

## A NEW CHARACTERIZATION OF CONVEX $\varphi$ -FUNCTIONS WITH A PARAMETER

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**Abstract.** We show that, under some additional assumptions, all projection operators onto latticially closed subsets of the Orlicz-Musielak space generated by  $\Phi$  are isotonic if and only if  $\Phi$  is convex with respect to its second variable. A dual result of this type is also proven for antiprojections. This gives the positive answer to the problem presented in Opuscula Mathematica in 2012.

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### 1. INTRODUCTION

The isotonicity of projection operators onto closed (or latticially closed) subsets of ordered metric spaces has been investigated since 1986, mainly in Hilbert spaces and Banach spaces (for details and possible applications see [1, 3–5] and the references given there), and in modular spaces [2, 9]. Moreover, antiprojection operators have also been examined from this point of view in [6, 7, 9].

It turns out that the monotonic properties of projections and antiprojections in various function spaces can be determined by two special functional inequalities, named “the properties of four elements”, which were defined in [1, 6], and further investigated in [2, 5, 7, 9]. In particular, it has been shown that, under some additional assumption denoted in [9] as (\*), both these inequalities are valid in an Orlicz-Musielak space if and only if this space is generated by a convex  $\varphi$ -function [9, Theorem 3.5].

In this note we prove a new, even stronger theorem stating that, under some new assumption on the  $\varphi$ -function  $\Phi$ , all projections onto latticially closed subsets of the Orlicz-Musielak space  $L^\Phi$  are isotonic (and all antiprojections onto such sets are antiisotonic) if and only if  $\Phi$  is convex with respect to its second variable (Theorem 3.4,

see also Remark 3.5 and Theorem 3.6). This strengthens the results published in [2, 7], and gives the affirmative answer to the question presented as Problem 3.8 in [9].

## 2. PRELIMINARIES

In this section we review some of the standard facts on Orlicz-Musielak spaces. In particular, we set up notation and terminology used in the following part of the text. For a more detailed summary of the modular spaces theory we refer the reader to [10, 11].

Let us first recall the notion of an Orlicz-Musielak space.

**Definition 2.1.** Let  $(\Omega, \Sigma, \mu)$  be a space with nonzero measure. Then  $\Phi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a  $\varphi$ -function with a parameter if it satisfies the following properties:

1. for every  $t \in \Omega$ ,  $\varphi_t(\cdot) = \Phi(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nondecreasing, continuous function such that  $\varphi_t(0) = 0$  and  $\varphi_t(x) > 0$  for  $x > 0$ ,
2. for every  $x \in \mathbb{R}_+$ ,  $\Phi(\cdot, x) : \Omega \rightarrow \mathbb{R}_+$  is a  $\Sigma$ -measurable function.

**Definition 2.2.** Let  $\Phi$  be a  $\varphi$ -function with a parameter and denote by  $M(\Omega; \mathbb{R})$  the set of all real  $\Sigma$ -measurable functions defined on  $\Omega$ , with equality  $\mu$ -almost everywhere. For  $f \in M(\Omega; \mathbb{R})$  define

$$\rho_\Phi(f) = \int_{\Omega} \Phi(t, |f(t)|) d\mu(t).$$

Then  $\rho_\Phi$  is the *Orlicz-Musielak modular* generated by  $\Phi$ . The corresponding modular space

$$L^\Phi = \left\{ f \in M(\Omega; \mathbb{R}) : \lim_{\alpha \rightarrow 0} \rho_\Phi(\alpha f) = 0 \right\}$$

is the *Orlicz-Musielak space*. If the function  $\Phi$  is independent of  $t$ , the modular  $\rho_\Phi$  is said to be the *Orlicz modular*. Every Orlicz-Musielak space can be ordered in the natural way by the cone of nonnegative functions ( $f \geq g$  if and only if  $f - g \geq 0$ ).

**Remark 2.3.** By the Lebesgue dominant convergence theorem, it is easy to show that

$$L^\Phi = \{ f \in M(\Omega; \mathbb{R}) : \rho_\Phi(\alpha f) < \infty \text{ for some } \alpha > 0 \}.$$

The following definition of the projection and the antiprojection onto a subset of an Orlicz-Musielak space agrees with the one widely used in metric space theory.

**Definition 2.4.** Let  $D$  be any subset of  $L^\Phi$  and choose  $f \in L^\Phi$ . Then

$$P_D(f) = \left\{ y \in D : \rho_\Phi(f - y) = \inf_{d \in D} \rho_\Phi(f - d) \right\}$$

and

$$P_D^a(f) = \left\{ z \in D : \rho_\Phi(f - z) = \sup_{d \in D} \rho_\Phi(f - d) \right\}.$$

The sets  $P_D(f)$  and  $P_D^a(f)$  (both of which can be empty) are called the *projection of  $f$  onto  $D$*  and the *antiprojection of  $f$  onto  $D$* , respectively.

