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DECISIVENESS OF THE SPECTRAL GAPS OF PERIODIC SCHRÖDINGER OPERATORS ON THE DUMBBELL-LIKE METRIC GRAPH

Hiroaki Niikuni

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Abstract. In this paper, we consider periodic Schrödinger operators on the dumbbell-like metric graph, which is a periodic graph consisting of lines and rings. Let one line and two rings be in the basic period. We see the relationship between the structure of graph and the band-gap spectrum.

Keywords: quantum graph, spectral gap, band structure, Hill operator.

Mathematics Subject Classification: 34L05, 34L15, 34B45.

1. INTRODUCTION AND MAIN RESULTS

Quantum graph is a metric graph equipped with a differential operator on its edges and vertex conditions on its vertices. There are a lot of papers on the spectral theory for quantum graphs as seen in textbook [1]. In the book, quantum graphs related to physical models are introduced. Especially, Korotyaev and Lobanov [7] investigated the spectra of periodic Schrödinger operators on a zigzag nanotube, which is in a class of carbon nanotube. In order to analyze the spectrum of the Hamiltonians, they gave the unitarily equivalence between the operator and the direct sum of its corresponding Hamiltonians on a quasi-1D periodic metric graph, which has a necklace structure. Namely, they reduced the problem to Hamiltonians on the metric graph consisted of lines and rings (see also [9] for the analysis of the spectrum of carbon nano-structures). The model is called the degenerate zigzag nanotube. On the other hand, the quasi-1D periodic graph consisting of only rings is considered by Duclos, Exner and Turek [2].

In this paper, we periodically add rings to the degenerate zigzag nanotube and see the relationship between the structure of graph and the band-gap structure of the spectrum. We consider the graph Γ seen in the following Figure 1 in the next page.

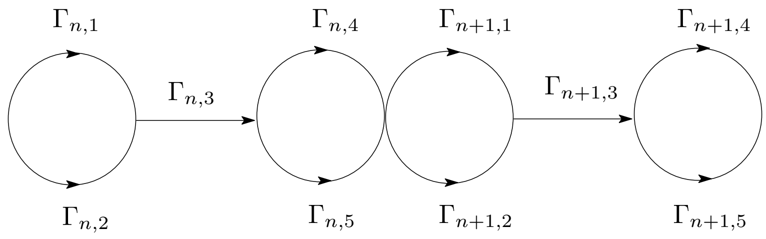

Fig. 1. The graph Γ

Let $r_0 = \frac{1}{\pi}$, $T = \frac{4}{\pi} + 1$, $\mathbb{J} = \{1, 2, 3, 4, 5\}$ and $\mathcal{Z} = \mathbb{Z} \times \mathbb{J}$. For $(n, j) \in \mathcal{Z}$, we define the segments $\Gamma_{n,j} \subset \mathbb{R}^2$ as follows:

$$\begin{aligned}
\Gamma_{n,1} &= \left\{(x, y) \,|\, \{x - (r_0 + Tn)\}^2 + y^2 = r_0^2,\ y > 0\right\},\\
\Gamma_{n,2} &= \left\{(x, y) \,|\, \{x - (r_0 + Tn)\}^2 + y^2 = r_0^2,\ y < 0\right\},\\
\Gamma_{n,3} &= \{(x, 0) \,|\, 2r_0 + Tn < x < 2r_0 + Tn + 1\},\\
\Gamma_{n,4} &= \left\{(x, y) \,|\, \{x - (3r_0 + 1 + Tn)\}^2 + y^2 = r_0^2,\ y > 0\right\},\\
\Gamma_{n,5} &= \left\{(x, y) \,|\, \{x - (3r_0 + 1 + Tn)\}^2 + y^2 = r_0^2,\ y < 0\right\}.
\end{aligned}$$

Each segment $\Gamma_{n,j}$ has the orientation from the minimum of x to the maximum of x in $\Gamma_{n,j}$. For a function y defined on Γ, we abbreviate $y_\alpha = y|_{\Gamma_\alpha}$ for $\alpha := (n, j) \in \mathcal{Z}$. Since the length of each segment is 1, we identify every $\Gamma_{n,j}$ as $(0, 1)$ below. Owing to this identification, each y_α can be identified with a function on the interval $(0, 1)$ through the local coordinate $x \in (0, 1)$. Let $\Gamma = \cup_{(n,j)\in\mathcal{Z}} \Gamma_{n,j}$ and consider the Hilbert space $\mathcal{H} = \oplus_{(n,j)\in\mathcal{Z}} L^2(\Gamma_{n,j})$ equipped with the inner product $\langle \psi, \varphi \rangle_{\mathcal{H}} = \sum_{\alpha\in\mathcal{Z}} \langle \psi_\alpha, \varphi_\alpha \rangle_{L^2(\Gamma_\alpha)}$ for $\psi, \varphi \in L^2(\Gamma)$.

For a real-valued function $q \in L^2(0, 1)$, we define

$$(Hf_\alpha)(x) = -f_\alpha''(x) + q(x)f_\alpha(x), \quad x \in (0, 1),$$

$$\mathrm{Dom}(H) = \left\{ \bigoplus_{\alpha\in\mathcal{Z}} f_\alpha \in \mathcal{H} \;\middle|\; \begin{array}{l}
\bigoplus_{\alpha\in\mathcal{Z}} (-f_\alpha'' + qf_\alpha) \in \mathcal{H},\\
f_{n,1}(1) = f_{n,2}(1) = f_{n,3}(0),\\
-f_{n,1}'(1) - f_{n,2}'(1) + f_{n,3}'(0) = 0,\\
f_{n,3}(1) = f_{n,4}(0) = f_{n,5}(0),\\
-f_{n,3}'(1) + f_{n,4}'(0) + f_{n,5}'(0) = 0,\\
f_{n,4}(1) = f_{n,5}(1) = f_{n+1,1}(0) = f_{n+1,2}(0),\\
-f_{n,4}'(1) - f_{n,5}'(1) + f_{n+1,1}'(0) + f_{n+1,2}'(0) = 0\\
\text{for} \quad n \in \mathbb{Z}
\end{array} \right\}.$$

Here, for $\alpha \in \mathcal{Z}$, $f_\alpha'(1)$ and $f_\alpha'(0)$ imply $f_\alpha'(1 - 0)$ and $f_\alpha'(+0)$, respectively. The vertex condition appearing in the definition of $\mathrm{Dom}(H)$ is called the Kirchhoff vertex condition. The self-adjointness of the operator H is shown in a similar way to [7].

In order to analyze the spectrum of H, we need the spectral theory of the related Hill operator $H_0 := -d^2/dx^2 + q$ in $L^2(\mathbb{R})$, where $q \in L^2(0, 1)$ is extended to the periodic function on $\mathbb{R}$ with period 1, that is, $q(x)$ satisfies $q(x+1) = q(x)$ for almost every $x \in \mathbb{R}$. Floquet–Bloch theory [3, 10, 13] gives us the band structure of $\sigma(H_0)$. Namely, $\sigma(H_0)$ is purely absolutely continuous and consists of infinitely many closed intervals. The intervals are characterized as follows. We consider the Schrödinger equation corresponding to H_0:

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbb{R}, \lambda \in \mathbb{C}. \tag{1.1}$$

Let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions to (1.1) subject to the initial conditions

$$\theta(0,\lambda) = 1, \quad \theta'(0,\lambda) = 0 \quad \text{and} \quad \varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1,$$

respectively. It is known that $\theta(x,\lambda)$, $\theta'(x,\lambda)$, $\varphi(x,\lambda)$, $\varphi'(x,\lambda)$ are entire in $\lambda \in \mathbb{C}$. The function $\Delta(\lambda) := (\theta(1,\lambda) + \varphi'(1,\lambda))/2$ is called the discriminant of the spectrum of H_0 or the Lyapunov function for (1.1). The function $\Delta(\lambda) \pm 1$ has infinitely many real zeroes $\lambda_0^+, \lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+, \dots$. They can be arranged such that $\lambda_0^+ < \lambda_1^- \le \lambda_1^+ < \lambda_2^- \le \lambda_2^+ < \cdots$. The spectrum of H_0 is given by the Lyapunov function $\Delta(\lambda)$ and the zeroes of $\Delta(\lambda) \pm 1$ as follows:

$$\sigma(H_0) = \{\lambda \in \mathbb{R} \,|\, |\Delta(\lambda)| \le 1\} = \bigcup_{j=1} [\lambda_{j-1}^+, \lambda_j^-].$$

For $j \in \mathbb{N}$, the interval $B_j := [\lambda_{j-1}^+, \lambda_j^-]$ is called the jth band of $\sigma(H_0)$. The consecutive bands B_j and B_{j+1} are separated by the open interval $G_j := (\lambda_j^-, \lambda_j^+)$. The sequence $\{\lambda_{2j}^+\}_{j=0}^\infty \cup \{\lambda_{2j}^-\}_{j=1}^\infty$ is the spectrum of the equation $-y'' + qy = \lambda y$ satisfying the periodic boundary condition of the period 1: $y(x+1) = y(x)$ on $\mathbb{R}$. On the other hand, $\{\lambda_{2j-1}^+\}_{j=1}^\infty \cup \{\lambda_{2j-1}^-\}_{j=1}^\infty$ is the spectrum of the equation $-y'' + qy = \lambda y$ with the 1 anti-periodic boundary condition: $y(x+1) = -y(x)$ on $\mathbb{R}$. If there exists some $j \in \mathbb{N}$ such that $\lambda_j^- = \lambda_j^+$ is valid, then the jth spectral gap is degenerate, i.e., $G_j = \emptyset$. This implies that B_j and B_{j+1} merge, or there exists an eigenvalue whose multiplicity is 2, of $-y'' + qy = \lambda y$ subject to the periodic or anti-periodic boundary conditions. Let $\sigma_D(H_0) := \{\mu_n\}_{n=1}^\infty$ be the Dirichlet spectrum, namely, the spectrum of the eigenvalue problem $-y'' + qy = \lambda y$ with $y(0) = y(1) = 0$. Note that $\sigma_D(H_0) = \{\lambda \in \mathbb{R} \,|\, \varphi(1,\lambda) = 0\}$ and $\mu_n \in [\lambda_n^-, \lambda_n^+]$ for each $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ (see [12]).

Let us describe our theorems. For that purpose, we define

$$D(\lambda) = \Delta(\lambda)\left(4\Delta^2(\lambda) + \frac{\theta(1,\lambda)\varphi'(1,\lambda)}{2} - \frac{7}{2}\right),$$

which is a discriminant of the spectrum of H. Moreover, let $\sigma_\infty(H)$ be the set of all eigenvalues of H with infinite multiplicities.

Theorem 1.1. *We have $\sigma(H) = \sigma_\infty(H) \cup \sigma_{ac}(H)$, where*

$$\sigma_\infty(H) = \sigma_D(H_0) \qquad \text{and} \qquad \sigma_{ac}(H) = \{\lambda \in \mathbb{R} \,|\, D(\lambda) \in [-1, 1]\}.$$

The discriminant $D(\lambda)$ has the following properties:

Theorem 1.2.

(i) *We have* $\lim_{\lambda\to-\infty} D(\lambda) = \infty$.
(ii) *For* $c \in (-1, 1)$, $D(\lambda) - c$ *has only real simple zeroes.*
(iii) *The function* $D'(\lambda)$ *has only real simple zeroes* $\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}, \dots,$ *which are separated by the simple zeroes* $\eta_{0,1}, \eta_{0,2}, \eta_{0,3}, \dots,$ *of* $D(\lambda)$. *Namely, we have*

$$\eta_{0,1} < \lambda_{0,1} < \eta_{0,2} < \lambda_{0,2} < \eta_{0,3} < \lambda_{0,3} < \cdots . \tag{1.2}$$

Furthermore, we have

$$D(\lambda_{0,2n}) \geq 1 \quad \text{and} \quad D(\lambda_{0,2n-1}) \leq -1 \quad \text{for any} \quad n \in \mathbb{N}.$$

(iv) *The function* $D(\lambda) - 1$ *has only real zeroes. Let* $z_0^+, z_1^-, z_1^+, z_2^-, z_2^+, \dots$ *be its zeroes counted with multiplicities. Then, we have*

$$z_0^+ < z_1^- < z_1^+ < z_2^- < z_2^+ < z_3^- \leq z_3^+ < z_4^- < z_4^+ < z_5^- < z_5^+ < z_6^- \leq z_6^+ < \cdots .$$

(v) *The function* $D(\lambda) + 1$ *has only real zeroes. Let* $x_1^-, x_1^+, x_2^-, x_2^+, \dots$ *be its zeroes counted with multiplicities. Then, we have*

$$x_1^- < x_1^+ < x_2^- \leq x_2^+ < x_3^- < x_3^+ < x_4^- < x_4^+ < x_5^- \leq x_5^+ < x_6^- \leq x_6^+ < x_7^- < x_7^+ \cdots .$$

(vi) *We have the following inequality:*

$$z_0^+ < x_1^- < x_1^+ < z_1^- < z_1^+ < x_2^- \leq x_2^+ < z_2^- < z_2^+ < x_3^- < x_3^+ < z_3^- \leq z_3^+ < \cdots .$$

Next, we describe the spectral properties of H. For that purpose, we prepare notations. We put

$$q_0 = \int_0^1 q(x)dx, \quad \hat{q}_{s,n} = \int_0^1 q(x) \sin 2n\pi x dx, \quad \hat{q}_n = \int_0^1 q(x)e^{2\pi i n x}dx,$$

$$\alpha_1^{\pm} = \arccos\Big(\pm\frac{1}{3}\Big), \quad \alpha_2^{\pm} = \arccos\Big(\pm\frac{2}{3}\Big),$$

$$v_{n,1}^+ = \alpha_1^+ + 2n\pi, \quad v_{n,1}^- = \alpha_2^+ + 2n\pi, \quad v_{n,2}^{\pm} = (2n-1)\pi, \quad v_{n,3}^+ = 2n\pi - \alpha_2^+,$$

$$v_{n,3}^- = 2n\pi - \alpha_1^+, \quad u_{n,1}^- = \alpha_1^- + 2(n-1)\pi, \quad u_{n,1}^+ = \alpha_2^- + 2(n-1)\pi,$$

$$u_{n,2}^- = 2n\pi - \alpha_2^-, \quad u_{n,2}^+ = 2n\pi - \alpha_1^-, \quad u_{n,3}^{\pm} = 2n\pi$$

for $n \in \mathbb{N}$. We define the operators H_p and H_{ap} in $\mathcal{H}$ as

$$(H_p f_\alpha)(x) = -f_\alpha''(x) + q(x)f_\alpha(x), \quad x \in (0, 1), \ \alpha \in \mathcal{Z},$$

$$\mathrm{Dom}(H_p) = \left\{ \bigoplus_{\alpha\in\mathcal{Z}} f_\alpha \in \mathcal{H} \;\middle|\; \begin{array}{l} \bigoplus_{\alpha\in\mathcal{Z}}(-f_\alpha'' + qf_\alpha) \in \mathcal{H}, \\ f_{n,1}(1) = f_{n,2}(1) = f_{n,3}(0), \\ -f_{n,1}'(1) - f_{n,2}'(1) + f_{n,3}'(0) = 0, \\ f_{n,3}(1) = f_{n,4}(0) = f_{n,5}(0), \\ -f_{n,3}'(1) + f_{n,4}'(0) + f_{n,5}'(0) = 0, \\ f_{n,4}(1) = f_{n,5}(1) = f_{n+1,1}(0) = f_{n+1,2}(0), \\ -f_{n,4}'(1) - f_{n,5}'(1) + f_{n+1,1}'(0) + f_{n+1,2}'(0) = 0, \\ f_{n,j}(x) = f_{n+1,j}(x) \quad \text{and} \quad f_{n,j}'(x) = f_{n+1,j}'(x) \\ \text{for} \quad (n,j)\in\mathcal{Z} \quad \text{and} \quad x\in(0,1) \end{array} \right\}$$

and

$$(H_{ap}f_\alpha)(x) = -f_\alpha''(x) + q(x)f_\alpha(x), \quad x\in(0,1),\ \alpha\in\mathcal{Z},$$

$$\mathrm{Dom}(H_{ap}) = \left\{ \bigoplus_{\alpha\in\mathcal{Z}} f_\alpha \in \mathcal{H} \;\middle|\; \begin{array}{l} \bigoplus_{\alpha\in\mathcal{Z}}(-f_\alpha'' + qf_\alpha) \in \mathcal{H}, \\ f_{n,1}(1) = f_{n,2}(1) = f_{n,3}(0), \\ -f_{n,1}'(1) - f_{n,2}'(1) + f_{n,3}'(0) = 0, \\ f_{n,3}(1) = f_{n,4}(0) = f_{n,5}(0), \\ -f_{n,3}'(1) + f_{n,4}'(0) + f_{n,5}'(0) = 0, \\ f_{n,4}(1) = f_{n,5}(1) = f_{n+1,1}(0) = f_{n+1,2}(0), \\ -f_{n,4}'(1) - f_{n,5}'(1) + f_{n+1,1}'(0) + f_{n+1,2}'(0) = 0, \\ f_{n,j}(x) = -f_{n+1,j}(x) \quad \text{and} \quad f_{n,j}'(x) = -f_{n+1,j}'(x) \\ \text{for} \quad (n,j)\in\mathcal{Z} \quad \text{and} \quad x\in(0,1) \end{array} \right\}.$$

Furthermore, let $\lambda_{2,0}^+, \lambda_{2,2}^-, \lambda_{2,2}^+, \lambda_{2,4}^-, \lambda_{2,4}^+, \dots$ (respectively, $\lambda_{2,1}^-, \lambda_{2,1}^+, \lambda_{2,3}^-, \lambda_{2,3}^+, \dots$) be the spectrum of H_p (respectively, H_{ap}). Then, we obtain the followings.

