Opuscula Math. 35, no. 1 (2015), 71–83
http://dx.doi.org/10.7494/OpMath.2015.35.1.71 Opuscula Mathematica

CONTROLLABILITY OF SEMILINEAR SYSTEMS WITH FIXED DELAY IN CONTROL

Surendra Kumar and N. Sukavanam

Communicated by Alexander Gomilko

Abstract. In this paper, different sufficient conditions for exact controllability of semilinear systems with a single constant point delay in control are established in infinite dimensional space. The existence and uniqueness of mild solution is also proved under suitable assumptions. In particular, local Lipschitz continuity of a nonlinear function is used. To illustrate the developed theory some examples are given.

Keywords: first order delay system, mild solution, fixed point, exact controllability.

Mathematics Subject Classification: 93B05.

1. INTRODUCTION

Let $Z = L_2([0,\tau];V)$, $Y = L_2([0,\tau];\hat{V})$ be the function spaces corresponding to Hilbert spaces V and $\hat{V}$, respectively. Let $C([-h,0],V)$ be the Banach space of all continuous functions from $[-h,0]$ to V with the supremum norm.

Consider the following semilinear system with delay in control:

$$\begin{aligned} x'(t) &= Ax(t) + B_0u(t) + B_1u(t-h) + f(t,x_t), \quad && t \in (0,\tau], \\ x(t) &= \phi(t), \quad u(t) = 0, && t \in [-h,0], \end{aligned} \tag{1.1}$$

where the state $x(\cdot)$ takes its value in space V, the control function $u(\cdot)$ takes its value in space $\hat{V}$, $A : D(A) \subseteq V \to V$ is a closed linear operator with dense domain $D(A)$ generating a C_0-semigroup $T(t)$, B_0 and B_1 are bounded linear operators from $\hat{V}$ to V, and the operator $f : [0,\tau] \times C([-h,0],V) \to V$ is nonlinear. If $x : [-h,\tau] \to V$ is a continuous function, then $x_t : [-h,0] \to V$ is defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$ and $\phi \in C([-h,0],V)$.

If $f \equiv 0$, then the system (1.1) is called the corresponding linear system and is denoted by $(1.1)^*$.

Controllability is a qualitative property of dynamical control systems and is of particular importance in control theory. In infinite dimensional spaces, controllability results for abstract linear systems are well-developed and extensively investigated in the literature [2]. Many results on exact controllability in infinite dimensional spaces are summarized by Balachandran *et al.* [1]. Controllability of a linear system with fixed delay in control is proved by Klamka [9, 10]. In [4] relative controllability and minimum energy control of linear systems with distributed delay in control is studied in a finite dimensional space. Stochastic relative exact and approximate controllability of linear stochastic time variable systems is shown by Klamka [8] with a single time variable point delay in control. Sufficient conditions for exact controllability and null controllability of linear systems with delay in both state and control are obtained by Davies *et al.* [3]. Controllability of linear time varying systems with multiple time delay and impulsive effect is shown in [11]. Using Schauder's fixed point theorem, Klamka [7] discussed the controllability of semilinear and nonlinear systems in a finite dimensional space. However, to the best of author's knowledge, in infinite dimensional space exact controllability of the semilinear system with fixed delay in control is an untreated topic in the literature so far and this fact is the motivation of the present paper.

This paper has two main objectives. The first objective is to obtain existence and uniqueness of the mild solution using a technique similar to that of [13] with suitable modification. To prove the results, we assume that the nonlinear function is locally Lipschitz continuous in the second argument and satisfies the linear growth condition. Our second objective is to show exact controllability of the semilinear system (1.1) under suitable conditions. For this, first we prove exact controllability of the corresponding linear system $(1.1)^*$ using the method of steps then the results are extended for a class of semilinear systems.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, the existence and uniqueness of the mild solution is proved. Exact controllability of a semilinear system is shown in Section 4. The paper is ended with some examples in Section 5.

2. PRELIMINARIES

In this section some basic definitions, which are useful for further developments, are given.

