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STRONGLY INCREASING SOLUTIONS OF CYCLIC SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH POWER-TYPE NONLINEARITIES

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Abstract. We consider n-dimensional cyclic systems of second order differential equations

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' = q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1}, \quad i=1,\dots,n, \quad (x_{n+1}=x_1) \tag{$*$}$$

under the assumption that the positive constants α_i and β_i satisfy $\alpha_1\dots\alpha_n > \beta_1\dots\beta_n$ and $p_i(t)$ and $q_i(t)$ are regularly varying functions, and analyze positive strongly increasing solutions of system $(*)$ in the framework of regular variation. We show that the situation for the existence of regularly varying solutions of positive indices for $(*)$ can be characterized completely, and moreover that the asymptotic behavior of such solutions is governed by the unique formula describing their order of growth precisely. We give examples demonstrating that the main results for $(*)$ can be applied to some classes of partial differential equations with radial symmetry to acquire accurate information about the existence and the asymptotic behavior of their radial positive strongly increasing solutions.

Keywords: systems of differential equations, positive solutions, asymptotic behavior, regularly varying functions.

Mathematics Subject Classification: 34C11, 26A12.

1. INTRODUCTION

We consider the cyclic system of second order nonlinear differential equations

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' = q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1}, \quad i=1,2,\dots,n, \quad (x_{n+1}=x_1), \tag{1.1}$$

for which the following conditions are always assumed to hold:

(a) α_i and β_i, $i=1,2,\dots,n$, are positive constants such that

$$\alpha_1\alpha_2\dots\alpha_n > \beta_1\beta_2\dots\beta_n; \tag{1.2}$$

(b) $p_i, q_i : [a,\infty) \to (0,\infty),\ a > 0,\ i = 1,2,\dots,n$, are continuous functions;
(c) all $p_i(t)$ simultaneously satisfy either

$$\int_a^\infty p_i(t)^{-\frac{1}{\alpha_i}}\,dt = \infty, \quad i = 1,2,\dots,n, \tag{1.3}$$

or

$$\int_a^\infty p_i(t)^{-\frac{1}{\alpha_i}}\,dt < \infty, \quad i = 1,2,\dots,n. \tag{1.4}$$

Systems of the form (1.1) arise, for example, in the study of positive radial solutions in the exterior domain for the system of p-Laplacian equations

$$\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) = f_i(|x|)|u_{i+1}|^{\gamma_i - 1}u_{i+1}, \quad i = 1,\dots,n,\ (u_{n+1} = u_1),$$

where $p > 1$ and $\gamma_i > 0$ are constants, $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^N$ and $f_i(t)$, $i = 1,\dots,n$, are positive continuous functions on $[a,\infty)$.

By a positive solution of the ordinary differential system (1.1) we mean a vector function $(x_1(t),\dots,x_n(t))$ consisting of positive continuous components $x_i(t)$, $i = 1,\dots,n$, which are continuously differentiable together with $p_i(t)|x_i'(t)|^{\alpha_i-1}x_i'(t)$ on an interval of the form $[T,\infty)$ and satisfy the system of differential equations (1.1) there. We are interested in the asymptotic behavior of positive solutions of (1.1) as $t \to \infty$. It should be noticed that a positive solution $(x_1(t),\dots,x_n(t))$ of (1.1) may exhibit a variety of asymptotic behaviors as $t \to \infty$ because if (1.3) holds each component $x_i(t)$ is either increasing and satisfies

$$\lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \infty \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \text{const} > 0,$$

where $P_i(t) = \int_a^t p_i(s)^{-1/\alpha_i}\,ds$, or is decreasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \text{const} > 0 \quad \text{or} \quad \lim_{t\to\infty} x_i(t) = 0,$$

while if (1.4) holds each component $x_i(t)$ is either increasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \infty \quad \text{or} \quad \lim_{t\to\infty} x_i(t) = \text{const} > 0,$$

or is decreasing and satisfies

$$\lim_{t\to\infty} x_i(t) = \text{const} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = \text{const} > 0 \quad \text{or} \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = 0,$$

where $\pi_i(t) = \int_t^\infty p_i(s)^{-1/\alpha_i}\,ds$.