Theorem 1.3.

(i) *We have* $D(\lambda_{2,n}^\pm) = (-1)^n$ *for any* n*, and the inequality*

$$\begin{aligned} \lambda_{2,0}^+ &< \lambda_{2,1}^- < \lambda_{2,1}^+ < \lambda_{2,2}^- < \lambda_{2,2}^+ < \lambda_{2,3}^- \le \lambda_{2,3}^+ \\ &< \lambda_{2,4}^- < \lambda_{2,4}^+ < \lambda_{2,5}^- < \lambda_{2,5}^+ < \lambda_{2,6}^- \le \lambda_{2,6}^+ < \cdots . \end{aligned}$$

(ii) *The absolutely continuous spectrum of* H *has the band structure. Namely, we have*

$$\sigma_{ac}(H) = \bigcup_{j=1}^{\infty} [\lambda_{2,j-1}^+, \lambda_{2,j}^-].$$

(iii) *For* $j\in\mathbb{N}$*, we call* $\gamma_j = (\lambda_{2,j}^-, \lambda_{2,j}^+)$ *the* j*th gap of* $\sigma(H)$*. Then,* $\gamma_{3n-2} \neq \emptyset$ *and* $\gamma_{3n-1} \neq \emptyset$ *for any* $n\in\mathbb{N}$*.*

(iv) *We have the following asymptotics:*

$$\lambda^{\pm}_{2,6n-5} = (v^{\pm}_{n-1,1})^2 + q_0 + o\left(\frac{1}{n}\right), \tag{1.3}$$

$$\lambda^{\pm}_{2,6n-4} = (u^{\pm}_{n,1})^2 + q_0 + o\left(\frac{1}{n}\right), \tag{1.4}$$

$$\lambda^{\pm}_{2,6n-3} = (v^{\pm}_{n,2})^2 + q_0 \pm \sqrt{|\hat{q}_{2n-1}|^2 - \frac{(\hat{q}_{s,2n-1})^2}{10}} + \mathcal{O}\left(\frac{1}{n}\right), \tag{1.5}$$

$$\lambda^{\pm}_{2,6n-2} = (u^{\pm}_{n,2})^2 + q_0 + o\left(\frac{1}{n}\right), \tag{1.6}$$

$$\lambda^{\pm}_{2,6n-1} = (v^{\pm}_{n,3})^2 + q_0 + o\left(\frac{1}{n}\right), \tag{1.7}$$

$$\lambda^{\pm}_{2,6n} = (u^{\pm}_{n,3})^2 + q_0 \pm \sqrt{|\hat{q}_{2n}|^2 - \frac{(\hat{q}_{s,2n})^2}{10}} + \mathcal{O}\left(\frac{1}{n}\right) \tag{1.8}$$

as $n \to \infty$.

Let us compare our results with the classical results and the one provided by Korotyaev and Lobanov [7]. First of all, we consider the classical case. In the basic period cell of H_0, there is no ring. In this case, it is known as described above that every spectral gap of H_0 has a possibility to be degenerate. Next, let us add a ring to the basic period cell of H_0. Namely, we consider the metric graph $\Gamma^1 = \cup_{(n,j)\in\mathcal{Z}_1}\Gamma^1_{n,j}$, where $\mathbb{J}_1 = \{1,2,3\}$, $\mathcal{Z}_1 = \mathbb{Z}\times\mathbb{J}_1$, and $\Gamma^1_{n,j}$ and its orientation is defined as seen in Figure 2 for $(n,j)\in\mathcal{Z}_1$. Let $\mathcal{H}_1 = L^2(\Gamma^1) = \oplus_{(n,j)\in\mathcal{Z}_1} L^2(\Gamma^1_{n,j})$ and consider the operator H_1 defined as follows:

$$(H_1 f_{n,j})(x) = -f''_{n,j}(x) + q(x) f_{n,j}(x), \quad x\in(0,1),\ (n,j)\in\mathcal{Z}_1,$$

$$\mathrm{Dom}(H_1) = \left\{ \bigoplus_{(n,j)\in\mathcal{Z}_1} f_{n,j} \in L^2(\Gamma^1) \;\middle|\; \begin{array}{l} \bigoplus_{\alpha\in\mathcal{Z}_1}(-f''_\alpha + q f_\alpha) \in L^2(\Gamma^1), \\ -f'_{n,1}(1) + f'_{n,2}(0) - f'_{n,3}(1) = 0, \\ f_{n,2}(0) = f_{n,1}(1) = f_{n,3}(1), \\ f'_{n+1,1}(0) - f'_{n,2}(1) + f'_{n,3}(0) = 0, \\ f_{n,2}(1) = f_{n+1,1}(0) = f_{n,3}(0) \quad \text{for} \quad n\in\mathbb{Z} \end{array} \right\}.$$

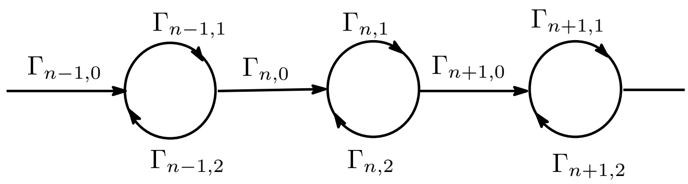

Fig. 2. The graph Γ^1

Korotyaev and Lobanov [7] proved that $\sigma(H_1) = \sigma_D(H_0) \cup \sigma_{ac}(H_1)$ and $\sigma_{ac}(H_1)$ has the band structure. Letting $[\lambda_{1,j-1}^+, \lambda_{1,j}^-]$ be the jth band of $\sigma_{ac}(H_1)$ for $j \in \mathbb{N}$, they obtained the inequality

$$\lambda_{1,0}^+ < \lambda_{1,1}^- < \lambda_{1,1}^+ < \lambda_{1,2}^- \leq \lambda_{1,2}^+ < \lambda_{1,3}^- < \lambda_{1,3}^+ < \lambda_{1,4}^- \leq \lambda_{1,4}^+ < \cdots$$

and concluded that every odd-numbered spectral gap is not degenerate, i.e., $\gamma_{1,2n-1} := (\lambda_{1,2n-1}^-, \lambda_{1,2n-1}^+) \neq \emptyset$ for any $n \in \mathbb{N}$. Furthermore, we call a ring and a line *parts* of our graph. For a general potential, we call a spectral gap *indecisive* if the gap has the possibility to be degenerate or nondegenerate. On the other hand, we call a spectral gap is *decisive* if the gap has no possibility to be degenerate. Then, the arrangement of parts of graphs is closely related to the decisiveness of spectral gaps of corresponding quantum graphs. The basic period cell of the real line is consisted of a line and every spectral gap G_j is indecisive for $n \in \mathbb{N}$. On the other hand, the basic period cell of the graph Γ^1 consists of a ring and a line. In this case, $\gamma_{1,2n} := (\lambda_{1,2n}^-, \lambda_{1,2n}^+)$ is indecisive and $\gamma_{1,2n-1}$ is decisive for $n \in \mathbb{N}$. Moreover, since γ_{3n-1} and γ_{3n-2} are decisive and γ_{3n} is indecisive in the case of our quantum graph H, whose metric graph consists of two rings and one line, we can say that the arrangement of parts closely relates to the decisiveness of the spectral gaps. Thus, we are also interested in the quantum graph whose metric graph has three rings and one line in the basic period cell. As numerical results, we can see a relationship between the arrangement of parts and decisiveness of its spectral gaps, although we do not have a mathematical proof of the relationship now.

Let us see the proof of Theorems 1.1–1.3 below.

2. PROOFS OF THEOREMS 1.1 AND 1.2.

Let us start with the proof of Theorem 1.1.

Proof of Theorem 1.1. We first pick $\lambda \in \sigma_D(H_0)$, arbitrarily. We make eigenfunctions corresponding to $\lambda \in \sigma_D(H_0)$. Putting

$$\Psi_{0,1}(x,\lambda) = \varphi(x,\lambda), \quad \Psi_{0,2}(x,\lambda) = -\varphi(x,\lambda),$$

$\Psi_{n,j}(x,\lambda) = 0$ for $(n,j) \in (\mathbb{Z} \times \mathbb{J}) \setminus \{(0,1),(0,2)\}$, we define $\Psi_n = (\Psi_{m-n,j})_{(m,j)\in\mathcal{Z}_1}$, which belongs to $\mathrm{Dom}(H)$. For any $n \in \mathbb{N}$, we make sure that Ψ_n is an eigenfunction corresponding to λ. Thus, we see that $\sigma_D(H_0) \subset \sigma_\infty(H)$.

We use a direct integral decomposition for H (see [4, 13]). For $\mu \in [0, 2\pi)$, we define a fiber operator H_μ in the Hilbert space $\mathcal{H}_\mu = \oplus_{j=1}^5 L^2(\Gamma_{0,j})$ as follows:

$$(H_\mu f_j)(x) = -f_j''(x) + q(x) f_j(x), \quad x \in (0,1),\ j \in \mathbb{J},$$

$$\mathrm{Dom}(H_\mu) = \left\{ \bigoplus_{j=1}^{5} f_j \in \mathcal{H}_\mu \;\middle|\; \begin{array}{l} \bigoplus_{j=1}^{5}(-f_j'' + qf_j) \in \mathcal{H}_\mu, \\ f_1(1) = f_2(1) = f_3(0), \\ -f_1'(1) - f_2'(1) + f_3'(0) = 0, \\ f_3(1) = f_4(0) = f_5(0), \\ -f_3'(1) + f_4'(0) + f_5'(0) = 0, \\ f_4(1) = f_5(1) = e^{i\mu} f_1(0) = e^{i\mu} f_2(0), \\ -f_4'(1) - f_5'(1) + e^{i\mu} f_1'(0) + e^{i\mu} f_2'(0) = 0 \end{array} \right\}.$$

Furthermore, we consider the Hilbert space

$$\mathcal{H} = \int_{[0,2\pi)}^{\oplus} \mathcal{H}_\mu \frac{d\mu}{2\pi} = L^2\left([0,2\pi), \mathcal{H}_\mu, \frac{d\mu}{2\pi}\right)$$

and the unitary operator $U : L^2(\Gamma^1) \to \mathcal{H}$ defined as

$$(Uf)(\mu) = \sum_{n\in\mathbb{Z}} e^{in\mu} f_n, \quad f = (f_n)_{n\in\mathbb{Z}} = (f_{n,j})_{(n,j)\in\mathcal{Z}_1} \in L^2(\Gamma^1).$$

Then, we have the direct integral representation of H:

$$UHU^{-1} = \int_{[0,2\pi)}^{\oplus} H(\mu) \frac{d\mu}{2\pi}.$$

Since $H(\mu)$ acts on the finite graph $\cup_{j=1}^{5}\Gamma_{0,j}$, the spectrum of $H(\mu)$ consists of the discrete spectrum. For $\mu \in [0, 2\pi)$, let $\{E_n(\mu)\}_{n\in\mathbb{N}}$ stand for the increasing sequence of the eigenvalues of $H(\mu)$ counted with multiplicities. Let $\mathcal{N}$ be the set of natural numbers n such that $E_n(\mu)$ does depend on $\mu \in [0, 2\pi)$. Then, we have $\sigma(H) = \sigma_\infty(H) \cup \sigma_{ac}(H)$, where $\sigma_\infty(H) = \{E_n(\mu) \,|\, E_n(\mu) \text{ is independent of } \mu \in [0, 2\pi)\}$ and

$$\sigma_{ac}(H) = \bigcup_{n\in\mathcal{N}} \bigcup_{\mu\in[0,2\pi)} \{E_n(\mu)\}.$$

Since we have $\sigma_D(H_0) \subset \sigma_\infty(H)$, we next investigate $\sigma(H) \setminus \sigma_D(H_0)$. We pick $\lambda \notin \sigma_D(H_0)$, arbitrarily. We consider the characteristic equation $H_\mu f = \lambda f$ for $0 \neq f \in \mathrm{Dom}(H_\mu)$, that is, we consider the following system of 7 equations:

$$-f_j''(x) + q(x)f_j(x) = \lambda f_j(x), \quad x \in (0,1), j \in \mathbb{J}, \tag{2.1}$$
$$f_1(1) = f_2(1) = f_3(0), \tag{2.2}$$
$$-f_1'(1) - f_2'(1) + f_3'(0) = 0, \tag{2.3}$$
$$f_3(1) = f_4(0) = f_5(0), \tag{2.4}$$
$$-f_3'(1) + f_4'(0) + f_5'(0) = 0, \tag{2.5}$$
$$f_4(1) = f_5(1) = e^{i\mu} f_1(0) = e^{i\mu} f_2(0), \tag{2.6}$$
$$-f_4'(1) - f_5'(1) + e^{i\mu} f_1'(0) + e^{i\mu} f_2'(0) = 0. \tag{2.7}$$

Using the fundamental solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$, any solution y to (1.1) is given by

$$y(x,\lambda) = w(x,\lambda)y(0,\lambda) + \frac{\varphi(x,\lambda)}{\varphi(1,\lambda)}y(1,\lambda),$$

where

$$w(x,\lambda) = \theta(x,\lambda) - \frac{\theta(1,\lambda)}{\varphi(1,\lambda)}\varphi(x,\lambda).$$

Thus, it follows by (2.1), (2.2), (2.4) and (2.6) that

$$f_1(x,\lambda) = Xw(x,\lambda) + \frac{Y}{\varphi(1,\lambda)}\varphi(x,\lambda), \tag{2.8}$$

$$f_2(x,\lambda) = Xw(x,\lambda) + \frac{Y}{\varphi(1,\lambda)}\varphi(x,\lambda), \tag{2.9}$$

$$f_3(x,\lambda) = Yw(x,\lambda) + \frac{Z}{\varphi(1,\lambda)}\varphi(x,\lambda), \tag{2.10}$$

$$f_4(x,\lambda) = Zw(x,\lambda) + \frac{e^{i\mu}X}{\varphi(1,\lambda)}\varphi(x,\lambda), \tag{2.11}$$

$$f_5(x,\lambda) = Zw(x,\lambda) + \frac{e^{i\mu}X}{\varphi(1,\lambda)}\varphi(x,\lambda), \tag{2.12}$$

where $X = f_1(0,\lambda)$, $Y = f_1(1,\lambda)$ and $Z = f_5(0,\lambda)$. Substituting (2.8), (2.9) and (2.10) for (2.3), we obtain

$$-2w'(1,\lambda)X - \frac{2\varphi'(1,\lambda)+\theta(1,\lambda)}{\varphi(1,\lambda)}Y + \frac{Z}{\varphi(1,\lambda)} = 0. \tag{2.13}$$

We also obtain

$$\frac{2e^{i\mu}}{\varphi(1,\lambda)}X - w'(1,\lambda)Y - \frac{\varphi'(1,\lambda)+2\theta(1,\lambda)}{\varphi(1,\lambda)}Z = 0, \tag{2.14}$$

$$-\frac{2e^{i\mu}(\varphi'(1,\lambda)+\theta(1,\lambda))}{\varphi(1,\lambda)}X + \frac{2e^{i\mu}}{\varphi(1,\lambda)}Y - 2w'(1,\lambda)Z = 0 \tag{2.15}$$

by substituting (2.10)-(2.12) for (2.5) and (2.7). Since $\Delta(\lambda) = (\theta(1,\lambda)+\varphi'(1,\lambda))/2$, we obtain the system

$$M(\lambda,\mu)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0,$$

where

$$M(\lambda,\mu) = \begin{pmatrix} -2w'(1,\lambda) & -\frac{2\Delta(\lambda)+\varphi'(1,\lambda)}{\varphi(1,\lambda)} & \frac{1}{\varphi(1,\lambda)} \\ \frac{2e^{i\mu}}{\varphi(1,\lambda)} & -w'(1,\lambda) & -\frac{2\Delta(\lambda)+\theta(1,\lambda)}{\varphi(1,\lambda)} \\ -\frac{4e^{i\mu}\Delta(\lambda)}{\varphi(1,\lambda)} & \frac{2e^{i\mu}}{\varphi(1,\lambda)} & -2w'(1,\lambda) \end{pmatrix}.$$

Since f is non-trivial, this system has a non-trivial solution, that is, we have det $M(\lambda,\mu) = 0$. By virtue of $\theta(1,\lambda)\varphi'(1,\lambda) - \theta(1,\lambda)\varphi(1,\lambda) = 1$, we have $\varphi(1,\lambda)w'(1,\lambda) = -1$. Thus, we obtain

$$\begin{aligned}0 &= \varphi^3(1,\lambda)\text{det } M(\lambda,\mu)\\ &= 4 - 4e^{i\mu}\Delta(\lambda)(2\Delta(\lambda)+\varphi'(1,\lambda))(2\Delta(\lambda)+\theta(1,\lambda)) + 4e^{2i\mu} + 28e^{i\mu}\Delta(\lambda),\end{aligned}$$

which implies

$$0 = 1 - e^{i\mu}\Delta(\lambda)(8\Delta^2(\lambda)+\theta(1,\lambda)\varphi'(1,\lambda)) + e^{2i\mu} + 7e^{i\mu}\Delta(\lambda).$$

Multiplying $e^{-i\mu}$, we have

$$0 = e^{-i\mu} + e^{i\mu} + \Delta(\lambda)(7 - 8\Delta^2(\lambda) - \theta(1,\lambda)\varphi'(1,\lambda).$$

So, we derive

$$2\cos\mu = \Delta(\lambda)(8\Delta^2(\lambda)+\theta(1,\lambda)\varphi'(1,\lambda) - 7).$$

Since the left hand side depends on μ, $\{E_n(\mu)\}_{n\in\mathcal{N}}$ solves this equation for each $\mu \in [0,2\pi)$. Thus, we have

$$\sigma_{ac}(H) \setminus \sigma_D(H_0) = \{\lambda \in \mathbb{R} \mid D(\lambda) \in [-1,1]\} \setminus \sigma_D(H_0).$$

Since $\sigma_{ac}(H)$ is a closed set, we obtain $\sigma_{ac}(H) = \{\lambda \in \mathbb{R} \mid D(\lambda) \in [-1,1]\}$. □

We next examine properties of $D(\lambda)$.