Definition 2.1. A function $x(\cdot) \in C([-h, \tau]; V)$ is said to be the mild solution of (1.1) if it satisfies

$$x(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s)B_0 u(s)ds \\ + \int_0^t T(t-s)B_1 u(s-h)ds + \int_0^t T(t-s)f(s, x_s)ds, & t \in [0, \tau], \\ \phi(t), & t \in [-h, 0]. \end{cases} \tag{2.1}$$

Let $x(\tau, \phi(0), u)$ be the state value of system (1.1) at time τ corresponding to the control u. The system (1.1) is said to be exactly controllable in time interval $[0, \tau]$, if for every desired final state x_τ there exists a control function $u(\cdot) \in Y$ such that the mild solution $x(t)$ given by (2.1) satisfies $x(\tau) = x_\tau$.

To prove our main results we impose the following conditions:

[H1] The linear control system without delay (when $B_1 \equiv 0$ and $f \equiv 0$) is controllable.

[H2] The nonlinear function $f : [0,\tau] \times C([-h,0];V) \to V$ is locally Lipschitz continuous in x and uniform in $t \in [0,\tau]$ i.e. there exists a positive number $L(r)$ such that

$$\|f(t,x_1) - f(t,x_2)\|_V \le L(r)\|x_1 - x_2\|_C,$$

holds for all $x_j \in C([-h,0];V)$ with $\|x_j\| < r$, $j = 1,2$ and $t \in [0,\tau]$.

[H3] There exists a real number k such that

$$\|f(t,x)\|_V \le k[1 + \|x\|_C]$$

holds for all $x \in C([-h,0];V)$ and $t \in [0,\tau]$.

3. EXISTENCE AND UNIQUENESS OF MILD SOLUTION

The existence and uniqueness of the mild solution is proved using the technique similar to [13].

Theorem 3.1. *Under assumptions* [H2] *and* [H3] *the system* (2.1) *admits a unique mild solution in* $C([-h,\tau];V)$ *for each control function* $u \in Y$.

Proof. Let $l_f = \max_{0\le t\le\tau} \|f(t,0)\|$ and $\max\{\|B_0\|, \|B_1\|\} \le M_B$. Again let $M \ge 1$ be a constant such that $\|T(t)\| \le M$.

Define the mapping $\Phi : C([-h,t_1];V) \to C([-h,t_1];V)$ as

$$(\Phi x)(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s)B_0u(s)ds \\ + \int_0^t T(t-s)B_1u(s-h)ds + \int_0^t T(t-s)f(s,x_s)ds, & t \in [0,t_1], \\ \phi(t), & t \in [-h,0]. \end{cases}$$

Now, if we are able to show that Φ has a fixed point in the space $C([-h,t_1];V)$, then (2.1) is the mild solution on $[-h,t_1]$.

Let for $r_0 > 0$,

$$B_{r_0} = \{x(\cdot) \in C([-h,t_1];V) : \|x\|_{C([-h,t_1];V)} \le r_0,\ x(0) = \phi(0)\}.$$

Clearly, B_{r_0} is a bounded and closed subset of $C([-h,t_1];V)$. For any $x(\cdot) \in B_{r_0}$ and $0 \le s \le t_1$, we have

$$\|x_s\|_C = \max_{-h\le\theta\le0} \|x(s+\theta)\| \le \max_{-h\le\eta\le t_1} \|x(\eta)\| \le r_0.$$

Thus

$$\|(\Phi x)(t)\| \le M\|\phi(0)\| + MM_B\left[\int_0^t \|u(s)\|ds + \int_{-h}^{t-h}\|u(\sigma)\|d\sigma\right]$$
$$+ ML(r_0)\int_0^t \|x_s\|ds + Ml_f\int_0^t ds$$
$$\le M\left[\|\phi(0)\| + 2M_B\sqrt{\tau}\|u\|_Y + \{L(r_0)r_0 + l_f\}t_1\right].$$

Now let $r_0 = 2M\big[\|\phi(0)\| + 2M_B\sqrt{\tau}\|u\|_Y\big] + 1$ and $0 < t_1 < \tau$ is small enough such that

$$\{L(r_0)r_0 + l_f\}t_1 \le \big[\|\phi(0)\| + 2M_B\sqrt{\tau}\|u\|_Y\big] + 1.$$

Therefore, Φ maps the ball B_{r_0} of radius r_0 into itself.