**Definition 2.5.** We say that the projection operator  $P_D$  is *isotonic* if, for any  $x, y \in L^\Phi$  such that  $x \leq y$  and both  $P_D(x), P_D(y)$  are non-empty, there exist  $w \in P_D(x)$  and  $v \in P_D(y)$  satisfying  $w \leq v$ . In particular, if  $x \leq y$  and  $P_D(x), P_D(y)$  are both singletons, then  $P_D(x) \leq P_D(y)$ . The antiprojection operator  $P_D^a$  is *antiisotonic* if, for any  $x, y \in L^\Phi$  such that  $x \leq y$  and both  $P_D^a(x), P_D^a(y)$  are non-empty, there exist  $w \in P_D^a(x)$  and  $v \in P_D^a(y)$  with  $w \geq v$ . In the case when  $x \leq y$  and  $P_D^a(x), P_D^a(y)$  are both singletons, we get  $P_D^a(x) \geq P_D^a(y)$ .

**Definition 2.6.** A set  $D \subset L^\Phi$  is called *latticeclosed* if  $\min(x, y) \in D$  and  $\max(x, y) \in D$  for all  $x, y \in D$ .

Various examples of such sets are given in [7, Ex. 2.4] and [8, Ex. 1.4].

### 3. MAIN RESULTS

From the results proved in [2, 7], the following theorem can be derived.

**Theorem 3.1.** *If  $\Phi$  is convex with respect to its second variable, then:*

- (1) *for each latticeclosed set  $D \subset L^\Phi$ , the projection operator  $P_D$  is isotonic,*
- (2) *for each latticeclosed set  $D \subset L^\Phi$ , the antiprojection operator  $P_D^a$  is antiisotonic.*

Moreover, it is known that the analogue of the above theorem does not have to be true if  $\Phi$  is not a convex function of  $x$ , see [9, Ex. 3.7] for the counterexample.

Now we are going to show that the assumption of convexity in Theorem 3.1 is, in fact, the essential one. We will need the following additional (but not too restrictive) assumption.

**Assumption (\*\*).** For any  $y_1, y_2, y_3, y_4 > 0, t \in \Omega$  and  $\varepsilon > 0$ , there exist two disjoint sets  $\Omega_1, \Omega_2 \in \Sigma$  such that:

- (a)  $\mu(\Omega_1) = \mu(\Omega_2) \in (0, \infty)$ ,
- (b)  $|\Phi(w, y_i) - \Phi(t, y_i)| < \varepsilon$  for  $w \in \Omega_1 \cup \Omega_2$  and  $1 \leq i \leq 4$ .

**Remark 3.2.** The condition (\*\*) is satisfied in various important cases, the examples of which are listed below.

1. (the Orlicz case) If  $\Phi(t, x) = \Phi(x)$ , then for all  $y_1, y_2, y_3, y_4, t$  and  $\varepsilon$ , we can choose the same pair of disjoint sets  $\Omega_1, \Omega_2 \in \Sigma$  with equal, positive and finite measure.
2. (the sequence case with some additional assumptions) Let  $\Omega = \mathbb{N}$  with the counting measure and assume that, given  $t \in \mathbb{N}$  and  $M > 0$ , we may find  $t_1 \neq t$  such that

$$|\Phi(t_1, y) - \Phi(t, y)| < \varepsilon \text{ for all } y \in [0, M].$$

Then we put  $\Omega_1 = \{t\}, \Omega_2 = \{t_1\}$ , where  $t_1$  is chosen for  $M = \max_i y_i$ .

3. (the continuous atomless case) Suppose that  $\Omega \subset \mathbb{R}^n$  is an open set with positive Lebesgue measure and that, for each  $x \in \mathbb{R}_+$ , the function  $\Phi(\cdot, x)$  is continuous. Then there exist positive numbers  $\varepsilon_i$  with

$$|\Phi(w, y_i) - \Phi(t, y_i)| < \varepsilon \text{ for } w \in B(t, \varepsilon_i) \text{ and } 1 \leq i \leq 4,$$

where  $B$  denotes the closed ball in  $\mathbb{R}^n$ . In this case, we choose any disjoint sets  $\Omega_1, \Omega_2 \subset B(t, \min_i \varepsilon_i)$  which satisfy (a).

Moreover, without loss of generality we can assume that  $\mu(\Omega_1) = \mu(\Omega_2) = 1$  (replacing the measure  $\mu$  with  $\mu/\mu(\Omega_1)$ , if necessary).