In our previous paper [8] we have studied the problem of existence and asymptotic behavior of positive solutions $(x_1(t), \ldots, x_n(t))$ of (1.1) all components of which are decreasing and satisfy

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, \ldots, n, \quad \text{in case (1.3) holds,}$$

or

$$\lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = 0, \quad i = 1, \ldots, n, \quad \text{in case (1.4) holds,}$$

in the framework of regular variation, and have shown that almost complete analysis can be made of regularly varying solutions of (1.1) having negative regularity indices, by which we mean those solutions $(x_1(t), \ldots, x_n(t))$ of (1.1) where all components are regularly varying functions of negative indices. (See Section 2 for the definition of regularly varying functions.)

A question naturally arises: Is it possible to analyze the existence and asymptotic behavior of positive solutions $(x_1(t), \ldots, x_n(t))$ of (1.1) all components of which are increasing and satisfy

$$\lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \infty, \quad i = 1, \ldots, n, \quad \text{in case (1.3) holds,} \tag{1.5}$$

or

$$\lim_{t\to\infty} x_i(t) = \infty, \quad i = 1, \ldots, n, \quad \text{in case (1.4) holds,} \tag{1.6}$$

in the same spirit as in [8]? Such solutions are referred to as *strongly increasing solutions* of (1.1). It is clear that a positive solution $(x_1(t), \ldots, x_n(t))$ of (1.1) is strongly increasing if and only if

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} p_i(t)|x_i'(t)|^{\alpha_i - 1} x_i'(t) = \infty, \quad i = 1, \ldots, n. \tag{1.7}$$

The aim of this paper is to give an affirmative answer to the above question by showing that if we limit ourselves to the case where $p_i(t)$ and $q_i(t)$ are regularly varying, then with the help of the theory of regular variation we can characterize the situation in which (1.1) possesses strongly increasing solutions $(x_1(t), \ldots, x_n(t))$ where all the components are regularly varying functions of positive indices, and moreover determine the unique precise growth law which governs the asymptotic behavior of all such solutions of (1.1).

The main results of this paper will be presented in Section 4. Under the assumption that $p_i(t)$ and $q_i(t)$ are regularly varying the existence of strongly increasing regularly varying solutions of (1.1) is proved by solving the system of integral equations

$$x_i(t) = c_i + \int_T^t \left(\frac{1}{p_i(s)} \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds, \quad i = 1, \ldots, n, \tag{1.8}$$

for some positive constants c_i and $T > a$ with the help of fixed point techniques combined with basic properties of regularly varying functions. Furthermore, the asymptotic behavior of the obtained solutions is shown to obey the unique law describing their order of growth accurately. For this purpose essential use is made of the fact that one can acquire thorough knowledge of strongly increasing regularly varying solutions of the following system of asymptotic relations associated with (1.1):

$$x_i(t) \sim \int_T^t \left(\frac{1}{p_i(s)} \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds, \quad t \to \infty, \quad i = 1, \ldots, n, \tag{1.9}$$

which may be regarded as an "approximation" of (1.8). Here and hereafter the symbol $\sim$ is used to mean the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

The exposition of the analysis of system (1.9) by means of regularly varying functions is given in Section 3, which is preceded by Section 2 in which the definition and some basic properties of regularly varying functions are summarized for the reader's benefit. In the final Section 5 it is shown that our main results for (1.1) can be effectively applied to some classes of partial differential equations with radial symmetry including metaharmonic equations and systems involving p-Laplace operators on exterior domains in $\mathbb{R}^N$.

Since the publication of the book [14] of Marić in the year 2000 there has been an increasing interest in the study of differential equations in the framework of regularly varying functions and as a consequence the theory of regular variation has proved to be a powerful tool in the asymptotic analysis of differential equations, giving rise to detailed and accurate information about the existence, the asymptotic behavior and the structure of positive solutions of various types of ordinary differential equations. See, for example, the papers [6, 7, 9–13].

2. REGULARLY VARYING FUNCTIONS

For the reader's convenience we summarize here the definition and some basic properties of regularly varying functions.

Definition 2.1. A measurable function $f : [0, \infty) \to (0, \infty)$ is called *regularly varying* (*at infinity*) *of index* $\rho \in \mathbb{R}$ (written $f \in \mathrm{RV}(\rho)$) if for all $\lambda > 0$

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho.$$

The simplest example of a regularly varying function is the power function ct^ρ where $c > 0$ is constant and $\rho \in \mathbb{R}$, or, more generally, the function $ct^\rho(1 + \varepsilon(t))$ where $\varepsilon(t)$ is a measurable function on $(0, \infty)$ such that $\varepsilon(t) \to 0$ as $t \to \infty$.