Next, we show that Φ is a contraction on B_{r_0}. For this, let us take $x_1,\ x_2 \in\ B_{r_0}$, then we get

$$\|(\Phi x_1)(t) - (\Phi x_2)(t)\| \le M\int_0^t \|f(s,(x_1)_s) - f(s,(x_2)_s)\|ds$$
$$\le ML(r_0)\int_0^t \|(x_1)_s - (x_2)_s\|ds$$
$$\le ML(r_0)t\|x_1 - x_2\|_{C([-h,t_1];V)}.$$

By repeating this process, we get

$$\|(\Phi^m x_1)(t) - (\Phi^m x_2)(t)\| \le \frac{M^m L^m(r_0)t^m}{m!}\|x_1 - x_2\|_{C([-h,t_1];V)}$$
$$\le \frac{M^m L^m(r_0)\tau^m}{m!}\|x_1 - x_2\|_{C([-h,t_1];V)}.$$

Hence we have

$$\|\Phi^m x_1 - \Phi^m x_2\|_{C([-h,t_1];V)} \le \frac{M^m L^m(r_0)\tau^m}{m!}\|x_1 - x_2\|_{C([-h,t_1];V)}.$$

Therefore, Φ^m is a contraction mapping for a large integer m. By Banach's fixed point theorem, we conclude that Φ has a fixed point in B_{r_0}, so (2.1) is the mild solution on $[-h, t_1]$. Similarly, we can prove that (2.1) is the mild solution on an interval $[t_1, t_2]$, $t_1 < t_2$. Repeating the above process, we can show that (2.1) is the mild solution with a maximal existence interval $[-h, t^*)$, $t^* \le \tau$. Next, we show that the mild solution is bounded. If $t \in [-h, 0]$, then $x(t) = \phi(t)$. Hence it is bounded. If $t \in [0, \tau]$, then

$$\begin{aligned}\|x(t)\|_V &\leq M\|\phi(0)\| + MM_B \int_0^t \|u(s)\|ds + MM_B \int_0^t \|u(s-h)\|ds \\ &\quad + Mk \int_0^t [1 + \|x_s\|_C]\, ds \\ &\leq M\|\phi(0)\| + 2MM_B\sqrt{\tau}\|u(s)\|_Y + Mk\tau + Mk \int_0^t \|x_s\|_C ds.\end{aligned}$$

Then Gronwall's inequality implies that

$$\|x(t)\|_V \leq \|x_t\|_C \leq \left[M\|\phi(0)\| + 2MM_B\sqrt{\tau}\|u(s)\|_Y + Mk\tau\right] \exp(Mk\tau).$$

This implies that $x(t)$ is bounded in the interval $[-h, t^*)$. Thus we conclude that $x(\cdot)$ is well defined on $[-h, \tau]$.

Finally, we prove the uniqueness of mild solutions. For this, let x_1 and x_2 be any two solutions of (2.1). If $t \in [-h, 0]$, then $x_1(t) = x_2(t) = \phi(t)$ implies the uniqueness of mild solutions in $[-h, 0]$. Next, if $t \in [0, \tau]$, let

$$r^\star = \max\{\|x_1\|_{C([-h,\tau];V)},\ \|x_2\|_{C([-h,\tau];V)}\}.$$

Then

$$\begin{aligned}\|x_1(t) - x_2(t)\|_V &\leq M \int_0^t \|f(s, (x_1)_s) - f(s, (x_2)_s)\|ds \\ &\leq ML(r^\star) \int_0^\tau \|(x_1)_s - (x_2)_s\|_C ds.\end{aligned}$$

Therefore,

$$\|(x_1)_t - (x_2)_t\|_C \leq ML(r^\star) \int_0^\tau \|(x_1)_s - (x_2)_s\|_C ds.$$

Hence, Gronwall's inequality implies that $(x_1)_t = (x_2)_t$ for all $t \in [0, \tau]$ and consequently $x_1 = x_2$. This completes the proof. □

4. CONTROLLABILITY RESULTS

In this section, using the method of steps, first we prove exact controllability of linear systems $(1.1)^*$. Then, exact controllability of the semilinear system (1.1) is shown.