**Remark 3.3.** It is also worth mentioning that our new assumption (\*\*) implies properties (1) and (3) required in (\*) (see [9, p. 172]), with  $A_{t,\varepsilon}^{y_i} = \Omega_1$  (or  $\Omega_2$ ). Additionally, in all of the cases presented in the previous remark the remaining condition (2) is also satisfied.

Our new result, which improves Theorem 3.1, reads as follows.

**Theorem 3.4.** *Let  $\Phi: \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a  $\varphi$ -function with a parameter such that (\*\*) is satisfied. Then the following conditions are equivalent:*

- (i) for each  $t \in \Omega$ ,  $\varphi_t$  is a convex function,
- (ii) for each  $t \in \Omega$ ,  $\varphi_t$  satisfies the Lim inequality: if  $a, b \geq 0$  and  $c \geq a$ , then

$$\varphi_t(a) + \varphi_t(b + c) \geq \varphi_t(a + b) + \varphi_t(c),$$

- (iii) for any latticially closed set  $D \subset L^\Phi$ , the projection operator  $P_D$  is isotonic,
- (iv) for any latticially closed set  $D \subset L^\Phi$ , the antiprojection operator  $P_D^a$  is antiisotonic.

*Proof.* Let us first point out that the equivalence (i)  $\Leftrightarrow$  (ii) has already been observed and used in [9, Theorem 3.5], while (i)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (iv) hold by Theorem 3.1. Consequently, we only need to prove that both (iii) and (iv) imply (ii).

Suppose, on the contrary, that  $\varphi_t$  does not satisfy the Lim inequality for some  $t \in \Omega$ . This gives

$$\varphi_t(a) + \varphi_t(b + c) < \varphi_t(a + b) + \varphi_t(c) - 4\varepsilon \tag{3.1}$$

with appropriately chosen  $a, b, \varepsilon > 0$  and  $c > a$ . By (\*\*) and the following remark, for such  $t, \varepsilon$  and  $y_1 = a, y_2 = b + c, y_3 = a + b, y_4 = c$  we can find the corresponding sets  $\Omega_1, \Omega_2$  of measure 1.

Set  $A = \Omega_1 \cup \Omega_2, d = c/a > 1$  and define four simple functions on  $\Omega$ :

$$\begin{aligned} x &= (a + b + c) \cdot I_{\Omega_2}, & y &= \frac{c(d-1)}{d} \cdot I_{\Omega_1} + (b + 2c) \cdot I_{\Omega_2}, \\ z &= c \cdot I_A, & w &= (b + c) \cdot I_A, \end{aligned}$$

where  $I_D$  denotes the characteristic function of  $D$ . Then it is easy to observe that all these mappings are elements of the Orlicz-Musielak space  $L^\Phi$  and that  $x \leq y, z(t) < w(t)$  for  $t \in A$ .

Moreover, by (\*\*) and (3.1),

$$\begin{aligned} \rho_{\Phi}(x - z) &= \int_{\Omega_1} \Phi(w, c) dw + \int_{\Omega_2} \Phi(w, a + b) dw \\ &\geq \int_{\Omega_1} [\Phi(t, c) - \varepsilon] dw + \int_{\Omega_2} [\Phi(t, a + b) - \varepsilon] dw = \varphi_t(c) + \varphi_t(a + b) - 2\varepsilon \\ &> \varphi_t(b + c) + \varphi_t(a) + 2\varepsilon = \int_{\Omega_1} [\Phi(t, b + c) + \varepsilon] dw + \int_{\Omega_2} [\Phi(t, a) + \varepsilon] dw \\ &\geq \int_{\Omega_1} \Phi(w, b + c) dw + \int_{\Omega_2} \Phi(w, a) dw = \rho_{\Phi}(x - w) \end{aligned}$$

and

$$\begin{aligned} \rho_{\Phi}(y - z) &= \int_{\Omega_1} \Phi(w, a) dw + \int_{\Omega_2} \Phi(w, b + c) dw \\ &\leq \int_{\Omega_1} [\Phi(t, a) + \varepsilon] dw + \int_{\Omega_2} [\Phi(t, b + c) + \varepsilon] dw = \varphi_t(a) + \varphi_t(b + c) + 2\varepsilon \\ &< \varphi_t(a + b) + \varphi_t(c) - 2\varepsilon = \int_{\Omega_1} [\Phi(t, a + b) - \varepsilon] dw + \int_{\Omega_2} [\Phi(t, c) - \varepsilon] dw \\ &\leq \int_{\Omega_1} \Phi(w, a + b) dw + \int_{\Omega_2} \Phi(w, c) dw = \rho_{\Phi}(y - w). \end{aligned}$$