As seen in [12], the fundamental solutions $\theta(x,\lambda)$, $\varphi(x,\lambda)$ and their derivatives behave as follows:

$$\theta(x,\lambda) = \cos\sqrt{\lambda}x + \frac{1}{2\sqrt{\lambda}}\int_0^x(\sin\sqrt{\lambda}x + \sin\sqrt{\lambda}(x-2t))q(t)dt + \mathcal{O}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{|\lambda|}\right), \tag{2.16}$$

$$\theta'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \frac{1}{2}\int_0^x(\cos\sqrt{\lambda}x + \cos\sqrt{\lambda}(x-2t))q(t)dt + \mathcal{O}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{|\lambda|^{1/2}}\right), \tag{2.17}$$

$$\varphi(x,\lambda) = \frac{\sin x\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1}{2\lambda}\int_0^x(-\cos\sqrt{\lambda}x + \cos\sqrt{\lambda}(x-2t))q(t)dt + \mathcal{O}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{|\lambda|^{3/2}}\right), \tag{2.18}$$

$$\varphi'(x,\lambda) = \cos\sqrt{\lambda}x + \frac{1}{2\sqrt{\lambda}}\int_0^x(\sin\sqrt{\lambda}x + \sin\sqrt{\lambda}(x-2t))q(t)dt + \mathcal{O}\left(\frac{e^{|\mathrm{Im}\sqrt{\lambda}|x}}{|\lambda|}\right) \tag{2.19}$$

as $|\lambda| \to \infty$, uniformly on bounded sets of $[0,1] \times L^2(0,1)$. Because of (2.16) and (2.19), we obtain

$$\Delta(\lambda) = \cos\sqrt{\lambda} + \frac{q_0 \sin\sqrt{\lambda}}{2\sqrt{\lambda}} + \frac{S(\lambda)}{2\sqrt{\lambda}} + \mathcal{O}\left(\frac{e^{|\mathrm{Im}\ \sqrt{\lambda}|}}{|\lambda|}\right), \tag{2.20}$$

where

$$S(\lambda) = \int_0^1 \sin\sqrt{\lambda}(1-2t)q(t)dt.$$

We notice that $S(\lambda) \to 0$ as $|\lambda| \to \infty$. It turns out by (2.16), (2.19) and (2.20) that

$$D(\lambda) = D_0(\lambda) + \frac{q_0 \sin\sqrt{\lambda}}{4\sqrt{\lambda}}(27\cos^2\sqrt{\lambda} - 7) + \frac{S(\lambda)}{4\sqrt{\lambda}}(9\cos^2\sqrt{\lambda} - 7) + \mathcal{O}\left(\frac{e^{3|\mathrm{Im}\ \sqrt{\lambda}|}}{|\lambda|}\right), \tag{2.21}$$

where $D_0(\lambda) = \cos\sqrt{\lambda}(\frac{9}{2}\cos^2\sqrt{\lambda} - \frac{7}{2})$ is the discriminant in the unperturbed case: $q \equiv 0$.

Proof of Theorem 1.2 (i). Owing to (2.21), we notice that the statement of Theorem 1.2 (i) is valid. □

In order to capture the behaviors of $D(\lambda)$, we first state the information on the zeroes of $D(\lambda)$. Since our discriminant is factorized into $\Delta(\lambda)$ and $d(\lambda) := 4\Delta^2(\lambda) + \frac{\theta(1,\lambda)\varphi'(1,\lambda)}{2} - \frac{7}{2}$, we describe the statements on zeroes of two functions. For the purpose, we prepare notations

$$\Omega(a,r) = \{\lambda \in \mathbb{C} \,|\, |\sqrt{\lambda} - a| < r\} \quad \text{and} \quad C(a,r) = \{\lambda \in \mathbb{C} \,|\, |\sqrt{\lambda} - a| = r\},$$

where $a \in \mathbb{C}$ and $r > 0$.

Lemma 2.1. *There exists some $n_0 \in \mathbb{N}$ such that $\Delta(\lambda)$ has exactly n_0 zeroes, counted with multiplicities, in the domain $\Omega(0, n_0\pi)$ and for each $n > n_0$, exactly one simple zero in the domain $\Omega(\frac{\pi}{2} + n\pi, \frac{\pi}{4})$. There are no other zeroes.*

This might be well-known, but we give a proof of this lemma all over again just to satisfy ourselves. In order to prove this and ensuing lemmas, we quote the following lemma from [12].

Lemma 2.2. *If $|z - n\pi| \geq \frac{\pi}{4}$ for all integers n, then*

$$e^{|\mathrm{Im}\ z|} < 4|\sin z|.$$

Proof of Lemma 2.1. We put $\sqrt{\lambda} = a + bi$ and $a, b \in \mathbb{R}$, where $\lambda \in \Omega(\frac{\pi}{2} + n\pi, \frac{\pi}{4})$. Because of $\{a - (\frac{\pi}{2} + n\pi)\}^2 + b^2 = \frac{\pi^2}{16}$, we have

$$|\sqrt{\lambda} - (\frac{\pi}{2} + n\pi) - m\pi|^2 \geq \pi^2\Big(|m| - \frac{1}{4}\Big)^2 \geq \frac{\pi^2}{16}$$

for any $m \in \mathbb{Z}$. This combined with Lemma 2.2, we have $e^{|\mathrm{Im}\ \sqrt{\lambda}|} < 4|\cos\sqrt{\lambda}|$ on $C(\frac{\pi}{2}+n\pi, \frac{\pi}{4})$. This together with (2.20) implies

$$\frac{|\Delta(\lambda)-\cos\sqrt{\lambda}|}{|\cos\sqrt{\lambda}|} = \left|\frac{q_0 \sin\sqrt{\lambda}}{2\sqrt{\lambda}\cos\sqrt{\lambda}}\right| + \frac{\mathcal{O}(e^{|\mathrm{Im}\ \sqrt{\lambda}|})}{|\lambda||\cos\sqrt{\lambda}|} \to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

Since $\Delta(\lambda)$ is entire, we can utilize Rouché's theorem and conclude that the numbers of zeroes of $\Delta(\lambda)$ and $\cos\sqrt{\lambda}$ are the same for large enough n. Since $\cos\sqrt{\lambda}$ has a simple zero $\sqrt{\lambda} = \frac{\pi}{2}+n\pi$ in $\Omega(\frac{\pi}{2}+n\pi, \frac{\pi}{4})$, $\Delta(\lambda)$ has a simple zero in $\Omega(\frac{\pi}{2}+n\pi, \frac{\pi}{4})$ for $n > n_0$.

On the other hand, it follows by $|\sqrt{\lambda} - \frac{\pi}{2} - m\pi| \geq \frac{\pi}{4}$ for any $m \in \mathbb{Z}$ and $\lambda \in C(0, n\pi)$ and Lemma 2.2 that $e^{|\mathrm{Im}\ \sqrt{\lambda}|} < 4|\cos\sqrt{\lambda}|$. Thus, we have $|\Delta(\lambda) - \cos\sqrt{\lambda}| < o(1)|\cos\sqrt{\lambda}|$ as $|\lambda| \to \infty$. So, Rouché's theorem again leads us to our goal. □

We next give the statements on the zeroes of $d(\lambda)$.

Lemma 2.3. *There exists some $n_0 \in \mathbb{N}$ such that $d(\lambda)$ has exactly one simple zero in $\Omega(n\pi+\frac{\pi}{8}, \frac{\pi}{8})$ and $\Omega(n\pi-\frac{\pi}{8}, \frac{\pi}{8})$ for $n > n_0$, respectively.*

Proof. Putting $\Delta_-(\lambda) = \frac{1}{2}(\varphi'(1,\lambda) - \theta(1,\lambda))$, we have $\theta(1,\lambda)\varphi(1,\lambda) = \Delta^2(\lambda) - \Delta_-^2(\lambda)$. Thus, we see that

$$d(\lambda) = \frac{9}{2}\Delta^2(\lambda) - \frac{1}{2}\Delta_-^2(\lambda) - \frac{7}{2}. \tag{2.22}$$

Using (2.16) and (2.19), we obtain

$$\Delta_-(\lambda) = -\frac{S(\lambda)}{\sqrt{\lambda}} + \frac{\mathcal{O}(e^{|\mathrm{Im}\ \sqrt{\lambda}|})}{|\lambda|} \quad \text{as} \quad |\lambda| \to \infty. \tag{2.23}$$

Let $d_0(\lambda) = \frac{9}{2}\cos^2\sqrt{\lambda} - \frac{7}{2}$. Substituting (2.20) and (2.23) for (2.22), we have

$$d(\lambda) = d_0(\lambda) + \mathcal{O}\left(|\lambda|^{-1/2}\right) \quad \text{as} \quad |\lambda| \to \infty.$$

We see that there exists some $n_0 \in \mathbb{N}$ such that $d(\lambda)$ has exactly one simple zero in $C(n\pi - \frac{\pi}{8}, \frac{\pi}{8})$ for $n > n_0$. We pick $\lambda \in C(n\pi - \frac{\pi}{8}, \frac{\pi}{8})$, arbitrarily. Then, there exists some $\theta \in [0, 2\pi)$ such that $\sqrt{\lambda} = n\pi - \frac{\pi}{8} + \frac{\pi}{8}e^{i\theta}$. Since

$$\cos^2\sqrt{\lambda} = \cos^2\Big(-\frac{\pi}{8}+\frac{\pi}{8}e^{i\theta}\Big) = \frac{1}{2}\Big\{1+\cos\Big(-\frac{\pi}{4}+\frac{\pi}{4}e^{i\theta}\Big)\Big\},$$

we have

$$\begin{aligned} d_0(\lambda) &= \frac{9}{4\sqrt{2}}\Big\{\cos\Big(\frac{\pi}{4}e^{i\theta}\Big) + \sin\Big(\frac{\pi}{4}e^{i\theta}\Big)\Big\} - \frac{5}{4} \\ &= \frac{9}{4\sqrt{2}}\{\cos(\beta+i\alpha) + i\sin(\beta+i\alpha)\} - \frac{5}{4} \\ &= \frac{9}{4\sqrt{2}}\{\cosh\alpha\cos\beta + \cosh\alpha\sin\beta + i(-\sinh\alpha\sin\beta + \sinh\alpha\cos\beta)\} - \frac{5}{4}, \end{aligned}$$

where $\alpha = \frac{\pi}{4}\sin\theta$ and $\beta = \frac{\pi}{4}\cos\theta$. Thus, we have

$$|d_0(\lambda)| = A(\theta)^2 + B(\theta)^2 =: f(\theta),$$

where $A(\theta) = \frac{9}{4\sqrt{2}}\cosh\alpha(\cos\beta + \sin\beta) - \frac{5}{4}$ and $B(\theta) = \sinh\alpha(\cos\beta - \sin\beta)$. Let us show that a constant $C > 0$ satisfying $|d_0(\lambda)| \geq C > 0$ on $C(n\pi - \frac{\pi}{8}, \frac{\pi}{8})$. Since $f(\theta)$ is 2π-periodic and continuous on $[0, 2\pi]$, there exists the minimum C of $f(\theta)$. In order to show $C > 0$, it suffices to show there does not exist $\theta \in [0, 2\pi)$ satisfying $A(\theta) = B(\theta) = 0$. First, let us suppose that $\sinh\alpha = 0$. This implies $\alpha = 0$, which means $\theta = 0, \pi$. If $\theta = 0$, then it turns out by $\cos\beta = \sin\beta = \frac{1}{\sqrt{2}}$ that $A(0) = 1 \neq 0$. If $\theta = \pi$, then it follows by $\cos\beta = \frac{1}{\sqrt{2}}$ and $\sin\beta = -\frac{1}{\sqrt{2}}$ that $A(\pi) = -\frac{5}{4} \neq 0$. Next, we suppose that $\cos\beta = \sin\beta$. Since this implies $\theta = 0$, we have $A(0) = 1 \neq 0$. Thus, it turns out that there exists a constant $C > 0$ such that $|d_0(\lambda)| \geq C$.

Therefore, we see that $d(\lambda) - d_0(\lambda) = d_0(\lambda)\mathcal{O}\left(|\lambda|^{-1/2}\right)$ as $|\lambda| \to \infty$. This combined Rouché's theorem means that the number of zeroes in $\Omega(n\pi - \frac{\pi}{8}, \frac{\pi}{8})$ of $d(\lambda)$ and $d_0(\lambda)$ is the same. In a similar way, we also see that the number of zeroes in $\Omega(n\pi + \frac{\pi}{8}, \frac{\pi}{8})$ of $d(\lambda)$ and $d_0(\lambda)$ is the same. Since the zeroes of $d_0(\lambda)$ are given by

$$\sqrt{\lambda} = \gamma_+ + 2n\pi, (2\pi - \gamma_+) + 2n\pi, \gamma_- + 2n\pi, (2\pi - \gamma_-) + 2n\pi, \quad n = 0, 1, 2, 3, \ldots,$$

where $\gamma_\pm = \arccos\frac{\pm\sqrt{7}}{3}$. It follows by $\frac{1}{\sqrt{2}} < \frac{\sqrt{7}}{3}$ that $0 < \gamma_+ < \frac{\pi}{4}$ and $\frac{3}{4}\pi < \gamma_- < \pi$. Thus, we see that $\Omega(2n\pi + \frac{\pi}{8}, \frac{\pi}{8})$ only includes $(\gamma_+ + 2n\pi)^2$, $\Omega(2n\pi - \frac{\pi}{8}, \frac{\pi}{8})$ only includes $\{(2\pi - \gamma_+) + 2n\pi\}^2$, $\Omega(2n\pi + \pi + \frac{\pi}{8}, \frac{\pi}{8})$ only includes $\{(2\pi - \gamma_-) + 2n\pi\}^2$ and $\Omega(2n\pi + \pi - \frac{\pi}{8}, \frac{\pi}{8})$ only includes $(\gamma_- + 2n\pi)^2$ for every $n \in \mathbb{N} \cup \{0\}$. This implies our assertion. □

Let $C(n) = C_1(n) - C_2(n) - C_3(n) + C_4(n)$, where

$$\begin{aligned}
C_1(n) &= \{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = n\pi + in\pi t,\ -1 \leq t \leq 1\},\\
C_2(n) &= \{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = n\pi t + in\pi,\ -1 \leq t \leq 1\},\\
C_3(n) &= \{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = -n\pi + in\pi t,\ -1 \leq t \leq 1\},\\
C_4(n) &= \{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = n\pi t - in\pi,\ -1 \leq t \leq 1\}
\end{aligned}$$

for $n \in \mathbb{N}$. Moreover, let $\Omega(n)$ be the domain surrounded by $C(n)$ for $n \in \mathbb{N}$. In order to get further properties on the zeroes of $D(\lambda)$ and prove the ensuing lemmas, we need the following inequality on $C(n)$.

Lemma 2.4. *For a fixed $p \in (-1, 1)$, there exist some constant $M_p > 0$ and $n_0(p) \in \mathbb{N}$ such that*

$$e^{|\mathrm{Im}\ \sqrt{\lambda}|} < M_p|\cos\sqrt{\lambda} + p|$$

on $C(n)$ for any $n > n_0(p)$. The constant M_p and $n_0(p)$ depend on p, but does not depend on n.

Proof. First, we show that our inequality is valid on $C_1(n)$. Let $\sqrt{\lambda} = n\pi + in\pi t$, where $-1 \le t \le 1$. Since $\cos\sqrt{\lambda} = (-1)^n \cosh n\pi t$, we have

$$|\cos\sqrt{\lambda}+p| = \begin{cases} \cosh\pi t + p \ge 1+p > 0 & \text{if } n \text{ is even,} \\ -(-\cosh n\pi t + p) \ge 1-p > 0 & \text{if } n \text{ is odd.} \end{cases}$$

So, $|\cos\sqrt{\lambda}+p| \ge \min\{1+p, 1-p\} > 0$. Furthermore, we have

$$\frac{e^{2|\mathrm{Im}\ \sqrt{\lambda}|}}{|\cosh\sqrt{\lambda}+p|^2} = \begin{cases} \frac{e^{2|n\pi t|}}{(\cosh n\pi t+p)^2} & \text{if } n \text{ is even,} \\ \frac{e^{2|n\pi t|}}{(\cosh n\pi t-p)^2} & \text{if } n \text{ is odd.} \end{cases}$$

Since $f(x) := \frac{e^{2x}}{(\cosh x \pm p)^2}$ is continuous on $\mathbb{R}$, $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 4$, there exists some $C_p > 0$ such that $f(x) \le C_p$ on $\mathbb{R}$. Thus, we have

$$e^{|\mathrm{Im}\ \sqrt{\lambda}|} < 4|\cos\sqrt{\lambda}+p|$$

on $C_1(n)$. In a similar way, we have the same inequality on $C_3(n)$.