Lemma 4.1. *Under assumption* [H1] *the linear control system* $(1.1)^*$ *with delay in control is exactly controllable.*

Proof. Consider the linear delay system $(1.1)^*$ given by

$$\begin{cases} x'(t) = Ax(t) + B_0u(t) + B_1u(t-h), & t \in (0,\tau], \\ x(t) = \phi(t),\ u(t) = 0, & t \in [-h,0]. \end{cases}$$

To prove controllability of $(1.1)^*$, we use the method of steps, which is based on searching for the mild solution of the system $(1.1)^*$ in succeeding intervals whose length depends on the delay occurring in the system.

Now consider the following system in an interval $[0,h]$

$$\begin{cases} y'(t) = Ay(t) + B_0u(t), & t \in [0,h], \\ y(0) = y_0 = \phi(0). \end{cases} \tag{4.1}$$

Since $u(t-h) = 0$ for $0 \le t \le h$, we conclude that the mild solution $x(t)$ of $(1.1)^*$ and the mild solution $y(t)$ of (4.1) coincide in the interval $[0,h]$. Hence

$$x(h) = y(h) = y_h = T(h)\phi(0) + \int_0^h T(h-s)B_0u(s)ds,$$

The controllability of system (4.1) on interval $[0,h]$ implies that there exists a control function $u_1(\cdot) \in L_2([0,h];\hat{V})$ (say) that steers the system from initial state $y(0)$ to the state $y(h)$. Define

$$v_1(t) = \begin{cases} 0, & t \in [-h,0], \\ u_1(t), & t \in [0,h]. \end{cases}$$

Then $v_1(\cdot) \in L_2([-h,h];\hat{V})$ and steers the system $(1.1)^*$ from $x(0)$ to $x(h)$. This shows that the system $(1.1)^*$ is controllable in the interval $[0,h]$.

In the next step, consider the system in an interval $[h,2h]$ as

$$\begin{cases} y'(t) = Ay(t) + B_0u(t), & t \in [h,2h], \\ y(h) = y_h + y_1, \end{cases} \tag{4.2}$$

where $y_1 = \int_h^{2h} T(h-s)B_1v_1(s-h)ds$ is known from the previous step since $(s-h) \in [0,h]$ and the control function is known. The mild solution of system (4.2) in the interval $[h,2h]$ is given by

$$y(t) = T(t-h)[y_h + y_1] + \int_h^t T(t-s)B_0u(s)ds.$$

At $t = 2h$, we get

$$x(2h) = y(2h) = y_{2h} = T(h)y_h + T(h)y_1 + \int_h^{2h} T(2h-s)B_0u(s)ds.$$

The controllability of system (4.2) implies that there exists a control $u_2(\cdot) \in L_2([h,2h];\hat{V})$ steers the system from $y(h)$ to $y(2h)$. Define

$$v_2(t) = \begin{cases} v_1(t), & t \in [0,h], \\ u_2(t), & t \in [h,2h]. \end{cases}$$

Then $v_2(\cdot) \in L_2([0,2h];\hat{V})$ and steers the system $(1.1)^*$ from $x(h)$ to $x(2h)$. If we continue in the same manner then at the n-th step, we have the following system in an interval $t \in [(n-1)h, nh]$:

$$\begin{cases} y'(t) = Ay(t) + B_0u(t), & t \in [(n-1)h, nh], \\ y((n-1)h) = y_{(n-1)h} + y_{(n-1)}, \end{cases} \tag{4.3}$$

where $y_{(n-1)} = \int_{(n-1)h}^{nh} T((n-1)h - s)B_1v_{n-1}(s-h)ds$ is known from the previous step. The mild solution of the system (4.3) in the interval $[(n-1)h, nh]$ is given by

$$y(t) = T(t-(n-1)h)[y_{(n-1)h} + y_{(n-1)}] + \int_{(n-1)h}^{t} T(t-s)B_0u(s)ds.$$