We have shown that, for the latticially closed set  $D = \{z, w\}$ ,

$$P_D(x) = \{w\}, \quad P_D(y) = \{z\}, \quad \text{and so } P_D(x)(t) > P_D(y)(t) \text{ for } t \in A,$$

and

$$P_D^a(x) = \{z\}, \quad P_D^a(y) = \{w\}, \quad \text{which gives } P_D^a(x) < P_D^a(y) \text{ on } A.$$

Therefore, the projection  $P_D$  is not isotonic, while the antiprojection  $P_D^a$  is not antiisotonic. In consequence, neither (iii) nor (iv) can be valid. This completes the proof of our theorem.  $\square$

**Remark 3.5.** We have actually proved a little more, namely: If there exist  $t \in \Omega$  with  $\varphi_t$  being non-convex, and  $\Omega_1, \Omega_2 \in \Sigma$  satisfying (a) and such that, for  $w \in \Omega_1 \cup \Omega_2$ ,  $\varphi_w$  is “similar to  $\varphi_t$ ” in the sense of (b), then (iii) and (iv) cannot hold.

Similar considerations lead us to the following additional result for the sequence case.

**Theorem 3.6.** Let  $\Phi: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a  $\varphi$ -function with a parameter and assume that there exist natural numbers  $t_1 \neq t_2$  such that the system of inequalities

$$\begin{cases} \varphi_{t_2}(a+b) - \varphi_{t_2}(a) > \varphi_{t_1}(b+c) - \varphi_{t_1}(c) \\ \varphi_{t_1}(a+b) - \varphi_{t_1}(a) > \varphi_{t_2}(b+c) - \varphi_{t_2}(c) \end{cases} \quad (3.2)$$

has a solution  $(a, b, c)$  with  $a, b, c > 0$ ,  $c > a$ . Then the conditions (iii) and (iv) from Theorem 3.4 are not satisfied.

*Proof.* Set  $\Omega_1 = \{t_1\}$ ,  $\Omega_2 = \{t_2\}$ ,  $A = \Omega_1 \cup \Omega_2$  and define simple functions  $x, y, z, w \in L^\Phi$  as in the proof of Theorem 3.4. Then, by (3.2),

$$\rho_\Phi(x-z) = \varphi_{t_1}(c) + \varphi_{t_2}(a+b) > \varphi_{t_1}(b+c) + \varphi_{t_2}(a) = \rho_\Phi(x-w)$$

and

$$\rho_\Phi(y-z) = \varphi_{t_1}(a) + \varphi_{t_2}(b+c) < \varphi_{t_1}(a+b) + \varphi_{t_2}(c) = \rho_\Phi(y-w).$$

Putting  $D = \{z, w\}$  and reasoning as above, we show that the operators  $P_D$  and  $P_D^a$  do not possess the required properties, which proves the theorem.  $\square$

**Remark 3.7.** If the system (3.2) has the desired solution, then the function  $\varphi_{t_1} + \varphi_{t_2}$  cannot be convex, and so at least one of its components is also a non-convex function.

Our final example shows that the assumptions of Theorem 3.6 can be satisfied even in the case when (\*\*) is not valid (and, in consequence, Theorem 3.4 does not work).

**Example 3.8.**

1. Suppose that there exist two distinct numbers  $t_1, t_2 \in \mathbb{N}$  such that  $\varphi_{t_1} = \varphi_{t_2}$  is a non-convex function. Then it does not satisfy the Lim inequality, and consequently (3.2) has a solution required in the previous theorem. Let us observe that in this case Remark 3.5 could also be applied.
2. Let  $\varphi_{t_1}(x) = \sqrt{x}$  and  $\varphi_{t_2}(x) = x$ . Then (3.2) leads to the system of inequalities

$$\begin{cases} b > \sqrt{b+c} - \sqrt{c}, \\ \sqrt{a+b} - \sqrt{a} > b, \end{cases}$$

which is solved by any positive numbers  $a, b, c$  such that

$$\sqrt{a+b} + \sqrt{a} < 1 < \sqrt{b+c} + \sqrt{c}.$$

3. Define  $\varphi_{t_1}(x) = \ln(x+1)$  and  $\varphi_{t_2}(x) = x/2$ . Then (3.2) can be rewritten as

$$\ln \frac{b+c+1}{c+1} < b/2 < \ln \frac{a+b+1}{a+1},$$

and this problem also has infinitely many solutions, e.g.  $a < d$ ,  $b = 2$ ,  $c > d$  for  $d = \frac{3-e}{e-1}$ .

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