Next, we show our inequality on $C_2(n)$. Putting $\sqrt{\lambda} = n\pi t + in\pi$, where $t \in [-1,1]$, then we have

$$|\cos\sqrt{\lambda}+p|^2 = (\cos n\pi t \cosh n\pi + p)^2 + \sin^2 n\pi t \sinh^2 n\pi.$$

Thus, we have

$$\frac{e^{2|\mathrm{Im}\ \sqrt{\lambda}|}}{|\cos\sqrt{\lambda}+p|^2} = \frac{1}{g_n(t)} \le \left|\frac{1}{g_n(t)} - 4\right| + 4 = \frac{4}{g_n(t)}\left|\frac{1}{4} - g_n(t)\right| + 4, \tag{2.24}$$

where

$$g_n(t) = \left(\cos n\pi t \frac{\cosh n\pi}{e^{n\pi}} + \frac{p}{e^{n\pi}}\right)^2 + \sin^2 n\pi t \frac{\sinh^2 n\pi}{e^{2n\pi}}.$$

Putting $h(x) = \frac{e^{2x}}{(\cosh x - |p|)^2 - 1}$, we see that $h'(x) < 0$ for any $x > x_0(p)$, where $x_0(p) > 0$ is a constant depending on p and is a large enough number. Moreover, $\lim_{x\to\infty} h(x) = 4$ holds true. Thus, there exists some constant $n_0(p) \in \mathbb{N}$ such that

$$\begin{aligned} &|g_n(t)| \\ &= \frac{e^{2n\pi}+e^{-2n\pi}}{4e^{2n\pi}} + \frac{1}{e^{2n\pi}} \cdot \frac{\cos^2 n\pi t - \sin^2 n\pi t}{2} + 2p\cos\pi t \frac{\cosh n\pi}{e^{2n\pi}} + \frac{p^2}{e^{2n\pi}} \\ &\ge \frac{e^{2n\pi}+e^{-2n\pi}}{4e^{2n\pi}} - \frac{1}{2e^{2n\pi}} - 2|p|\frac{\cosh n\pi}{e^{2n\pi}} + \frac{p^2}{e^{2n\pi}} \\ &= \frac{(\cosh n\pi - |p|)^2 - 1}{e^{2n\pi}} > \frac{(\cosh n_0(p)\pi - |p|)^2 - 1}{e^{2n_0(p)\pi}} \end{aligned}$$

for any $n > n_0(p)$. This combined with

$$\begin{aligned}
&\left|g_n(t) - \frac{1}{4}\right| \\
&= \left|\frac{\cos^2 n\pi t}{e^{2n\pi}} \cdot \frac{e^{2n\pi} + 2 + e^{-2n\pi}}{4} + 2p\cos n\pi t \cdot \frac{\cosh \pi}{e^{2n\pi}}\right. \\
&\quad \left. + \frac{p^2}{e^{2n\pi}} + \frac{\sin^2 n\pi t}{e^{2n\pi}} \cdot \frac{e^{2n\pi} - 2 + e^{-2n\pi}}{4} - \frac{1}{4}\right| \\
&\le \frac{2 + e^{-2n\pi}}{4e^{2n\pi}} + 2|p|\frac{\cosh n\pi}{e^{2n\pi}} + \frac{p^2}{e^{2n\pi}} + \frac{|-2 + e^{-2n\pi}|}{4e^{2n\pi}},
\end{aligned}$$

the right hand side of (2.24) is uniformly bounded on t and goes to 4 as $n \to \infty$. Thus, there exists some $M_p > 0$ such that

$$\frac{e^{2|\mathrm{Im}\ \sqrt{\lambda}|}}{|\cos\sqrt{\lambda} + p|^2} < M_p^2$$

on $C_2(n)$ for any $n > n_0(p)$. Similarly, we obtain the results stated in this lemma on $C_4(n)$. □

Lemma 2.5. *There exists some $n_0 > 1$ satisfying $D(\lambda)$ has $3n_0$ zeroes, counted with multiplicities, in $\Omega(n_0)$ and exactly simple zero in $\Omega(n\pi + \frac{\pi}{8}, \frac{\pi}{8})$, $\Omega(n\pi + \frac{7\pi}{8}, \frac{\pi}{8})$ and $\Omega(n\pi + \frac{\pi}{2}, \frac{\pi}{4})$ for any $n > n_0$, respectively. There are no other zeroes.*

Proof. It suffices to show that there exists some $n_0 \in \mathbb{N}$ such that $D(\lambda)$ has $3n_0$ zeroes in $C(n_0)$. First, let us show that there exists a constant $C > 0$ satisfying $|\cos\sqrt{\lambda}| \ge C > 0$ on $C(n)$. On $C_1(n) \cup C_3(n)$, we notice that $|\cos\sqrt{\lambda}| = |(-1)^n \cosh n\pi t| \ge 1$. On the other hand, it follows on $C_2(n)$ and $C_4(n)$ that

$$\begin{aligned}
|\cos^2\sqrt{\lambda}| &= \cos^2 n\pi t \cosh^2 n\pi + \sin^2 n\pi t \sinh^2 n\pi \\
&= \cos^2 n\pi t \cosh^2 n\pi + (\cosh^2 n\pi - 1)(1 - \cos^2 n\pi t) \\
&= \cosh^2 n\pi - \sin^2 n\pi t \\
&\ge \cosh^2 \pi - 1.
\end{aligned}$$

This is why our desired constant C exists.

By virtue of Lemma 2.5, it follows by

$$D_0(\lambda) = \frac{9}{2}\cos\sqrt{\lambda}\Big(\cos\sqrt{\lambda} - \frac{\sqrt{7}}{3}\Big)\Big(\cos\sqrt{\lambda} + \frac{\sqrt{7}}{3}\Big)$$

that

$$e^{|3\mathrm{Im}\ \sqrt{\lambda}|} < \frac{2}{9} M_0 M_{\frac{\sqrt{7}}{3}} M_{-\frac{\sqrt{7}}{3}} D_0(\lambda),$$

where M_p is the constant appearing in Lemma 2.4 for $p \in (-1, 1)$. Using (2.21) we have

$$|D(\lambda) - D_0(\lambda)| = \mathcal{O}\left(\frac{e^{3|\mathrm{Im}\ \sqrt{\lambda}|}}{|\lambda|^{1/2}}\right) + \frac{S(\lambda)(9\cos^2\sqrt{\lambda} - 7)}{4\sqrt{\lambda}}.$$

These two equations combined with the result of the first paragraph of this proof gives us

$$\left|\frac{D(\lambda)-D_0(\lambda)}{D_0(\lambda)}\right| = \frac{\mathcal{O}\left(\frac{e^{3|\mathrm{Im}\,\sqrt{\lambda}|}}{|\lambda|^{1/2}}\right)}{D_0(\lambda)} + \frac{S(\lambda)}{2\sqrt{\lambda}\cos\sqrt{\lambda}} = \mathcal{O}\left(\frac{1}{|\sqrt{\lambda}|^{1/2}}\right),$$

because of $S(\lambda) \to 0$ as $|\lambda| \to \infty$. Thus, the number of $D_0(\lambda)$ and $D(\lambda)$ in the domain $\Omega(n_0)$ are the same for large enough $n_0 \in \mathbb{N}$, owing to Rouché's theorem. The zeroes in $\Omega(1)$ of $D_0(\lambda)$ are $\sqrt{\lambda} = \frac{\pi}{2}, \gamma_+, \gamma_-$, while the zeroes in $\Omega(2) \setminus \Omega(1)$ of $D_0(\lambda)$ are $\sqrt{\lambda} = \frac{3}{2}\pi, 2\pi - \gamma_+, 2\pi - \gamma_-$. Since $D_0(\lambda)$ is 2π-periodic in $\sqrt{\lambda}$, we notice that $D(\lambda)$ has $3n_0$ zeroes in $\Omega(n_0)$, counted with multiplicities. These results combined with Lemmas 2.1 and 2.3 imply the other statements of this lemma. □

The following lemma will be a key to find that $\gamma_{3n-1} \neq \emptyset$ and $\gamma_{2n-2} \neq \emptyset$.

Lemma 2.6.

(i) *If λ satisfies $\Delta(\lambda) = \frac{1}{2}$, then we have $D(\lambda) \leq -\frac{19}{16}$.*

(ii) *If λ satisfies $\Delta(\lambda) = -\frac{1}{2}$, then we have $D(\lambda) \geq \frac{19}{16}$.*

(iii) *There exists some $n_0 \in \mathbb{N}$ such that both $\Delta(\lambda) + \frac{1}{2}$ and $\Delta(\lambda) - \frac{1}{2}$ has a real simple zero in $\Omega(n\pi + \frac{\pi}{2}, \frac{\pi}{4})$ for any $n > n_0$ and $2n_0$ zeroes in $\Omega(2n_0)$, counted with multiplicities. There are no other zeroes.*

Proof. We prove statement (i). It follows by $\Delta(\lambda) = \frac{1}{2}$ that

$$\theta(1,\lambda)\varphi'(1,\lambda) = \frac{1}{2} - \frac{\theta(1,\lambda)^2 + \varphi'(1,\lambda)^2}{2}.$$

Since $\frac{\theta(1,\lambda)^2+\varphi'(1,\lambda)^2}{2} = \Delta^2(\lambda) + \Delta_-^2(\lambda)$, we have

$$D(\lambda) = \Delta(\lambda)\left(4\Delta^2(\lambda) + \frac{1}{4} - \frac{\Delta^2(\lambda) + \Delta_-^2(\lambda)}{2} - \frac{7}{2}\right).$$

Substituting $\Delta(\lambda) = \frac{1}{2}$ for this, we have

$$D(\lambda) = -\frac{19}{16} - \frac{\Delta_-^2(\lambda)}{4} \leq -\frac{19}{16}.$$

In a similar way, we obtain statement (ii).

Next, we show the third statement. As proved in the first paragraph of Lemma 2.5, there exists a constant $C > 0$ such that $|\cos\sqrt{\lambda}| \geq C$ on $\Omega(2n)$. Moreover, it follows by Lemma 2.4 that $e^{|\mathrm{Im}\,\sqrt{\lambda}|} < M_0|\cos\sqrt{\lambda}|$. Thus, (2.20) implies that

$$\frac{|\Delta(\lambda) - \cos\sqrt{\lambda}|}{|\cos\sqrt{\lambda}|} \leq \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right).$$

Thus, the number of zeroes of $\Delta(\lambda)$ and $\cos\sqrt{\lambda}$ in $\Omega(2n)$ are the same for large enough $n \in \mathbb{N}$. We notice that $\cos\sqrt{\lambda}$ has $2n$ zeroes in $\Omega(2n)$.

Let us show that there exists some constant $C > 0$ such that $|\Delta(\lambda) + \frac{1}{2}| \geq C$ on $C(n\pi + \frac{\pi}{2}, \frac{\pi}{4})$. Put $\sqrt{\lambda} - (n\pi + \frac{\pi}{2}) = \frac{\pi}{4} e^{i\theta}$, where $\theta \in [0, 2\pi)$. Putting $\alpha = \frac{\pi}{4} \sin\theta$ and $\beta = \frac{\pi}{4} \cos\theta$, we have

$$\cos\sqrt{\lambda} = (-1)^{n+1}(\cosh\alpha \sin\beta + i \sinh\alpha \cos\beta).$$

Thus, we obtain

$$\left|\cos\sqrt{\lambda} + \frac{1}{2}\right|^2 = A(\theta)^2 + B(\theta)^2,$$

where

$$A(\theta) = (-1)^{n+1} \cosh\alpha \sin\beta + \frac{1}{2} \quad \text{and} \quad B(\theta) = \sinh\alpha \cos\beta.$$

We shall show that there does not exist θ such that $A(\theta) = B(\theta) = 0$. First, we suppose that $\sinh\alpha = 0$, which implies $\theta = 0, \pi$. If $\theta = 0$ (respectively, $\theta = \pi$), then we have $A(0) = \frac{(-1)^{n+1}}{\sqrt{2}} + \frac{1}{2} \neq 0$ (respectively $A(\pi) = \frac{(-1)^n}{\sqrt{2}} + \frac{1}{2} \neq 0$). On the other hand, we notice that $\cos\beta = 0$ does not happen because of $|\beta| \leq \frac{\pi}{4}$. Thus, we see that there exists some constant $C > 0$ such that $|\Delta(\lambda) + \frac{1}{2}| \geq C$ on $C(n\pi + \frac{\pi}{2}, \frac{\pi}{4})$.

Therefore, we have

$$\frac{|\Delta(\lambda) + \frac{1}{2} - (\cos\sqrt{\lambda} + \frac{1}{2})|}{|\cos\sqrt{\lambda} + \frac{1}{2}|} = \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

and get our conclusion on $\Delta(\lambda) + \frac{1}{2}$. In a similar way, we obtain the one on $\Delta(\lambda) - \frac{1}{2}$. □

It follows by Theorem 1.2 (i) that there exist $\mu_0 := \min\{\lambda \in \mathbb{R} \,|\, D(\lambda) = 1\}$. We recall $\sigma_D(H_0) = \{\mu_n\}_{n=1}^{\infty}$ and put

$$\left\{\lambda \,\middle|\, \Delta(\lambda) - \frac{1}{2} = 0\right\} = \{\zeta_0^+, \zeta_2^-, \zeta_2^+, \zeta_4^-, \zeta_4^+, \dots\}$$

and

$$\left\{\lambda \,\middle|\, \Delta(\lambda) + \frac{1}{2} = 0\right\} = \{\zeta_1^-, \zeta_1^+, \zeta_3^-, \zeta_3^+, \dots\},$$

where $\zeta_0^+ < \zeta_2^- < \zeta_2^+ < \zeta_4^- < \zeta_4^+ < \dots$ and $\zeta_1^- < \zeta_1^+ < \zeta_3^- < \zeta_3^+ < \dots$. We notice that

$$\mu_0 < \zeta_0^+ < \zeta_1^- < \mu_1 < \zeta_1^+ < \zeta_2^- < \mu_2 < \zeta_2^+ < \zeta_3^- < \mu_3 < \zeta_3^+ < \cdots.$$

We can roughly grab the behavior of $D(\lambda)$ by the following lemma.

Lemma 2.7. *For any $n \in \mathbb{N}$, we have $D(\mu_0) = 1$ and*

$$(-1)^n D(\mu_n) \geq 1, \quad (-1)^{n+1} D(\zeta_n^-) > 1 \quad \text{and} \quad (-1)^{n+1} D(\zeta_n^+) > 1.$$

Proof. By the definition of μ_0, we have $D(\mu_0) = 1$. Owing to Lemma 2.6, it suffices to show $(-1)^n D(\mu_n) \geq 1$ for any $n \in \mathbb{N}$. Since $\varphi(1, \mu_n) = 0$, we have $\theta(1, \mu_n)\varphi'(1, \mu_n) = 1$ and hence $d(\mu_n) = 4\Delta^2(\mu_n) - 3 \geq 1$ because of $\mu_n \in [\lambda_n^-, \lambda_n^+]$, which implies $(-1)^n \Delta(\mu_n) \geq 1$. Thus, we obtained our assertion. □

Next, we prepare notations to get the statements on zeroes of $D(\lambda) \pm 1$. Let $\tilde{C}(n) = \tilde{C}_1(n) - \tilde{C}_2(n) - \tilde{C}_3(n) + \tilde{C}_4(n)$, where

$$\begin{aligned}
\tilde{C}_1(n) &= \Big\{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = 2n\pi + \frac{\pi}{2} + i\Big(2n\pi + \frac{\pi}{2}\Big)t,\ -1 \le t \le 1\Big\},\\
\tilde{C}_2(n) &= \Big\{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = 2n\pi t + \frac{\pi}{2} + i\Big(2n\pi + \frac{\pi}{2}\Big),\ -1 \le t \le 1\Big\},\\
\tilde{C}_3(n) &= \Big\{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = -(2n\pi + \frac{\pi}{2}) + i\Big(2n\pi + \frac{\pi}{2}\Big)t,\ -1 \le t \le 1\Big\},\\
\tilde{C}_2(n) &= \Big\{\lambda \in \mathbb{C} \,|\, \sqrt{\lambda} = -(2n\pi t + \frac{\pi}{2}) + i\Big(2n\pi + \frac{\pi}{2}\Big),\ -1 \le t \le 1\Big\}
\end{aligned}$$

for $n \in \mathbb{N}$. Then, we obtain the following inequality.

Lemma 2.8. *For a fixed* $p = \pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}$*, there exists some constant* $\tilde{M}_p > 0$ *and* $\tilde{n}_0(p) \in \mathbb{N}$ *such that*

$$e^{|\mathrm{Im}\ \sqrt{\lambda}|} < \tilde{M}_p |\cos\sqrt{\lambda} + p|$$

on $\tilde{C}(n)$ *for any* $n > \tilde{n}_0(p)$*. The constant* $\tilde{M}_p$ *and* $\tilde{n}_0(p)$ *depends on* p*, but does not depend on* n*.*

Proof. We consider $\lambda \in \tilde{C}_1(n)$, namely we put $\sqrt{\lambda} = 2n\pi + \frac{\pi}{2} + i(2n\pi + \frac{\pi}{2})t$, where $-1 \le t \le 1$. Since $\cos\sqrt{\lambda} = -i\sinh(2m\pi + \frac{\pi}{2})t$, we have

$$\frac{e^{2|\mathrm{Im}\ \sqrt{\lambda}|}}{|\cos\sqrt{\lambda} + p|^2} = \frac{e^{2|2n\pi + \frac{\pi}{2}||t|}}{p^2 + \sinh^2(2n\pi + \frac{\pi}{2})t}.$$

The function $f(y) := \frac{e^{2y}}{p^2 + \sinh^2 y}$ is continuous on $\mathbb{R}$ and we have $\lim_{y\to-\infty} = 0$ and $\lim_{y\to\infty} f(y) = 4$. This implies that there exists some $\tilde{M}_{1,p} > 0$ satisfying $e^{2|\mathrm{Im}\ \sqrt{\lambda}|} < \tilde{M}_{1,p}|\cos\sqrt{\lambda} + p|$ on $\tilde{C}_1(n)$. In a similar way, we get some constants $\tilde{M}_{3,p} > 0$ satisfying $e^{2|\mathrm{Im}\ \sqrt{\lambda}|} < \tilde{M}_{3,p}|\cos\sqrt{\lambda} + p|$ on $\tilde{C}_3(n)$.