The controllability of system (4.3) implies that there exists a control $u_n(\cdot) \in L_2([(n-1)h, nh];\hat{V})$ which steering the system from $y((n-1)h)$ to $y(nh)$. Define $v_n(t)$

$$v_n(t) = \begin{cases} v_{n-1}(t), & t \in [(n-2)h, (n-1)h], \\ u_n(t), & t \in [(n-1)h, nh]. \end{cases}$$

Then $v_n(\cdot) \in L_2([0,nh];\hat{V})$ and we have

$$\begin{aligned} x(nh) = y(nh) = y_{nh} = T(nh-(n-1)h)[y_{(n-1)h} + y_{(n-1)}] \\ + \int_{(n-1)h}^{nh} T(nh-s)B_0u_n(s)ds. \end{aligned} \tag{4.4}$$

Now we write (4.4) in terms of the initial condition, for this we use the properties of the C_0-semigroup and the results obtained for the mild solution in the previous steps. Thus we have

$$\begin{aligned} T(nh-(n-1)h)y_{(n-1)h} = T(nh)\phi(0) + \sum_{k=1}^{n-1} T(nh-kh) \int_{(k-1)h}^{kh} T(kh-s)B_0u_k(s)ds \\ + \sum_{k=1}^{n-1} T(nh-kh) \int_{(k-2)h}^{(k-1)h} T(kh-h-s)B_1v_{k-1}(s)ds. \end{aligned}$$

When we rewrite equation (4.4), we have

$$\begin{aligned} x(nh) &= y(nh) = y_{nh} \\ &= T(nh)\phi(0) + \sum_{k=1}^{n} T(nh-kh) \int_{(k-1)h}^{kh} T(kh-s)B_0 u_k(s)ds \\ &\quad + \sum_{k=1}^{n} T(nh-kh) \int_{(k-2)h}^{(k-1)h} T(kh-h-s)B_1 v_{k-1}(s)ds. \end{aligned}$$

Hence we can find the mild solution of $(1.1)^*$ in an interval $[nh,(n+1)h]$ with the initial condition $y_{nh}+y_n$, where $y_n = \int_{nh}^{\tau} T(nh-s)B_1 u(s-h)ds$, which is known from the previous step. Then we obtain

$$y(\tau,\phi(0),u) = y(\tau) \quad \text{for} \quad \tau \in [nh,(n+1)h].$$

Thus, the solution of $(1.1)^*$ at time $\tau > 0$ has the form

$$\begin{aligned} y(\tau,\phi(0),u) = x(\tau) &= T(\tau-nh)[y_{nh}+y_n] + \int_{nh}^{\tau} T(\tau-s)B_0 u(s)ds \\ &= T(\tau)\phi(0) + \sum_{k=1}^{n} T(nh-nk) \int_{(k-1)h}^{kh} T(kh-s)B_0 u_k(s)ds \\ &\quad + \sum_{k=1}^{n} T(nh-nk) \int_{(k-2)h}^{(k-1)h} T(kh-h-s)B_1 v_{k-1}(s)ds \\ &\quad + \int_{nh}^{\tau} T(\tau-s)B_0 u(s)ds + \int_{nh}^{\tau} T(\tau-s)B_1 u(s-h)ds. \end{aligned}$$

This shows that the system $(1.1)^*$ can be steered from $x(0)$ to $x(\tau)$. Therefore, the system $(1.1)^*$ is exact controllable. □

Remark 4.2. For a finite dimensional space, relative controllability and minimum energy control of linear time varying systems with time variable delays in control are proved by Klamka in [5].

Remark 4.3. In [6] relative controllability, absolute controllability and minimum energy control of linear time varying systems with lumped and distributed delays in the control function are examined.

