f_{\frac{t_i+t_j}{2}},$$

so the mapping $t \mapsto [u_0,u_1,u_2,u_3]f_t$ is n-exponentially convex in the Jensen sense. As this holds for all $n \in \mathbb{N}$, we see that the family Υ_1 satisfies the assumptions of

Corollary 4.8. For the remainder of this section we assume that $\mathbb{E}(f_t(X))$ and $\mathbb{E}(f_t(Y))$ are finite for all f_t given by (4.5). Hence, by Corollary 4.8, the mapping $t \mapsto \Lambda(f_t)$ is exponentially convex in the Jensen sense. It is straightforward to check that it is also continuous, so the mapping $t \mapsto \Lambda(f_t)$ is exponentially convex. An immediate consequence of Corollary 4.9 (i) is the following result.

Corollary 4.10. *Let $I \subset (0,\infty)$, $c \in I^\circ$, and let $X, Y : \Omega \to I$ be two random variables such that* (2.1) *and* (2.2) *hold. If $\mathbb{E}(Y^t - X^t) - \mathbb{E}^t(Y) + \mathbb{E}^t(X) \neq 0$ for some $t \in \mathbb{R}\backslash\{0,1,2\}$, then for all $r, s, t \in \mathbb{R}\backslash\{0,1,2\}$ such that $r < s < t$ we have*

$$\frac{\mathbb{E}(Y^t - X^t) - \mathbb{E}^t(Y) + \mathbb{E}^t(X)}{t(t-1)(t-2)} \geq \left(\frac{\mathbb{E}(Y^s - X^s) - \mathbb{E}^s(Y) + \mathbb{E}^s(X)}{s(s-1)(s-2)}\right)^{\frac{t-r}{s-r}} \cdot \left(\frac{\mathbb{E}(Y^r - X^r) - \mathbb{E}^r(Y) + \mathbb{E}^r(X)}{r(r-1)(r-2)}\right)^{\frac{s-t}{s-r}} > 0. \quad (4.6)$$

Applying Theorem 3.2 for the functions $f = f_t$ and $g = f_s$ given by (4.5) and defined on a segment $I = [a,b] \subset (0,\infty)$, we conclude that there exist $\xi \in I$ such that

$$\xi = \left(\frac{f_s'''}{f_t'''}\right)^{-1}\left(\frac{\Lambda(f_s)}{\Lambda(f_t)}\right) = \left(\frac{\Lambda(f_s)}{\Lambda(f_t)}\right)^{\frac{1}{s-t}}, \quad s \neq t.$$

Moreover, $\mu_{s,t}(\Upsilon_1)$ given by (4.4) for the family Υ_1 can be calculated in the limiting cases $s \to t$ as well and equal

$$\mu_{s,t}(\Upsilon_1) = \begin{cases} \left(\frac{\Lambda(f_s)}{\Lambda(f_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{2\Lambda(f_s f_0)}{\Lambda(f_0)} - \frac{3s^2-6s+2}{s(s-1)(s-2)}\right), & s = t \neq 0,1,2, \\ \exp\left(\frac{\Lambda(f_0^2)}{\Lambda(f_0)} + \frac{3}{2}\right), & s = t = 0, \\ \exp\left(\frac{\Lambda(f_0 f_1)}{\Lambda(f_1)}\right), & s = t = 1, \\ \exp\left(\frac{\Lambda(f_0 f_2)}{\Lambda(f_2)} - \frac{3}{2}\right), & s = t = 2. \end{cases}$$

By Corollary 4.9 (ii), $\mu_{s,t}(\Upsilon_1)$ are monotone in parameters s and t.

Remark 4.11. By applying Corollary 4.8 to the family of functions Υ_1 given by (4.5) and the pair of discrete random variables X and Y_1 or X and Y_2 from Remark 3.3 we conclude that the mapping $t \mapsto [u_0, u_1, u_2, u_3]f_t$ is exponentially convex which generalizes the result from [2] where the log-convexity of the mapping was proven. Also, the inequalities obtained in [2] can be recovered from Corollary 4.9 (i). Furthermore, $\mu_{s,t}(\Upsilon_1)$ applied for the same family of functions and random variables yield the Cauchy means obtained in [1].

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