The claim on $\tilde{C}_2(n)$ and $\tilde{C}_4(n)$ are shown in a similar way to the proof of the claim on $C_2(n)$ and $C_4(n)$ in Lemma 2.4. □

Let $\tilde{\Omega}(n)$ be the domain surrounded by $\tilde{C}(n)$ for $n \in \mathbb{N}$.

Lemma 2.9.

(i) *There exists some* $n_0 \in \mathbb{N}$ *such that* $D(\lambda) + 1$ *has 2 zeroes, counted with multiplicities, in* $\Omega(2n\pi + \frac{3\pi}{8}, \frac{\pi}{8})$*,* $\Omega(2n\pi + \frac{13\pi}{8}, \frac{\pi}{8})$ *and* $\Omega(2n\pi + \pi, \frac{\pi}{4})$*, respectively, for any* $n > n_0$ *and* $2 + 6n_0$ *zeroes, counted with multiplicities, in* $\tilde{\Omega}(n_0)$*. There are no other zeroes.*

(ii) *There exists some* $n_0 \in \mathbb{N}$ *such that* $D(\lambda) - 1$ *has 2 zeroes, counted with multiplicities, in* $\Omega(2n\pi + \frac{5\pi}{8}, \frac{\pi}{8})$*,* $\Omega(2n\pi, \frac{\pi}{4})$ *and* $\Omega(2n\pi + \frac{11\pi}{8}, \frac{\pi}{8})$*, respectively, for any* $n > n_0$ *and* $1 + 6n_0$ *zeroes, counted with multiplicities, in* $\tilde{\Omega}(n_0)$*. There are no other zeroes.*

Proof. We only show the statement (i). We first pick $\lambda \in C(2n\pi + \frac{3\pi}{8}, \frac{\pi}{8})$, arbitrarily. Let $\sqrt{\lambda} = 2n\pi + \frac{3\pi}{8} + \frac{\pi}{8}e^{i\theta}$, where $\theta \in [0, 2\pi)$. We notice that

$$D_0(\lambda) + 1 = \left(\cos\sqrt{\lambda} + 1\right)\left(\cos\sqrt{\lambda} - \frac{1}{3}\right)\left(\cos\sqrt{\lambda} - \frac{2}{3}\right).$$

We claim that for $p = 1, -\frac{1}{3}, -\frac{2}{3}$, there exists a constant $C_p > 0$ satisfying $|\cos\sqrt{\lambda} + p| \geq C_p$ on $C(2n\pi + \frac{3\pi}{8}, \frac{\pi}{8})$, where C_p depends on p only. Since

$$\cos\sqrt{\lambda} = \cos\left(\frac{3}{8}\pi + \beta\right)\cosh\alpha - i\sin\left(\frac{3}{8}\pi + \beta\right)\sinh\alpha,$$

we have

$$|\cos\sqrt{\lambda} + p|^2 = A(\theta)^2 + B(\theta)^2,$$

where $\alpha = \frac{\pi}{8}\sin\theta$, $\beta = \frac{\pi}{8}\cos\theta$, $A(\theta) = \cos(\frac{3}{8}\pi + \beta)\cosh\alpha + p$ and $B(\theta) = \sin(\frac{3}{8}\pi + \beta)\sinh\alpha$. It suffices to show that there does not exist $\theta \in [0, 2\pi)$ such that $A(\theta) = B(\theta) = 0$. First, we assume that $\sinh\alpha = 0$, which implies that $\theta = 0, \pi$. If $\theta = 0$, then we see that $\beta = \frac{\pi}{8}$ and hence $A(0) = p \neq 0$. If $\theta = \pi$, then we have $\beta = -\frac{\pi}{8}$ and hence $A = -\frac{1}{\sqrt{2}} + p \neq 0$. Second, we see that $\sin(\frac{3}{8}\pi + \beta) \neq 0$ because of $\frac{\pi}{4} \leq \frac{3\pi}{8} + \beta \leq \frac{\pi}{2}$. Thus, there exists some constant $C_p > 0$ satisfying $|D_0(\lambda)+1| \geq C_p > 0$. So, it follows by (2.21) that

$$\frac{|(D(\lambda) + 1) - (D_0(\lambda) + 1)|}{|D_0(\lambda) + 1|} \leq \mathcal{O}\left(\frac{1}{|\lambda|^{1/2}}\right)$$

and see that the number of zeroes of $D(\lambda) + 1$ and $D_0(\lambda) + 1$ in $\Omega(2n\pi + \frac{\pi}{4}, \frac{\pi}{4})$ are the same, counted with multiplicities. Since $\frac{\pi}{4} < \arccos\frac{2}{3} < \arccos\frac{1}{3} < \frac{\pi}{2}$, we see that $D_0(\lambda)+1$ has 2 zeroes in $\Omega(2n\pi+\frac{3\pi}{8}, \frac{\pi}{8})$. Thus, we conclude that $D(\lambda)+1$ has 2 zeroes in $\Omega(2n\pi + \frac{3\pi}{8}, \frac{\pi}{8})$, counted with multiplicities. In a similar way, we see that $D(\lambda) + 1$ has 2 zeroes, counted with multiplicities, in $\Omega(2n\pi + \frac{13\pi}{8}, \frac{\pi}{8})$ and $\Omega(2n\pi + \pi, \frac{\pi}{4})$, respectively. Similarly, we can make sure that there exists some $n_0 \in \mathbb{N}$ such that $D(\lambda) - 1$ has 2 zeroes, counted with multiplicities, in $\Omega(2n\pi + \frac{5\pi}{8}, \frac{\pi}{8})$, $\Omega(2n\pi, \frac{\pi}{4})$ and $\Omega(2n\pi + \frac{11\pi}{8}, \frac{\pi}{8})$, respectively.

Finally, it follows by (2.21) that

$$|(D(\lambda) \pm 1) - (D_0(\lambda) \pm 1)| = o(e^{3|\text{Im }\sqrt{\lambda}|})$$

as $|\lambda| \to \infty$. Since

$$D_0(\lambda) - 1 = \frac{9}{2}(\cos\sqrt{\lambda} - 1)\left(\cos\sqrt{\lambda} + \frac{1}{3}\right)\left(\cos\sqrt{\lambda} + \frac{2}{3}\right)$$

and

$$D_0(\lambda) + 1 = \frac{9}{2}(\cos\sqrt{\lambda} + 1)\left(\cos\sqrt{\lambda} - \frac{1}{3}\right)\left(\cos\sqrt{\lambda} - \frac{2}{3}\right),$$

we notice that $|(D(\lambda) \pm 1) - (D_0(\lambda) \pm 1)| = o(1)|D_0(\lambda) \pm 1|$ on $\tilde{C}(n)$ as $n \to \infty$ by virtue of Lemma 2.8. Thus, it follows by Rouché's theorem that the number in $\tilde{\Omega}(n)$ for enough large n of $D(\lambda) \pm 1$ and $D_0(\lambda) \pm 1$ are the same. Since $D_0(\lambda) - 1$ has $1 + 6n_0$ zeroes and $D_0(\lambda) + 1$ has $2 + 6n_0$ zeroes in $\tilde{\Omega}(n_0)$, we arrive at our goal. $\square$

Our next goal is to show that $D(\lambda) - c$ has only simple zeroes for $c \in (-1, 1)$. Basically, we rely on Rouché's theorem like above, but it is a little bit hard to find the solutions to $D_0(\lambda) - c = 0$. To overcome this difficulty, we utilize the method established by François Viète in the 16th century to find analytic solutions of cubic equations whose every solution is real. For constants $p < 0$ and $q \in \mathbb{R}$ and a cubic equation $z^3 + pz + q = 0$, we recall that the discriminant of this equation is $\mathcal{D} = -(4p^3 + 27q^2)$. If $\mathcal{D} > 0$, then the corresponding equation has three different real solutions. in the case where $\mathcal{D} > 0$, Vièta showed the three real roots of $z^3+pz+q = 0$ are given by

$$\alpha_k = 2\sqrt{-\frac{p}{3}} \cos\left\{\frac{1}{3} \arccos\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) - \frac{2\pi}{3}k\right\}, \quad k = 0, 1, 2.$$

We fix $c \in (-1, 1)$, arbitrarily. Since our equation $D_0(\lambda) - c = 0$ is $x^3 - \frac{7}{9}x - \frac{2}{9}c = 0$, where $x = \cos\sqrt{\lambda}$, we see that $\mathcal{D} = -(-\frac{1372}{729} + \frac{4}{3}c^2) > \frac{400}{729} > 0$ and hence $x^3 - \frac{7}{9}x - \frac{2}{9}c = 0$ has three different real zeroes

$$x = \cos\sqrt{\lambda} = \sqrt{\frac{28}{27}} \cos\alpha_1(c), \quad \sqrt{\frac{28}{27}} \cos\alpha_2(c) \quad \text{and} \quad \sqrt{\frac{28}{27}} \cos\alpha_3(c),$$

where

$$\alpha_k(c) = \frac{1}{3} \arccos\left(\frac{9\sqrt{3}}{7\sqrt{7}}c\right) + \frac{2}{3}\pi(k-1), \quad k = 1, 2, 3.$$

We have the properties among these three solutions as seen in the following lemma.

Lemma 2.10. *As functions in $c \in [-1, 1]$, $\cos\alpha_1(c)$ and $\cos\alpha_2(c)$ are strictly increasing and $\cos\alpha_3(c)$ is strictly decreasing. For any $c \in [-1, 1]$, we have*

$$-1 \leq \sqrt{\frac{28}{27}} \cos\alpha_2(c) < -\frac{1}{2} < \sqrt{\frac{28}{27}} \cos\alpha_3(c) < \frac{1}{2} < \sqrt{\frac{28}{27}} \cos\alpha_1(c) \leq 1.$$

Moreover, we see that $-1 = \sqrt{\frac{28}{27}} \cos\alpha_2(c)$ (respectively, $\sqrt{\frac{28}{27}} \cos\alpha_1(c) = 1$) is valid if and only if $c = -1$ (respectively, $c = 1$).

Proof. We notice that $\alpha_1(c)$, $\alpha_2(c)$ are $\alpha_3(c)$ strictly decreasing in $c \in [-1, 1]$ and $0 < \alpha_1(c) < \frac{\pi}{3}$, $\frac{2}{3}\pi < \alpha_2(c) < \pi$ and $\frac{4}{3}\pi < \alpha_3(c) < \frac{5}{3}\pi$ because of $|\frac{9\sqrt{3}}{7\sqrt{7}}c| < 1$. This is why we have $\cos\alpha_1(c)$ and $\cos\alpha_2(c)$ are strictly increasing and $\cos\alpha_3(c)$ is strictly decreasing. Moreover, we have $\frac{1}{2} = \cos\frac{\pi}{3} < \cos\alpha_1(c) < \cos 0 = 1$, $-1 < \cos\alpha_2(c) < -\frac{1}{2}$ and $-\frac{1}{2} < \cos\alpha_3(c) < \frac{1}{2}$. Namely, we obtained

$$-1 < \cos\alpha_2(c) < -\frac{1}{2} < \cos\alpha_3(c) < \frac{1}{2} < \cos\alpha_1(c) < 1.$$

Let us enhance the accuracy of this inequality up to our desired version.

For this purpose, we show the following equality:

$$\cos\left(\frac{1}{3} \arccos\frac{9\sqrt{3}}{7\sqrt{7}}\right) = \sqrt{\frac{27}{28}}.$$

Putting $\cos\left(\frac{1}{3}\arccos\frac{9\sqrt{3}}{7\sqrt{7}}\right) = p$ and $\arccos\frac{9\sqrt{3}}{7\sqrt{7}} = \theta$, we have $\cos\frac{\theta}{3} = p$. Since

$$\frac{9\sqrt{3}}{7\sqrt{7}} = \cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} = p(4p^2-3),$$

we obtain $4p^3 - 3p - \frac{9\sqrt{3}}{7\sqrt{7}} = 0$. This implies that $(p - \sqrt{\frac{27}{28}})(4p^2 + \frac{6\sqrt{3}}{\sqrt{7}}p + \frac{6}{7}) = 0$. The solutions to the quadratic equation $4p^2 + \frac{6\sqrt{3}}{\sqrt{7}}p + \frac{6}{7} = 0$ are $p = -\frac{\sqrt{3}}{\sqrt{7}}, -\frac{\sqrt{3}}{2\sqrt{7}}$. Since $\frac{1}{2} < \cos\alpha_1(c) < 1$, we notice $\frac{1}{2} < p < 1$. So, we obtain $p = \sqrt{\frac{27}{28}}$.

We also obtain

$$\cos\alpha_2(-1) = -\sqrt{\frac{27}{28}}$$

because

$$\begin{aligned}\cos\alpha_2(-1) &= \cos\left(\frac{1}{3}\arccos(-k) + \frac{2}{3}\pi\right) = \cos\left(\frac{1}{3}(\pi - \arccos k) + \frac{2}{3}\pi\right)\\ &= -\cos\left(\frac{1}{3}\arccos k\right), \quad \text{where} \quad k = \frac{9\sqrt{3}}{7\sqrt{7}}.\end{aligned}$$

Moreover, it follows by $\sin\frac{\theta}{3} = \frac{1}{\sqrt{28}}$ that

$$\cos\alpha_1(-1) = \cos\left(\frac{1}{3}\arccos(-k)\right) = \cos\frac{1}{3}\left(\pi - \arccos k\right) = \frac{\sqrt{3}}{\sqrt{7}}$$

and hence

$$\sqrt{\frac{28}{27}}\cos\alpha_1(-1) = \frac{2}{3} > \frac{1}{2}.$$

In a similar way, we obtain $\cos\alpha_2(1) = -\frac{\sqrt{3}}{\sqrt{7}}$, $\cos\alpha_3(1) = -\frac{\sqrt{3}}{2\sqrt{7}}$ and $\cos\alpha_3(-1) = \frac{\sqrt{3}}{2\sqrt{7}}$. So, our desired inequality is proved.

□

We prepare notations

$$u^+_{0,1}(c) = \arccos A_3(c), \quad u^+_{0,2}(c) = \arccos A_2(c) \quad \text{and} \quad u^+_{0,3}(c) = \arccos A_1(c),$$

where

$$A_1(c) = \sqrt{\frac{28}{27}}\cos\alpha_2(c), \quad A_2(c) = \sqrt{\frac{28}{27}}\cos\alpha_3(c), \quad A_3(c) = \sqrt{\frac{28}{27}}\cos\alpha_1(c),$$

although these notations might be a little bit confusing because of the different indices. Putting

$$u^\pm_{n,i} = u^\pm_{n,i}(c) = n\pi \pm u^+_{0,i}(c)$$

for $i = 1, 2, 3$ and $n \in \mathbb{N}$, we see that

$$\begin{aligned}u^+_{0,1} < u^+_{0,2} < u^+_{0,3} < u^-_{1,3} < u^-_{1,2} < u^-_{1,1} < u^+_{1,1} < u^+_{1,2} < u^+_{1,3}\\ < u^-_{2,3} < u^-_{2,2} < u^-_{2,1} < u^+_{2,1} < u^+_{2,2} < u^+_{2,3} < \cdots.\end{aligned}$$

The zeroes of $D_0(\lambda) - c$ are given by $\{(u^{\pm}_{n,i}(c))^2\}_{i=1,2,3,n\in\mathbb{N}}$. We see that $(u^-_{n,3})^2 \in \Omega(\frac{\pi}{6} + (2n-1)\pi, \frac{\pi}{6})$, $(u^-_{n,2})^2 \in \Omega(\frac{\pi}{2} + (2n-1)\pi, \frac{\pi}{6})$, $(u^-_{n,1})^2 \in \Omega(\frac{5\pi}{6} + (2n-1)\pi, \frac{\pi}{6})$, $(u^+_{n,1})^2 \in \Omega(\frac{\pi}{6} + 2n\pi, \frac{\pi}{6})$, $(u^+_{n,2})^2 \in \Omega(\frac{\pi}{2} + 2n\pi, \frac{\pi}{6})$ and $(u^+_{n,3})^2 \in \Omega(\frac{5\pi}{6} + 2n\pi, \frac{\pi}{6})$ for any $n \in \mathbb{N}$.

These preparations combined with Rouché's theorem and Lemma 2.4 lead us to the following lemma.

Lemma 2.11. *For a fixed $c \in (-1,0) \cup (0,1)$, there exists some $n_0 \in \mathbb{N}$ such that $D(\lambda) - c$ has one simple zero in $\Omega(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$, $\Omega(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$ and $\Omega(n\pi + \frac{5}{6}\pi, \frac{\pi}{6})$, respectively, for any $n > n_0$ and $3n_0$ zeroes, counted with multiplicities, in $\Omega(n_0)$. There are no other zeroes.*

Proof. First, we show the claim for $\Omega(n_0)$. Let us pick $\lambda \in C(n)$, arbitrarily. We notice that

$$D_0(\lambda) - c = \frac{9}{2}(\cos\sqrt{\lambda} - A_1(c))(\cos\sqrt{\lambda} - A_2(c))(\cos\sqrt{\lambda} - A_3(c)).$$

This combined with Lemma 2.4 and 2.10 means that

$$e^{3|\mathrm{Im}\ \sqrt{\lambda}|} < \frac{2}{9} M_{A_1(c)} M_{A_2(c)} M_{A_3(c)} |D_0(\lambda) - c|$$

on $C(n)$ for large enough n. By virtue of (2.21), we have

$$|(D(\lambda) - C) - (D_0(\lambda) - c)| = \mathcal{O}\left(\frac{e^{3|\mathrm{Im}\ \sqrt{\lambda}|}}{\sqrt{\lambda}}\right) = o(1)|D_0(\lambda) - c|$$

on $C(n)$ as $n \to \infty$. Thus, we conclude that the number of zeroes of $D(\lambda) - c$ is $3n_0$ for large enough $n_0 \in \mathbb{N}$.