To prove exact controllability of system (1.1), we define two new operators similar to [14]. For any $t_1, t_2 \in [0, \tau]$ with $t_2 > t_1$, $E : L_2([t_1, t_2]; \hat{V}) \to V$ and $N : L_2([t_1, t_2]; \hat{V}) \to V$ are defined as

$$E(t_1, t_2)u = \int_{t_1}^{t_2} T(t_2 - s)B_0 u(s)ds + \int_{t_1}^{t_2} T(t_2 - s)B_1 u(s - h)ds,$$

$$N(t_1, t_2)u = \int_{t_1}^{t_2} T(t_2 - s)f(s, x_s)ds,$$

where $x(\cdot)$ is the mild solution of (1.1) with the control function $u(\cdot) \in L_2([t_1, t_2]; \hat{V})$ in the definition of $N(t_1, t_2)$. Now we are able to prove sufficient conditions for exact controllability of the semilinear system (1.1).

Theorem 4.4. *Under assumptions* [H1]–[H3] *the semilinear control system* (1.1) *is exactly controllable if there exists a function* $Q(\cdot) \in L_1([0, \tau])$ *such that*

$$\|f(t, \psi)\| \le Q(t) \quad \textit{for all} \quad (t, \psi) \in [0, \tau] \times C.$$

Proof. Since $Q(\cdot) \in L_1([0, \tau])$, we can select an increasing sequence $t_n \in [0, \tau]$ such that $t_n \to \tau$ and

$$\int_{t_n}^{\tau} Q(t)dt \to 0, \quad \text{as} \quad n \to \infty.$$

Since, by assumption [H1] and Lemma 4.1, the linear system $(1.1)^*$ is exact controllable on $[0, \tau]$, for any $x_\tau \in V$ there exists a control function $\bar{u}_0 \in Y$ such that

$$x_\tau = T(\tau)\phi(0) + E(0, \tau)\bar{u}_0.$$

Let $x_1 = x(t_1, \phi(0), \bar{u}_0)$. Again, the controllability of $(1.1)^*$ on $[t_1, \tau]$ implies that there exists a control function $\bar{u}_1 \in L_2([t_1, \tau]; \hat{V})$ such that

$$x_\tau = T(\tau - t_1)x_1 + E(t_1, \tau)\bar{u}_1.$$

Define

$$\bar{v}_1(t) = \begin{cases} \bar{u}_0(t), & t \in [0, t_1], \\ \bar{u}_1(t), & t \in [t_1, \tau]. \end{cases}$$

Then $\bar{v}_1 \in Y$. If we continue in the same manner then we get three sequences x_n, $\bar{u}_n$ and $\bar{v}_n$ such that $\bar{u}_n(\cdot) \in L_2([t_n, \tau]; \hat{V})$, $\bar{v}_n(\cdot) \in Y$,

$$\bar{v}_n(t) = \begin{cases} \bar{u}_{n-1}(t), & t \in [0, t_n], \\ \bar{u}_n(t), & t \in [t_n, \tau] \end{cases}$$

and $x_n = x(t_n, \phi(0), \bar{u}_{n-1})$ with

$$x_\tau = T(\tau - t_n)x_n + E(t_n, \tau)\bar{u}_n.$$

Thus the mild solution of system (1.1) with the control function $\bar{v}_n$ is given by

$$\begin{aligned} x(t;\bar{v}_n) &= T(t-t_n)[T(t_n)\phi(0) + E(0,t_n)\bar{v}_n + N(0,t_n)\bar{v}_n] \\ &\quad + E(t_n,t)\bar{v}_n + N(t_n,t)\bar{v}_n \\ &= T(t-t_n)[T(t_n)\phi(0) + E(0,t_n)\bar{u}_{n-1} + N(0,t_n)\bar{u}_{n-1}] \\ &\quad + E(t_n,t)\bar{u}_n + N(t_n,t)\bar{u}_n \\ &= T(t-t_n)x_n + E(t_n,t)\bar{u}_n + N(t_n,t)\bar{u}_n. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x(\tau;\bar{v}_n) - x_\tau\| &\le \|T(t-t_n)x_n + E(t_n,t)\bar{u}_n - x_\tau\| + \|N(t_n,t)\bar{u}_n\| \\ &\le \int_{t_n}^{\tau} \|T(\tau-s)f(s,x_s)ds\| \le M\int_{t_n}^{\tau} \|Q(s)ds\| \to 0 \text{ as } n\to\infty. \end{aligned}$$