To prove the statement in $\Omega(n_0)$, we did not need to eliminate $c = 0$. Let us explain why we need to eliminate the case of $c = 0$. Since $\cos\alpha_3(c)$ is strictly decreasing, $\cos\alpha_3(0) = 0$ and hence $A_2(0)$, it follows by Lemma 2.10 that $A_1(c) \in (-1, -\frac{1}{2})$, $A_2(c) \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$ and $A_3(c) \in (\frac{1}{2}, 1)$ for $c \in (-1,0) \cup (0,1)$. So, it suffices to show that for a fixed $p \in (-1,1) \setminus \{-\frac{1}{2}, 0, \frac{1}{2}\}$, there exists some constant C_p such that

$$|\cos\sqrt{\lambda} - p| \geq C_p > 0 \tag{2.25}$$

on $C(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$, $C(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$ and $C(n\pi + \frac{5}{6}\pi, \frac{\pi}{6})$ because (2.25) implies that

$$|D_0(\lambda) - c| \geq \frac{9}{2} C_{A_1(c)} C_{A_2(c)} C_{A_3(c)} > 0 \tag{2.26}$$

on these cycles. In the case where $p = \pm\frac{1}{2}$, (2.25) does not hold true on $C(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$. In the case where $p = 0$, (2.25) does not hold true on $C(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$ and $C(n\pi + \frac{5\pi}{6}, \frac{\pi}{6})$. So, in order to prove (2.25), we need to eliminate $c = 0$, but the assertion of the zeroes of $D(\lambda)$ is in our possession in Lemma 2.5.

Let us show that $D(\lambda) - c$ has a simple zero in $\Omega(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$ for large $n \in \mathbb{N}$. Putting $\theta \in [0, 2\pi)$ and $\sqrt{\lambda} = \frac{\pi}{2} + n\pi + \frac{\pi}{6}e^{i\theta}$, we see that

$$|\cos\sqrt{\lambda} - p|^2 = A(\theta)^2 + B(\theta)^2,$$

where $\alpha = \frac{\pi}{6}\cos\theta$, $\beta = \frac{\pi}{6}\sin\theta$, $A(\theta) = (-1)^{n+1}\sin\alpha\cosh\beta - p$ and $B(\theta) = \cos\alpha\sinh\beta$. We show that there does not exist θ such that $A(\theta) = B(\theta) = 0$. First, we see that $\cos\alpha \neq 0$ because $-\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{6}$. On the other hand, we assume that $\sinh\beta = 0$, which implies $\theta = 0, \pi$. Since $A(0) = \frac{(-1)^{n+1}}{2} - p \neq 0$ and $A(\pi) = \frac{(-1)^n}{2} - p \neq 0$, (2.25) is shown on $C(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$. In a similar way, we can make sure (2.25) on $C(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$ and $C(n\pi + \frac{5}{6}\pi, \frac{\pi}{6})$ by using $p \neq 0$.

Therefore, on $C(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$, $C(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$ and $C(n\pi + \frac{5}{6}\pi, \frac{\pi}{6})$, we obtain

$$|(D(\lambda) - C) - (D_0(\lambda) - c)| = o(1)|D_0(\lambda) - c| \quad \text{as} \quad n \to \infty,$$

owing to (2.21). Since the number of zeroes of $D_0(\lambda) - c$ in $\Omega(n\pi + \frac{\pi}{6}, \frac{\pi}{6})$, $\Omega(n\pi + \frac{\pi}{2}, \frac{\pi}{6})$ and $\Omega(n\pi + \frac{5}{6}\pi, \frac{\pi}{6})$ is 1, the one of $D(\lambda) - c$ is also 1. □

Owing to Lemma 2.5, 2.7 and 2.11, the proof of Theorem 1.2 (ii) is finished. We next prove Theorem 1.2 (iii). For this purpose, we utilize Laguerre's theorem. So, we quote it from [14].

Definition 2.12. An entire function $f(z)$ is said to be of finite order if there is a positive number A such that

$$f(z) = \mathcal{O}(e^{r^A}) \quad \text{as} \quad |z| = r \to \infty.$$

The lower bound ρ of numbers A for which this is true is called the order of the function $f(z)$.

Theorem 2.13 (Laguerre, see Section 8.52 in [14])**.** *If $f(z)$ is an entire function, is real for a real z, of order less than 2, with real zeroes, then the zeroes of $f'(z)$ are also all real and are separated from each other by the zeroes of $f(z)$.*

Proof of Theorem 1.2 (iii), (iv), (v) *and* (vi). Let us make sure that $D(\lambda)$ satisfies the assumptions of Theorem 2.13. It follows by (2.21) that $D(\lambda) = \mathcal{O}(e^{3|\lambda|^{1/2}})$ and hence $D(\lambda) = \mathcal{O}(e^{|\lambda|})$ as $|\lambda| \to \infty$.

Let us show that the zeroes of $D(\lambda)$ are composed of only real zeroes. It turns out by Lemma 2.9 (i) and Lemma 2.5 that $D(\lambda) + 1$ has 2 zeroes in $\Omega(2n\pi + \frac{3\pi}{8}, \frac{\pi}{8})$ and $D(\lambda)$ has 1 zero in $\Omega(2n\pi + \frac{\pi}{8}, \frac{\pi}{8})$, counted with multiplicities, for $n \geq n_0$, where n_0 is enough large. Thus, we see that $-1 < D((2n\pi + \frac{\pi}{4})^2) < 0$. We recall that $\mu_0, \mu_1, \ldots, \mu_{2n_0} \in (-\infty, (2n\pi + \frac{\pi}{4})^2)$, $D(\mu_0) = 1$ and $(-1)^j D(\mu_j) \geq 1$ for $j = 1, 2, 3, \ldots, 2n_0$. These combined with $-1 < D((2n\pi + \frac{\pi}{4})^2) < 0$ and the intermediate value theorem implies that $D(\lambda)$ has at least $1 + 6n_0$ zeroes in $(-\infty, (2n\pi + \frac{\pi}{4})^2)$, counted with multiplicities. Since it follows by Lemma 2.5 that $D(\lambda)$ has exactly $1+6n_0$ zeroes in $\Omega(0, (2n\pi + \frac{\pi}{4}))$, we conclude that $D(\lambda)$ has only real zeroes. Therefore, (1.2) is established. This combined with Lemma 2.7 implies that $(-1)^n D(\lambda_{0,n}) \leq 1$ for any $n \in \mathbb{N}$. Thus, we obtain statement (iii).

We next show (iv), (v) and (vi). We pick $n_0 \in \mathbb{N}$ satisfying the statements of Lemma 2.6, 2.7 and 2.9. Since $\Delta(\lambda) + \frac{1}{2}$ (respectively, $\Delta(\lambda) - \frac{1}{2}$) has $2n_0$ zeroes in $\Omega(2n_0)$ and 1 zero in $\Omega(2n_0\pi + \frac{\pi}{2}, \frac{\pi}{4})$ by Lemma 2.6, we see that $\Delta(\lambda) + \frac{1}{2}$ (respectively, $\Delta(\lambda) - \frac{1}{2}$) has $2n_0 + 1$ zeroes in the interval $I = (-\infty, (2n_0\pi + \frac{3}{4}\pi)^2)$. Furthermore, since $\mu_{2n_0+1} \in ((2n_0\pi + \frac{3}{4}\pi)^2, (2n_0\pi + \frac{5}{4}\pi)^2)$ holds true for large enough $n_0 \in \mathbb{N}$, we see that

$$\mu_0, \zeta_0^+, \zeta_1^-, \mu_1, \zeta_1^+, \zeta_2^-, \mu_2, \zeta_2^+, \dots, \zeta_{2n_0}^-, \mu_{2n_0}, \zeta_{2n_0}^+, \zeta_{2n_0+1}^-$$

are in I and μ_{2n_0+1} is not in I.

Since $D(\lambda) - 1$ has 1 zero in $\Omega(2n_0\pi + \frac{5}{8}\pi, \frac{\pi}{8})$ and $\Omega(2n_0\pi + \frac{7}{8}\pi, \frac{\pi}{8})$ respectively, we see that $D((2n_0\pi + \frac{3}{4}\pi)^2) > 1$. Thus, we see that there exist 1 zero of $D(\lambda) + 1$ in the neighborhood of μ_0 and $\zeta_{2n_0+1}^-$, 2 zeroes of $D(\lambda) + 1$ in the neighborhood of

$$\mu_2, \mu_4, \dots, \mu_{2n_0}, \zeta_1^-, \zeta_1^+, \zeta_3^-, \zeta_3^+, \dots, \zeta_{2n_0-1}^-, \zeta_{2n_0-1}^+$$

and 2 zeroes of $D(\lambda) - 1$ in the neighborhood of

$$\mu_1, \mu_3, \dots \mu_{2n_0-1}, \zeta_0^+, \zeta_2^-, \zeta_2^+, \zeta_4^-, \zeta_4^+ \dots, \zeta_{2n_0}^-, \zeta_{2n_0}^+,$$

respectively and counted with multiplicities. Thus, it follows by the intermediate value theorem that at least $2+6n_0$ zeroes of $D(\lambda)+1$ and at least $2+6n_0$ zeroes of $D(\lambda)-1$ are in I.

It turns out by using Lemma 2.9 (i) that there are exactly $2 + 6n_0$ zeroes of $D(\lambda) + 1$ in $\Omega(0, 2n_0\pi + \frac{3}{4})$, counted with multiplicities. Moreover, it turns out by Lemma 2.9 (ii) that there are exactly $2 + 6n_0$ zeroes of $D(\lambda) - 1$ in $\Omega(0, 2n_0\pi + \frac{3}{4}\pi)$. Thus, we conclude that both $D(\lambda) + 1$ and $D(\lambda) - 1$ have only real zeroes. It follows by Lemma 2.7 that the inequalities stated in Theorem 1.2 (iv), (v) and (vi) hold true. □

3. PROOF OF THEOREM 1.3

In order to prove Theorem 1.3, we first prove the followings.

Lemma 3.1. (i) *The sequence* $\{z_n^\pm\}$ *satisfies the asymptotics*

$$\sqrt{z_{3n}^\pm} = u_{n,3}^\pm + \frac{q_0}{2u_{n,3}^\pm} + o\Big(\frac{1}{n}\Big), \tag{3.1}$$

$$\sqrt{z_{3n-1}^\pm} = u_{n,2}^\pm + \frac{q_0}{2u_{n,2}^\pm} + o\Big(\frac{1}{n^2}\Big), \tag{3.2}$$

$$\sqrt{z_{3n-2}^\pm} = u_{n,1}^\pm + \frac{q_0}{2u_{n,1}^\pm} + o\Big(\frac{1}{n^2}\Big) \quad as \quad n \to \infty. \tag{3.3}$$

(ii) *The sequence* $\{x_n^\pm\}$ *satisfies the asymptotics*

$$\sqrt{x_{3n-1}^\pm} = v_{n,2}^\pm + \frac{q_0}{2v_{n,2}^\pm} + o\Big(\frac{1}{n}\Big), \tag{3.4}$$

$$\sqrt{x_{3n}^\pm} = v_{n,3}^\pm + \frac{q_0}{2v_{n,3}^\pm} + o\Big(\frac{1}{n^2}\Big), \tag{3.5}$$

$$\sqrt{x_{3n+1}^\pm} = v_{n,1}^\pm + \frac{q_0}{2v_{n,1}^\pm} + o\Big(\frac{1}{n^2}\Big) \quad as \quad n \to \infty. \tag{3.6}$$

Proof. We give the proof of (i). We recall $\Delta_-(\lambda) = \frac{1}{2}(\varphi'(1,\lambda) - \theta(1,\lambda))$ and (2.23). Since the equation $D(\lambda) - 1$ is equivalent to $\Delta^3(\lambda) = \frac{1}{9}(\Delta_-^2(\lambda) + 7)\Delta(\lambda) + \frac{2}{9}$, which is a cubic equation with respect to $\Delta(\lambda)$. By using the method by François Viète, we see that the solutions to $D(\lambda) - 1$ are given by $\Delta(\lambda) = \Delta_1(\lambda), \Delta_2(\lambda), \Delta_3(\lambda)$, where

$$\Delta_j(\lambda) = 2\sqrt{\frac{\Delta_-^2(\lambda) + 7}{27}} \cos\left[\frac{1}{3}\arccos\left\{\frac{9\sqrt{3}}{(\Delta_-^2(\lambda) + 7)^{3/2}}\right\} + \frac{2\pi}{3}(j-1)\right]$$

for $j = 1, 2, 3$. First, we see that

$$\Delta_1(\lambda) \to \sqrt{\frac{28}{27}} \cos\left\{\frac{1}{3}\arccos\left(\frac{9\sqrt{3}}{7\sqrt{7}}\right)\right\} = 1 \quad \text{as} \quad \lambda \to \infty.$$

Next, we claim that $\Delta_2(\lambda) \to -\frac{2}{3}$ and $\Delta_3(\lambda) \to -\frac{1}{3}$ as $\lambda \to \infty$. Putting $\alpha = \frac{1}{3}\arccos\frac{9\sqrt{3}}{7\sqrt{7}}$, we have

$$\Delta_2(\lambda) \to \sqrt{\frac{28}{27}} \cos\left(\alpha + \frac{2\pi}{3}\right)$$

as $\lambda \to \infty$. Since $0 < \alpha < \frac{\pi}{6}$, we see that $\sin\alpha = \sqrt{\frac{1}{28}}$. Thus, we see that $\Delta_2(\lambda) \to -\frac{2}{3}$ as $\lambda \to \infty$. In a similar way, we notice that $\Delta_3(\lambda) \to -\frac{1}{3}$ as $\lambda \to \infty$.

We quote the following from [5]:

$$\Delta(\lambda) = \cos\chi(\lambda), \quad \chi(\lambda) = \sqrt{\lambda} - \frac{q_0}{2\sqrt{\lambda}} + \frac{o(1)}{\lambda}$$

as $|\lambda| \to \infty$. Putting $\lambda = z_{3n}^\pm$, we have

$$\chi(z_{3n}^\pm) = \sqrt{z_{3n}^\pm} - \frac{q_0}{2\sqrt{z_{3n}^\pm}} + \frac{o(1)}{z_{3n}^\pm}.$$

Putting $\chi(z_{3n}^\pm) = a_{3n}^\pm + b_{3n}^\pm$, $a_{3n}^\pm = \sqrt{z_{3n}^\pm} - \frac{q_0}{2\sqrt{z_{3n}^\pm}} + \frac{o(1)}{z_{3n}^\pm} \in \mathbb{R}$ and $b_{3n}^\pm = \frac{o(1)}{z_{3n}^\pm} \in \mathbb{R}$, we have

$$\Delta_1(z_{3n}^\pm) = \cos\chi(z_{3n}^\pm) = \cos a_{3n}^\pm \cosh b_{3n}^\pm - i\sin a_{3n}^\pm \sinh b_{3n}^\pm.$$

Since $\Delta_1(z_{3n}^\pm) \to 1$ as $n \to \infty$, we see that

$$\cos\left(\sqrt{z_{3n}^\pm} - \frac{q_0}{2\sqrt{z_{3n}^\pm}} + \frac{o(1)}{z_{3n}^\pm}\right) \to 1$$

as $n \to \infty$. Thus, we see that

$$\sqrt{z_{3n}^\pm} - \frac{q_0}{2\sqrt{z_{3n}^\pm}} + \frac{o(1)}{z_{3n}^\pm} \to 2\pi k$$

for some integer k. By virtue of Lemma 2.9 (i), we see that $\sqrt{z_{3n}^{\pm}} \in \Omega(2n\pi, \frac{\pi}{4})$, $\sqrt{z_{3n-1}^{\pm}} \in \Omega(\frac{5}{4}\pi + 2(n-1)\pi, \frac{\pi}{4})$ and $\sqrt{z_{3n-2}^{\pm}} \in \Omega(\frac{3}{4}\pi + 2(n-1)\pi, \frac{\pi}{4})$ for large enough $n \in \mathbb{N}$. Thus, we see that

$$|\sqrt{z_{3n}^{\pm}} - u_{n,3}^{\pm}| < \frac{\pi}{2}, \quad |\sqrt{z_{3n-1}^{\pm}} - u_{n,2}^{\pm}| < \frac{\pi}{2}, \quad |\sqrt{z_{3n-2}^{\pm}} - u_{n,1}^{\pm}| < \frac{\pi}{2}$$

for large enough $n \in \mathbb{N}$. Thus, we have $k = n$ and hence

$$\sqrt{z_{3n}^{\pm}} - u_{n,3}^{\pm} - \frac{q_0}{2\sqrt{z_{3n}^{\pm}}} \to 0$$

as $n \to \infty$. Putting

$$\epsilon_{3n}^{\pm} = \sqrt{z_{3n}^{\pm}} - u_{n,3}^{\pm} - \frac{q_0}{2u_{n,3}^{\pm}} + \frac{o(1)}{n^2},$$

we have $\epsilon_{3n}^{\pm} \to 0$ as $n \to \infty$. In a similar way, we have

$$\epsilon_{3n-1}^{\pm} := \sqrt{z_{3n-1}^{\pm}} - u_{n,2}^{\pm} - \frac{q_0}{u_{n,2}^{\pm}} + \frac{o(1)}{n^2} \to 0,$$
$$\epsilon_{3n-2}^{\pm} := \sqrt{z_{3n-2}^{\pm}} - u_{n,1}^{\pm} - \frac{q_0}{u_{n,1}^{\pm}} + \frac{o(1)}{n^2} \to 0$$

as $n \to \infty$.

It follows by Taylor's theorem that

$$\cos\chi(z_{3n}^{\pm}) = \cos(u_{n,3}^{\pm} + \epsilon_{3n}^{\pm}) = 1 - \frac{1}{2}(\epsilon_{3n}^{\pm})^2(1 + \mathcal{O}(\epsilon_{3n}^{\pm})) \tag{3.7}$$

as $n \to \infty$. On the other hand, we claim that

$$\cos\chi(z_{3n}^{\pm}) = 1 + o\Big(\frac{1}{n^2}\Big) \tag{3.8}$$

as $n \to \infty$. First, it follows by (2.23) that $\Delta_-^2(\lambda) = o\Big(\frac{1}{\lambda}\Big)$ as $|\lambda| \to \infty$ and hence

$$\sqrt{\frac{\Delta_-^2(\lambda) + 7}{27}} - \sqrt{\frac{7}{27}} = \frac{\Delta_-^2(\lambda)}{\sqrt{27}(\sqrt{\Delta_-^2(\lambda) + 7} + \sqrt{7})} = o\Big(\frac{1}{\lambda}\Big)$$

as $|\lambda| \to \infty$. Thus, we see that

$$\sqrt{\frac{\Delta_-^2(z_{3n}^{\pm}) + 7}{27}} = \sqrt{\frac{7}{27}} + o\Big(\frac{1}{n^2}\Big) \tag{3.9}$$

as $n \to \infty$. We put

$$\alpha(\lambda) = \arccos\frac{9\sqrt{3}}{(\Delta_-^2(\lambda) + 7)^{3/2}}, \quad \alpha_0 = \arccos\frac{9\sqrt{3}}{7\sqrt{7}} \quad \text{and} \quad \rho(\lambda) = \frac{1}{3}(\alpha(\lambda) - \alpha_0).$$

Then, we have

$$\cos\frac{\alpha(\lambda)}{3} = \cos\Big(\frac{\alpha_0}{3} + \rho(\lambda)\Big) = \cos\frac{\alpha_0}{3} - \Big(\sin\frac{\alpha_0}{3}\Big)\rho(\lambda)(1 + \mathcal{O}(\rho(\lambda))) \tag{3.10}$$

Since it follows by (2.23) that

$$\begin{aligned}
&\frac{1}{(\Delta_-^2(\lambda)+7)^{3/2}} - \frac{1}{7\sqrt{7}} \\
&= \left(\frac{1}{\sqrt{\Delta_-^2(\lambda)+7}} - \frac{1}{\sqrt{7}}\right)\left(\frac{1}{\Delta_-^2(\lambda)+7} + \frac{1}{\sqrt{\Delta_-^2(\lambda)+7}} + \frac{1}{7}\right) \\
&= \frac{\sqrt{7} - \sqrt{\Delta_-^2(\lambda)+7}}{\sqrt{7}\sqrt{\Delta_-^2(\lambda)+7}} \times \mathcal{O}(1) = \frac{\Delta_-^2(\lambda)}{\sqrt{7} + \sqrt{\Delta_-^2(\lambda)+7}} \times \mathcal{O}(1) = o\Big(\frac{1}{\lambda}\Big)
\end{aligned} \tag{3.11}$$

as $|\lambda| \to \infty$, we have

$$\alpha(\lambda) = \arccos\Big(a_0 + o\Big(\frac{1}{\lambda}\Big)\Big) \quad \text{as} \quad |\lambda| \to \infty,$$

where $a_0 = \frac{9\sqrt{3}}{7\sqrt{7}}$. Utilizing Taylor's theorem, there exists some $\theta \in (0,1)$ such that

$$\arccos(a+x) = \arccos a_0 - \frac{1}{1-a_0^2}x - \frac{\theta x + a}{\{1-(\theta x + a)^2\}^{3/2}}x^2.$$

So, we see that

$$\alpha(\lambda) = \arccos a_0 + o\Big(\frac{1}{\lambda}\Big) = \alpha_0 + o\Big(\frac{1}{\lambda}\Big)$$

and hence $\rho(\lambda) = o\Big(\frac{1}{\lambda}\Big)$ as $|\lambda| \to \infty$. Substituting this for (3.10), we see that $\cos\frac{\alpha(\lambda)}{3} = \cos\frac{\alpha_0}{3} + o(\frac{1}{\lambda})$ as $|\lambda| \to \infty$ and hence

$$\cos\frac{\alpha(z_{3n}^{\pm})}{3} = \sqrt{\frac{27}{28}} + o\Big(\frac{1}{n^2}\Big)$$

as $n \to \infty$. This combined with (3.9) and $\cos\chi(z_{3n}^{\pm}) = \Delta_1(z_{3n}^{\pm})$ implies (3.8). Comparing (3.7) and (3.8), we see that $\epsilon_{3n}^{\pm} = o(\frac{1}{n})$ as $n \to \infty$ and conclude that (3.1) is valid.

Next, we show (3.2). Since $\cos u_{n,2}^{-} = -\frac{2}{3}$, $\cos u_{n,2}^{+} = -\frac{1}{3}$, $\sin u_{n,2}^{-} = -\frac{\sqrt{5}}{3}$ and $\sin u_{n,2}^{+} = -\frac{2\sqrt{2}}{3}$, it follows by using Taylor's theorem for $\cos\chi(z_{3n-1}^{\pm}) = \cos(u_{n,2}^{\pm} + \epsilon_{3n-1}^{\pm})$ that

$$\cos\chi(z_{3n-1}^{+}) = -\frac{1}{3} + \frac{2\sqrt{2}}{3}\epsilon_{3n-1}^{+} + \frac{1}{6}(\epsilon_{3n-1}^{+})^2(1 + \mathcal{O}(\epsilon_{3n-1}^{+})), \tag{3.12}$$

$$\cos\chi(z_{3n-1}^{-}) = -\frac{2}{3} + \frac{\sqrt{5}}{3}\epsilon_{3n-1}^{-} + \frac{1}{6}(\epsilon_{3n-1}^{+})^2(1 + \mathcal{O}(\epsilon_{3n-1}^{-})). \tag{3.13}$$

Putting $\alpha_1 = \frac{\alpha_0}{3} + \frac{2}{3}\pi$, we have

$$\begin{aligned}\cos\Big(\frac{1}{3}\alpha(\lambda) + \frac{2\pi}{3}\Big) &= \cos(\alpha_1 + \rho(\lambda))\\ &= \cos\alpha_1 - (\sin\alpha_1)\rho(\lambda))(1 + \mathcal{O}(\rho(\lambda))\\ &= -\sqrt{\frac{3}{7}} - \frac{1}{7}\rho(\lambda)(1 + \mathcal{O}(\rho(\lambda)).\end{aligned}$$

Since we get

$$\sqrt{\frac{\Delta_-^2(z_{3n-1}^{\pm} + 7)}{27}} = \sqrt{\frac{7}{27}} + o\Big(\frac{1}{n^2}\Big)$$

in a similar way to (3.9), it turns out by $\cos\chi(z_{3n-1}^-) = \Delta_2(z_{3n-1}^-)$ that

$$\begin{aligned}\cos\chi(z_{3n-1}^-) &= \Big(\sqrt{\frac{28}{27}} + o\Big(\frac{1}{(u_{n,2}^-)^2}\Big)\Big)\Big(-\sqrt{\frac{3}{7}} - \frac{1}{7}\rho(\lambda)(1 + \mathcal{O}(\rho(z_{3n-1}^-))\Big)\\ &= -\frac{2}{3} - \frac{1}{7}\sqrt{\frac{28}{27}}\rho(z_{3n-1}^-)(1 + \mathcal{O}(\rho(z_{3n-1}^-))) + o\Big(\frac{1}{n^2}\Big)\end{aligned}$$

as $n \to \infty$. Since $\rho(z_{3n-1}^-) = o(\frac{1}{n^2})$, we have $\cos\chi(z_{3n-1}^-) = -\frac{2}{3} + o(\frac{1}{n^2})$ as $n \to \infty$. Similarly, we also obtain $\cos\chi(z_{3n-1}^+) = -\frac{1}{3} + o(\frac{1}{n^2})$ as $n \to \infty$. These combined with (3.12) and (3.13) implies that (3.2) and (3.3) hold true.

In a similar way, we obtain the statements of (ii). □

This lemma is a preparation for Theorem 1.3 (iii). We next make a preparation for Theorem 1.3 (i) and (ii). For this purpose, we define the monodromy matrix.

Definition 3.2. Let $\Theta(x,\lambda) = \{\Theta_\alpha(x,\lambda)\}_{\alpha\in\mathcal{Z}}$ and $\Phi(x,\lambda) = \{\Phi_\alpha(x,\lambda)\}_{\alpha\in\mathcal{Z}}$ be the solutions to the equations

$$-f_\alpha''(x,\lambda) + q(x)f_\alpha(x,\lambda) = \lambda f_\alpha(x,\lambda), \quad \alpha\in\mathcal{Z}, \tag{3.14}$$
$$f_{n,1}(1) = f_{n,2}(1) = f_{n,3}(0), \tag{3.15}$$
$$-f_{n,1}'(1) - f_{n,2}'(1) + f_{n,3}'(0) = 0, \tag{3.16}$$
$$f_{n,3}(1) = f_{n,4}(0) = f_{n,5}(0), \tag{3.17}$$
$$-f_{n,3}'(1) + f_{n,4}'(0) + f_{n,5}'(0) = 0, \tag{3.18}$$
$$f_{n,4}(1) = f_{n,5}(1) = f_{n+1,1}(0) = f_{n+1,2}(0), \tag{3.19}$$
$$-f_{n,4}'(1) - f_{n,5}'(1) + f_{n+1,1}'(0) + f_{n+1,2}'(0) = 0 \tag{3.20}$$

for $n \in \mathbb{Z}$ subject to the initial conditions

$$\Theta_{0,1}(0,\lambda) = 1, \quad \Theta_{0,1}'(0,\lambda) = 0,$$

and

$$\Phi_{0,1}(0,\lambda) = 0, \quad \Phi_{0,1}'(0,\lambda) = 1,$$

respectively. Then, we define the monodromy matrix with respect to H as follows:

$$\mathcal{M}(\lambda) = \begin{pmatrix} \Theta_{1,0}(0,\lambda) & \Phi_{1,0}(0,\lambda) \\ \Theta'_{1,0}(0,\lambda) & \Phi'_{1,0}(0,\lambda) \end{pmatrix}.$$

We recall the monodromy matrix

$$M(\lambda) = \begin{pmatrix} \theta(1,\lambda) & \varphi(1,\lambda) \\ \theta'(1,\lambda) & \varphi'(1,\lambda) \end{pmatrix}$$

for H_0. The components of the monodromy matrix are given by the fundamental solutions as follows, where $\theta_1 = \theta(1,\lambda)$, $\theta'_1 = \theta'(1,\lambda)$, $\varphi_1 = \varphi(1,\lambda)$ and $\varphi'_1 = \varphi'(1,\lambda)$.

Lemma 3.3. *We have:*

$$\begin{aligned}
\Theta_{1,1}(0) &= \frac{1}{2}(\varphi'_1 + 2\theta_1)\{(\theta_1 + 2\varphi')\theta_1 - 2\} - \frac{\theta_1}{2}, \\
\Theta'_{1,1}(0) &= \frac{\varphi'_1}{2\varphi_1}(\varphi'_1 + 2\theta_1)\{(\theta_1 + 2\varphi'_1)\theta_1 - 2\} - \frac{\theta_1\varphi'_1}{2\varphi_1} - \frac{(\theta_1 + 2\varphi'_1)\theta_1 - 2}{\varphi_1}, \\
\Phi_{1,1}(0) &= \frac{\varphi_1}{2}(\varphi'_1 + 2\theta_1)(\theta_1 + 2\varphi'_1) - \frac{\varphi_1}{2}, \\
\Phi'_{1,1}(0) &= \frac{\varphi'_1}{2}(\varphi'_1 + 2\theta_1)(\theta_1 + 2\varphi'_1) - \frac{\varphi'_1}{2} - (\theta_1 + 2\varphi'_1)
\end{aligned}$$

and

$$\mathcal{M}(\lambda) = \mathcal{R}^{-1}(\lambda)T(\lambda)\mathcal{R}(\lambda)M(\lambda),$$

where

$$\mathcal{R}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi(1,\lambda) \end{pmatrix}, \qquad T(\lambda) = \begin{pmatrix} T_{11}(\lambda) & T_{12}(\lambda) \\ T_{21}(\lambda) & T_{22}(\lambda) \end{pmatrix},$$

and

$$\begin{aligned}
T_{11}(\lambda) &= \frac{(\theta_1 + 2\varphi'_1)(\varphi'_1 + 2\theta_1)}{2} - \frac{1}{2} - \varphi'_1(\varphi'_1 + 2\theta_1), \\
T_{12}(\lambda) &= \varphi'_1 + 2\theta_1, \\
T_{21}(\lambda) &= \frac{\varphi'_1}{2}(\varphi'_1 + 2\theta_1)(\theta_1 + 2\varphi'_1) + \frac{3}{2}\varphi'_1 - \varphi'^2_1(\varphi'_1 + 2\theta_1) - (\theta_1 + 2\varphi'_1), \\
T_{22}(\lambda) &= \varphi'_1(\varphi'_1 + 2\theta_1) - 2.
\end{aligned}$$

Proof. Since any solution to $-f'' + qf = \lambda f$ is given by

$$f(x,\lambda) = f(0,\lambda)\theta(x,\lambda) + \frac{f(1,\lambda) - \theta(1,\lambda)f(0,\lambda)}{\varphi(1,\lambda)}\varphi(x,\lambda)$$

for $x \in (0,1)$, we obtain

$$f'_\alpha(0,\lambda) = \frac{f_\alpha(1,\lambda) - \theta(1,\lambda)f_\alpha(0,\lambda)}{\varphi(1,\lambda)}$$

and

$$f'_\alpha(1,\lambda) = \frac{\varphi'(1,\lambda)f_\alpha(1,\lambda) - f_\alpha(0,\lambda)}{\varphi(1,\lambda)}$$

for $\alpha \in \mathcal{Z}$. Substituting these for (3.16), (3.18) and (3.20) and using (3.15), (3.17) and (3.19), we obtain the following:

$$f_{0,3}(1) - (\theta_1 + 2\varphi'_1)f_{0,1}(1) + 2f_{0,1}(0) = 0, \tag{3.21}$$

$$-(\varphi'_1 + 2\theta_1)f_{0,3}(1) + f_{0,1}(1) + 2f_{1,1}(0) = 0, \tag{3.22}$$

$$f'_{1,1}(0) = \frac{\varphi'_1}{\varphi_1}f_{1,1}(0) - \frac{1}{\varphi_1}f_{0,3}(1). \tag{3.23}$$

Substituting $f = \Theta$ for (3.21) and using $\Theta_{0,1}(x,\lambda) = \theta(x,\lambda)$, we obtain

$$\Theta_{0,3}(1) = (\theta_1 + 2\varphi'_1)\theta_1 - 2. \tag{3.24}$$

Using this after substituting $f = \Theta$ for (3.22), we have

$$\Theta_{1,1}(0) = \frac{1}{2}(\varphi'_1 + 2\theta_1)\{(\theta_1 + 2\varphi'_1)\theta_1 - 2\} - \frac{\theta_1}{2}.$$

Using this and (3.24) after substituting $f = \Theta$ for (3.23), we obtain

$$\Theta'_{1,1}(0) = \frac{\varphi'_1}{2\varphi_1}(\varphi'_1 + 2\theta_1)\{(\theta_1 + 2\varphi'_1)\theta_1 - 2\} - \frac{\theta_1\varphi'_1}{2\varphi_1} - \frac{(\theta_1 + 2\varphi'_1)\theta_1 - 2}{\varphi_1}.$$

In a similar way, we see that $\Phi_{1,1}(0)$ and $\Phi'_{1,1}(0)$ are expressed as seen in the statement of this lemma. Moreover, straightforward calculations give us the latter part of the assertion of this lemma. □

Lemma 3.4. *We have*

$$\sigma(H_{ap}) = \{\lambda \in \mathbb{R} \,|\, D(\lambda) = -1\}$$

and

$$\sigma(H_p) = \{\lambda \in \mathbb{R} \,|\, D(\lambda) = 1\}.$$

Proof. It suffices to show that det $(\mathcal{M}(\lambda) \pm I) = 2(1 \pm D(\lambda))$. It follows by Lemma 3.3 and det $M(\lambda) = 1$ that

$$\det \mathcal{M}(\lambda) = \det T(\lambda) = 1.$$

Furthermore, we see that

$$\begin{aligned}
\operatorname{tr} \mathcal{M}(\lambda) &= \frac{\theta_1 + \varphi'_1}{2}(\varphi'_1 + 2\theta_1)(\theta_1 + 2\varphi'_1) - (\varphi'_1 + 2\theta_1) - \frac{\theta_1 + \varphi'_1}{2} - (\theta_1 + 2\varphi'_1)\\
&= \Delta(\lambda)(2\Delta(\lambda) + \theta_1)(2\Delta(\lambda) + \varphi'_1) - \Delta(\lambda) - 6\Delta(\lambda)\\
&= \Delta(\lambda)(8\Delta^2(\lambda) + \theta_1\varphi'_1 - 7) = 2D(\lambda).
\end{aligned}$$