This implies that $x(\tau;\bar{v}_n) = x_\tau$ for sufficiently large n. Hence, the semilinear system (1.1) is exactly controllable. □

5. EXAMPLES

Example 5.1. Let $V = L_2(0,\pi)$ and $A \equiv \frac{d^2}{dx^2}$ with $D(A)$ consisting of all $y \in V$ with $\frac{d^2y}{dx^2}$ and $y(0) = 0 = y(\pi)$. Put $e_n(x) = \sqrt{2/\pi}\sin(nx)$, $0 \le x \le \pi$, $n = 1,2,\ldots$. Then $\{e_n : n = 1,2,\ldots\}$ is an orthonormal base for V and e_n is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. Then the C_0-semigroup $T(t)$ generated by A has $\exp(\lambda_n t)$ as the eigenvalues and e_n as their corresponding eigenfunctions [12]. Define an infinite-dimensional space $\hat{V}$ by

$$\hat{V} = \left\{ u \,\middle|\, u = \sum_{n=2}^{\infty} u_n e_n \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.$$

The norm in $\hat{V}$ is defined by

$$\|u\|_{\hat{V}} = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{1/2}.$$

Define a continuous linear map B from $\hat{V}$ to V as

$$Bu = 2u_2e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n e_n \in \hat{V}.$$

We define the operator $B_0 : Y \to Z$ by $(B_0u)(t) = (Bu)(t)$.

Let us consider the following semilinear control system of the form

$$\begin{aligned}
&\frac{\partial y(t,x)}{\partial t} = \frac{\partial^2 y(t,x)}{\partial x^2} + B_0 u(t,x) + u(t-h,x) + f(t, y(t-h,x)), \quad t \in [0,\tau],\\
&y(t,0) = y(t,\pi) = 0, \quad t \in [0,\tau],\\
&y(t,x) = \phi(t,x), \quad t \in [-h,0], 0 < x < \pi
\end{aligned} \tag{5.1}$$

where $\phi(t,x)$ is continuous. The system (5.1) can be written in the abstract form given by (1.1) with $B_1 = I$. The control function $u(t,x) \in L_2([0,\tau];\hat{V}) = L_2([0,\tau]\times(0,\pi))$. If the conditions [H1] is satisfied, then controllability of the corresponding linear system to (5.1) follows from Lemma 4.1. Also if the nonlinear term f is considered as an operator satisfying Hypothesis [H2] and [H3] then exact controllability of system (5.1) follows from Theorem 4.4.

Example 5.2. Consider the controlled wave equation with a distributed control $u(\cdot) \in L_2([0,1])$:

$$\begin{aligned}
&\frac{\partial^2 y(t,x)}{\partial t^2} = \frac{\partial^2 y(t,x)}{\partial x^2} + u(t,x) + u(t-h,x) + f(t, y(t+\theta,x)), \quad t \in [0,\tau],\\
&y(t,0) = y(t,1) = 0, \quad t > 0,\\
&y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad 0 \le x \le 1,
\end{aligned} \tag{5.2}$$

where $y_0,\ y_1 \in L_2([0,1])$.

Proceeding in a similar way to that in [2], introduce the Hilbert space $V = D(A_0^{1/2}) \oplus L_2([0,1])$, endowed with the inner product

$$\langle r,s\rangle = \left\langle \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \right\rangle = \sum_{n=1}^{\infty} \{n^2\pi^2 \langle r_1, e_n\rangle\langle e_n, s_1\rangle + \langle r_2, e_n\rangle\langle e_n, s_2\rangle\},$$

where $e_n(x) = \sqrt{2}\sin(n\pi x)$ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L_2([0,1])$.