Thus, we obtain det $(\mathcal{M}(\lambda) \pm I) = 2(1 \pm D(\lambda))$. □

Recall that

$$\sigma(H_p) = \{\lambda_{2,0}^+, \lambda_{2,2}^-, \lambda_{2,2}^+, \lambda_{2,4}^-, \lambda_{2,4}^+, \dots\}$$

and

$$\sigma(H_{ap}) = \{\lambda_{2,1}^-, \lambda_{2,1}^+, \lambda_{2,3}^-, \lambda_{2,3}^+, \dots, \}.$$

Proof of Theorem 1.3. Lemma 3.4 and Theorem 1.2 (iv), (v), (vi) implies (i). Theorem 1.1 and Theorem 1.3 (i) implies (ii). Lemma 2.6 implies that (iii). Thus, it suffices to show (iv). It follows by Lemma 3.4 and Theorem 1.2 (vi) that $\lambda_{2,0}^+ = z_0^+$, $\lambda_{2,2n-1}^\pm = x_n^\pm$, $\lambda_{2,2n}^\pm = z_n^\pm$. So, we see that $\lambda_{2,6n-3}^\pm = x_{3n-1}^\pm$, $\lambda_{2,6n-1}^\pm = x_{3n}^\pm$, $\lambda_{2,6n-1}^\pm = x_{3n}^\pm$, $\lambda_{2,6n+1}^\pm = x_{3n+1}^\pm$, $\lambda_{2,6n}^\pm = z_{3n}^\pm$, $\lambda_{2,6n-2}^\pm = z_{3n-1}^\pm$, $\lambda_{2,6n-4}^\pm = z_{3n-2}^\pm$. These combined with Lemma 3.1 imply that

$$\lambda_{2,6n-5}^\pm = (v_{n-1,1}^\pm)^2 + q_0 + o\left(\frac{1}{n}\right), \tag{3.25}$$

$$\lambda_{2,6n-4}^\pm = (u_{n,1}^\pm)^2 + q_0 + o\left(\frac{1}{n}\right), \tag{3.26}$$

$$\lambda_{2,6n-3}^\pm = (v_{n,2}^\pm)^2 + q_0 + o(1), \tag{3.27}$$

$$\lambda_{2,6n-2}^\pm = (u_{n,2}^\pm)^2 + q_0 + o\left(\frac{1}{n}\right), \tag{3.28}$$

$$\lambda_{2,6n-1}^\pm = (v_{n,3}^\pm)^2 + q_0 + o\left(\frac{1}{n}\right), \tag{3.29}$$

$$\lambda_{2,6n}^\pm = (u_{n,3}^\pm)^2 + q_0 + o(1) \tag{3.30}$$

as $n \to \infty$. So, we obtain (1.3), (1.4), (1.6), (1.7). Thus, our final work is to show (1.5) and (1.8).

Let us prove (1.8). Since $\sqrt{\lambda_{2,6n}^\pm} = 2n\pi + \mathcal{O}(\frac{1}{n})$ as $n \to \infty$, we consider λ such that $\sqrt{\lambda} = 2n\pi + \mathcal{O}(\frac{1}{n})$ as $n \to \infty$. We see that

$$\frac{\partial S}{\partial \lambda}(\lambda) = \frac{1}{2\sqrt{\lambda}} \int_0^1 (1-2t)q(t)\cos\sqrt{\lambda}(1-2t)dt$$

and

$$\frac{\partial^2 S}{\partial \lambda^2}(\lambda) = -\frac{1}{2\lambda}\frac{\partial S}{\partial \lambda}(\lambda) - \frac{1}{4\lambda}\int_0^1 (1-2t)^2 q(t)\sin\sqrt{\lambda}(1-2t)dt.$$

Then, it follows by

$$\sin\sqrt{\lambda}(1-2t) = -\sin 4t\pi t + \mathcal{O}\Big(\frac{1}{n}\Big)$$

and

$$\cos\sqrt{\lambda}(1-2t) = \cos 4n\pi t + \mathcal{O}\Big(\frac{1}{n}\Big)$$

for $t \in (0,1)$ as $\sqrt{\lambda} = 2n\pi + \mathcal{O}(\frac{1}{n})$ that

$$S(\lambda) = -\Big(\hat{q}_{s,2n} + \mathcal{O}\Big(\frac{1}{n}\Big)\Big), \quad \frac{\partial S}{\partial \lambda}(\lambda) = \frac{\tilde{q}_{c,2n} + \mathcal{O}(\frac{1}{n})}{4n\pi}, \quad \frac{\partial^2 S}{\partial \lambda^2}(\lambda) = \frac{\tilde{\tilde{q}}_{s,2n} + \mathcal{O}(\frac{1}{n})}{(4n\pi)^2}$$

as $\sqrt{\lambda} = 2n\pi + \mathcal{O}(\frac{1}{n})$, where

$$\tilde{q}_{c,n} = \int_0^1 (1-2t)q(t)\cos 2n\pi t dt$$

and

$$\tilde{\tilde{q}}_{s,n} = \int_0^1 (1-2t)^2 q(t)\sin 2n\pi t dt$$

for $n \in \mathbb{N}$. These combined with (2.23) imply that

$$\begin{aligned} \Delta_-(\lambda) &= \frac{\hat{q}_{s,2n} + \mathcal{O}(\frac{1}{n})}{4n\pi}, \qquad \dot{\Delta}_-(\lambda) = -\frac{\tilde{q}_{c,2n} + \mathcal{O}(\frac{1}{n})}{(4n\pi)^2}, \\ \ddot{\Delta}_-(\lambda) &= -\frac{\tilde{\tilde{q}}_{s,2n} + \mathcal{O}(\frac{1}{n})}{(4n\pi)^3} \end{aligned} \tag{3.31}$$

as $\sqrt{\lambda} = 2n\pi + \mathcal{O}(\frac{1}{n})$, where $\dot{f}(\lambda)$ implies that the derivative of f with respect to λ. Furthermore, we obtain

$$\begin{aligned} \Delta(\lambda) &= 1 + \mathcal{O}(\frac{1}{n^2}), \qquad \dot{\Delta}(\lambda) = \mathcal{O}(\frac{1}{n^2}), \\ \ddot{\Delta}(\lambda) &= -\frac{1 + \mathcal{O}(\frac{1}{n})}{(4n\pi)^2}, \quad \dddot{\Delta}(\lambda) = \mathcal{O}(\frac{1}{n^4}) \end{aligned} \tag{3.32}$$

as $\sqrt{\lambda} = 2n\pi + \mathcal{O}(\frac{1}{n})$.

Since

$$\theta(1,\lambda)\varphi'(1,\lambda) = \theta'(1,\lambda)\varphi(1,\lambda) + 1 = \Delta^2(\lambda) - \Delta_-^2(\lambda),$$

we see that

$$D(\lambda) = \frac{9}{2}\Delta^3(\lambda) - \frac{\Delta(\lambda)\Delta_-^2(\lambda)}{2} - \frac{7}{2}\Delta(\lambda). \tag{3.33}$$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be the zeroes of $\dot{\Delta}(\lambda)$ such that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Then, we have

$$\dot{D}(\lambda_{2n}) = -\Delta(\lambda_{2n})\Delta_-(\lambda_{2n})\dot{\Delta}_-(\lambda_{2n}). \tag{3.34}$$

Since $\sqrt{\lambda_n} = n\pi + \mathcal{O}(\frac{1}{n})$ as $n \to \infty$, it follows by substituting (3.31) and (3.32) that

$$\begin{aligned} \dot{D}(\lambda_{2n}) &= -\Big(1 + \mathcal{O}\Big(\frac{1}{n^2}\Big)\Big)\left(\frac{\hat{q}_{s,2n} + \mathcal{O}(\frac{1}{n})}{4n\pi}\right)\left(-\frac{\tilde{q}_{c,2n} + \mathcal{O}(\frac{1}{n})}{(4n\pi)^2}\right) \\ &= \frac{\hat{q}_{s,2n} + \mathcal{O}(\frac{1}{n})}{4n\pi} \cdot \frac{\tilde{q}_{c,2n} + \mathcal{O}(\frac{1}{n})}{(4n\pi)^2} + \mathcal{O}\Big(\frac{1}{n^5}\Big) \end{aligned} \tag{3.35}$$

as $n \to \infty$.

On the other hand, it follows by Taylor's theorem that

$$\dot{D}(\lambda_{2n}) = \ddot{D}(\lambda_{2,6n})s_{2n}(1 + \mathcal{O}(s_n)), \tag{3.36}$$

where $\lambda_{2,6n} \in [\lambda_{2,6n}^-, \lambda_{2,6n}^+]$ be the value satisfying $\dot{D}(\lambda_{2,6n}) = 0$ and $s_{2n} = \lambda_{2n} - \lambda_{2,6n} \to 0$ as $n \to \infty$ because of $\sqrt{\lambda_n} = n\pi + \mathcal{O}(\frac{1}{n})$ as $n \to \infty$. By virtue of

$$\begin{aligned}\ddot{D}(\lambda) = \frac{27}{2}\ddot{\Delta}(\lambda)\Delta^2(\lambda) + 27(\dot{\Delta}(\lambda))^2\Delta(\lambda) - \frac{\ddot{\Delta}(\lambda)\Delta_-^2(\lambda)}{2} \\ - 2\dot{\Delta}(\lambda)\Delta_-(\lambda)\dot{\Delta}_-(\lambda) - \Delta(\lambda)(\dot{\Delta}_-(\lambda))^2 - \Delta(\lambda)\Delta_-(\lambda)\ddot{\Delta}_-(\lambda) - \frac{7}{2}\ddot{\Delta}(\lambda),\end{aligned}$$

we obtain

$$\ddot{D}(\lambda_{2,6n}) = -\frac{10}{(4n\pi)^2} + \mathcal{O}\Big(\frac{1}{n^3}\Big)$$

as $n \to \infty$ by using (3.31) and (3.32). This combined with (3.34) and (3.35) means that

$$s_{2n} = -\frac{(\hat{q}_{s,2n} + \mathcal{O}(\frac{1}{n}))(\tilde{q}_{c,2n} + \mathcal{O}(\frac{1}{n}))}{40n\pi}$$

as $n \to \infty$.

Let $\epsilon_{2,6n}^{\pm} = \lambda_{2,6n}^{\pm} - \lambda_{2n}$ and $\epsilon_{2n}^{\pm} = \lambda_{2n}^{\pm} - \lambda_{2n}$. Then, we have $\epsilon_{2,6n}^{\pm} \to 0$ as $n \to \infty$ because of (3.30) and $\lambda_{2n} = 4n^2\pi^2 + q_0 + \ell^2(n)$. On the other hand, we see that and $\epsilon_{2n}^{\pm} \to 0$ as $n \to \infty$, because $\lambda_n = (n\pi)^2 + q_0 + \mathcal{O}(\frac{1}{n})$ and $\lambda_n^{\pm} = n^2\pi^2 + q_0 \pm |\hat{q}_n| + \mathcal{O}(\frac{1}{n})$ as $n \to \infty$, which is quoted from [3] and [6]. It turns out by (3.32) and Taylor's theorem that

$$\begin{aligned}\Delta(\lambda_{2,6n}^{\pm}) &= \Delta(\lambda_{2n}) + \dot{\Delta}(\lambda_{2n})\epsilon_{2,6n}^{\pm} + \frac{\ddot{\Delta}(\lambda_{2n})}{2}(\epsilon_{2,6n}^{\pm})^2 + \frac{\dddot{\Delta}(\lambda_{2n})}{6}(\epsilon_{2,6n}^{\pm})^3(1 + \mathcal{O}(\epsilon_{2,6n}^{\pm})) \\ &= \Delta(\lambda_{2n}) + A_{2,2n}^{\pm},\end{aligned}$$

where

$$A_{2,2n}^{\pm} = \frac{\ddot{\Delta}(\lambda_{2n})}{2}(\epsilon_{2,6n}^{\pm})^2\Big(1 + \mathcal{O}\Big(\frac{\epsilon_{2,6n}^{\pm}}{n^2}\Big)\Big)$$

as $n \to \infty$. Furthermore, it follows by Taylor's theorem that

$$1 = \Delta(\lambda_{2n}^{\pm}) = \Delta(\lambda_{2n}) + A_{2n}^{\pm},$$

where

$$A_{2n}^{\pm} = \frac{\ddot{\Delta}(\lambda_{2n})}{2}(\epsilon_{2n}^{\pm})^2\Big(1 + \mathcal{O}\Big(\frac{\epsilon_{2n}^{\pm}}{n^2}\Big)\Big)$$

as $n \to \infty$. Thus, we see that

$$\Delta(\lambda_{2,6n}^{\pm}) = 1 + A_{0,2n}^{\pm} - A_{2n}^{\pm}. \tag{3.37}$$

It follows by (3.33) that

$$\Delta^3(\lambda_{2,6n}^{\pm}) = \frac{1}{9}(\Delta_-^2(\lambda_{2,6n}^{\pm}) + 7)\Delta(\lambda_{0,6n}^{\pm}) + \frac{2}{9}.$$

Inserting the first equality of (3.31) and (3.37), we obtain

$$(1+A^{\pm}_{2,2n}-A^{\pm}_{2n})^3=\frac{1}{9}\left\{\left(\frac{\hat{q}_{s,2n}+\mathcal{O}(\frac{1}{n})}{4n\pi}\right)^2+7\right\}(1+A^{\pm}_{2,2n}-A^{\pm}_{2n})+\frac{2}{9} \tag{3.38}$$

as $n\to\infty$. It follows by (3.32) that

$$A^{\pm}_{2,2n}=-\frac{1+\mathcal{O}(\frac{1}{n})}{2(4n\pi)^2}(\epsilon^{\pm}_{2,6n})^2\Big(1+\mathcal{O}\Big(\frac{\epsilon^{\pm}_{2,6n}}{n^2}\Big)\Big)$$

and

$$A^{\pm}_{2n}=-\frac{1+\mathcal{O}(\frac{1}{n})}{2(4n\pi)^2}(\epsilon^{\pm}_{2n})^2\Big(1+\mathcal{O}\Big(\frac{\epsilon^{\pm}_{2n}}{n^2}\Big)\Big)$$

as $n\to\infty$. Thus, we see that

$$A^{\pm}_{2,2n}-A^{\pm}_{2n}=B^{\pm}_{2n}+o\Big(\frac{1}{n^3}\Big)$$

as $n\to\infty$, where

$$B^{\pm}_{2n}=\frac{(\epsilon^{\pm}_{2n})^2-(\epsilon^{\pm}_{2,6n})^2}{2(4n\pi)^2}.$$

Since $\epsilon^{\pm}_{2n}\to 0$ and $\epsilon^{\pm}_{2,6n}\to 0$ as $n\to\infty$, we see that $A^{\pm}_{2,2n}-A^{\pm}_{2n}=o(\frac{1}{n^2})$ as $n\to\infty$. This combined with (3.38) means that

$$1+3B^{\pm}_{2n}+o\Big(\frac{1}{n^3}\Big)=\frac{1}{9}\left(\frac{\hat{q}_{s,2n}+\mathcal{O}(\frac{1}{n})}{4n\pi}\right)^2+1+\frac{7}{9}B^{\pm}_{2n}+o\Big(\frac{1}{n^3}\Big)$$

as $n\to\infty$. Thus, we obtain

$$\left(\frac{\hat{q}_{s,2n}+\mathcal{O}(\frac{1}{n})}{4n\pi}\right)^2=20B^{\pm}_{2n}+o\Big(\frac{1}{n^3}\Big)$$

and hence

$$(\hat{q}_{s,2n}+\mathcal{O}(\frac{1}{n}))^2=10\{(\epsilon^{\pm}_{2n})^2-(\epsilon^{\pm}_{2,6n})^2\}+o\Big(\frac{1}{n}\Big)$$

as $n\to\infty$. Since $\hat{q}_{s,n}=\text{Im }\hat{q}_n$ and $\hat{q}_n=\int_0^1 q(t)e^{2n\pi it}dt$, we have $|\hat{q}_{s,n}|=\mathcal{O}(\hat{q}_n)$ as $n\to\infty$. So, we have

$$(\hat{q}_{s,2n})^2+\frac{\mathcal{O}(|\hat{q}_{2n}|)}{n}=10\{(\epsilon^{\pm}_{2n})^2-(\epsilon^{\pm}_{2,6n})^2\}+o\Big(\frac{1}{n}\Big) \tag{3.39}$$

as $n\to\infty$. We refer $\lambda^{\pm}_n=n^2\pi^2+q_0\pm|\hat{q}_n|+\mathcal{O}(\frac{1}{n})$ from [11], and $\lambda_n=n^2\pi^2+q_0+\mathcal{O}(\frac{1}{n})$ from [6]. These imply that $\epsilon^{\pm}_{2n}=\pm|\hat{q}_{2n}|+\mathcal{O}\Big(\frac{1}{n}\Big)$ as $n\to\infty$. Substituting this for (3.39), we see that

$$(\epsilon^{\pm}_{2,6n})^2=|\hat{q}_{2n}|^2-\frac{(\hat{q}_{s,2n})^2}{10}+\mathcal{O}\Big(\frac{1}{n}\Big),$$

and hence

$$\epsilon_{2,6n}^{\pm} = \pm\sqrt{|\hat{q}_{2n}|^2 - \frac{(\hat{q}_{s,2n})^2}{10}} + \mathcal{O}\Big(\frac{1}{n}\Big)$$

as $n \to \infty$. This is why we get (1.8). The proof of (1.5) is similar to the one of (1.8). □

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Hiroaki Niikuni
niikuni@maebashi-it.ac.jp

Maebashi Institute of Technology
460-1 Kamisadori, Maebashi City
Gunma, 371-0816, Japan

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