Taking the operator

$$A = \begin{bmatrix} 0 & I \\ A_0 & 0 \end{bmatrix},$$

where $A_0 \equiv \frac{d^2}{dx^2}$ with domain $D(A_0) = \{\psi \in L_2([0,1]) : \psi, (d/dx)\psi$ are absolutely continuous, $(d^2/dx^2)\psi \in L_2([0,1])$ and $\psi(0) = \psi(1) = 0\}$. Then A is the infinitesimal generator of a semigroup $T(t)$ on V given by

$$T(t)\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \cos(n\pi t) & (n\pi)^{-1}\sin(n\pi t) \\ -(n\pi)\sin(n\pi t) & \cos(n\pi t) \end{bmatrix} \begin{bmatrix} r_1^n \\ r_2^n \end{bmatrix} e_n.$$

Then problem (5.2) can be formulated in the abstract form as

$$\begin{aligned}
\frac{dz(t,x)}{dt} &= Az(t,x) + Bu(t,x) + B_1 u(t-h,x) + Cf(t, z(t+\theta,x)),\\
z(0) &= z_0,
\end{aligned}$$

where

$$z = \begin{bmatrix} y \\ y_t \end{bmatrix}, \quad B = B_1 = C = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad z(0) = \begin{bmatrix} y(0) \\ y_t(0) \end{bmatrix}.$$

The controllability of the system (5.2) in the interval $[0, \tau]$ follows from Theorem 4.4, if all the assumptions are satisfied.

Let $f(t, z_t) = f(t, z_t(0)) = f(t, z(t)) = \frac{1}{t} + \sin z(t)$. It should be noted that

$$\|f(t, z_t)\| \leq \frac{1}{t} + 1 = Q(t).$$

Clearly, $Q(t) \notin L_1([0, \tau])$. Although it is easy to verify that the system (5.2) is exactly controllable as the nonlinear function is Lipschitz continuous in the second argument. This shows that Theorem 4.4 is only sufficient but not a necessary condition for exact controllability of the semilinear system.

REFERENCES

[1] K. Balachandran, J.P. Dauer, *Controllability of nonlinear systems in Banach spaces: a survey*, J. Optim. Theory Appl. **115** (2002), 7–28.

[2] R.F. Curtain, H.J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[3] I. Davies, P. Jackreece, *Controllability and null controllability of linear systems*, J. Appl. Sci. Environ. Manag. **9** (2005), 31–36.

[4] J. Klamka, *Relative controllability and minimum energy control of linear systems with distributed delays in control*, IEEE T. Automat. Contr. **21** (1976) 4, 594–595.

[5] J. Klamka, *Controllability of linear systems with time-variable delays in control*, Int. J. Control. **24** (1976) 6, 869–878.

[6] J. Klamka, *On the controllability of linear systems with delays in the control*, Int. J. Control. **25** (1977) 6, 875–883.

[7] J. Klamka, *Schauder's fixed-point theorem in nonlinear controllability problems,* Control Cybern. **29** (2000) 1, 153–165.

[8] J. Klamka, *Stochastic controllability of systems with variable delay in control*, Bull. Pol. Ac.: Tech. **56** (2008) 3, 279–284.

[9] J. Klamka, *Stochastic controllability and minimum energy control of systems with multiple delays in control*, Applied Math. Comput. **206** (2008), 704–715.

[10] J. Klamka, *Stochastic controllability of systems with multiple delays in control*, Int. J. Appl. Math. Comput. Sci. **19** (2009), 39–47.

[11] Y. Liu, S. Zhao, *Controllability analysis of linear time-varying systems with multiple time delay and impulsive effects*, Nonlinear Analysis: RWA. **13** (2012), 558–568.

[12] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim. **25** (1987), 715–722.

[13] L. Wang, *Approximate controllability for integrodifferential equations with multiple delays*, J. Optim. Theory Appl. **143** (2009), 185–206.

[14] L. Wang, *Approximate boundary controllability for semilinear delay differential equations*, J. Applied Math. **2011** (2011), Article ID 587890, 10 pp.

Surendra Kumar
mathdma@gmail.com

University of Delhi
Department of Mathematics
Delhi – 110007, India

N. Sukavanam
nsukvfma@iitr.ernet.in

Indian Institute of Technology, Roorkee
Department of Mathematics
Roorkee (Uttarakhand) – 247667, India

Received: May 29, 2013.
Revised: February 7, 2014.
Accepted: March 23, 2014.