We often use the symbol SV to denote RV(0) and call members of SV *slowly varying functions*. Typical representatives of the class SV are measurable functions on $[a, \infty)$ which have a (finite) positive limit at infinity, the logarithmic function, its powers $(\log t)^\beta$, $\beta \in \mathbb{R}$ and its iterates $\log_m t$ defined by $\log_m t = \log \log_{m-1} t$, $m = 2, 3, \ldots$. Another nontrivial example is

$$\exp\{(\log t)^{\beta_1} (\log_2 t)^{\beta_2} \ldots (\log_m t)^{\beta_m}\},$$

where $\beta_i \in (0, 1)$. Examples of oscillating slowly varying functions are $2+\sin(\log \log t)$, $t > e^e$, and

$$L(t) = \exp\Big\{(\log t)^\theta \cos\,(\log t)^\theta\Big\}, \quad \theta \in \Big(0, \frac{1}{2}\Big),$$

which satisfies

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} L(t) = 0.$$

It is easy to see that a regularly varying function $f(t)$ of index ρ can always be represented as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$, and so the class SV of slowly varying functions is of fundamental importance in the theory of regular variation.

One of the most important properties of regularly varying functions is the following *representation theorem.*

Proposition 2.2. *$f(t) \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \right\}, \quad t \geq t_0, \tag{2.1}$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If in particular $c(t) \equiv c_0$ in (2.1), then $f(t)$ is referred to as a *normalized* regularly varying function of index ρ.

The following result illustrates operations which preserve slow variation.

Proposition 2.3. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^\alpha$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following proposition.

Proposition 2.4. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^\varepsilon f(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

The following result called Karamata's integration theorem is of highest importance in handling slowly and regularly varying functions analytically, and will be used throughout the paper.

Proposition 2.5. *Let $L(t) \in \mathrm{SV}$. Then:*

(i) *if $\alpha > -1$,*

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \textit{and} \quad m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV},$$

provided $L(t)/t$ is integrable near the infinity in the latter case.

Definition 2.6. A vector function $(x_1(t), \ldots, x_n(t))$ is said to be regularly varying of index $(\rho_1, \ldots, \rho_n)$ if $x_i \in \mathrm{RV}(\rho_i)$ for $i = 1, \ldots, n$. If all ρ_i are positive (or negative), then $(x_1(t), \ldots, x_n(t))$ is called regularly varying of positive (or negative) index $(\rho_1, \ldots, \rho_n)$. The set of all regularly varying vector functions of index $(\rho_1, \ldots, \rho_n)$ is denoted by $\mathrm{RV}(\rho_1, \ldots, \rho_n)$.

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels [2]. See also Seneta [15]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [14].

3. ASYMPTOTIC RELATIONS ASSOCIATED WITH (1.1)

We assume that $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ and that they are expressed as

$$p_i(t) = t^{\lambda_i} l_i(t), \quad q_i(t) = t^{\mu_i} m_i(t), \quad l_i,\ m_i \in \mathrm{SV}, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

and seek positive increasing solutions $(x_1(t), \ldots, x_n(t))$ of system (1.1) consisting of components $x_i \in \mathrm{RV}(\rho_i)$, $\rho_i > 0$, represented in the form

$$x_i(t) = t^{\rho_i} \xi_i(t), \quad \xi_i \in \mathrm{SV}, \quad i = 1, \ldots, n. \tag{3.2}$$

We note that condition (1.3) is satisfied if either

$$\lambda_i < \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-\frac{1}{\alpha_i}} dt = \infty, \tag{3.3}$$

while condition (1.4) is satisfied if either

$$\lambda_i > \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-\frac{1}{\alpha_i}} dt < \infty. \tag{3.4}$$

In analyzing strongly increasing solutions of system (1.1) it is convenient to distinguish the case where $p_i(t)$ satisfy (1.3) from the case where $p_i(t)$ satisfy (1.4). For the case of (1.4), which is equivalent to (3.4) holding for $i = 1, \dots, n$, the solutions $(x_1(t), \dots, x_n(t))$ of (1.1) will be sought in the class $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > 0$, $i = 1, \dots, n$. For the case of (1.3), however, our attention will be focused on the two extreme cases:

$$\text{(a)} \quad \lambda_i = \alpha_i, \quad i = 1, \dots, n, \quad \text{and} \quad \text{(b)} \quad \lambda_i < \alpha_i, \quad i = 1, \dots, n,$$

which imply, respectively, that

$$P_i(t) = \int_a^t s^{-1} l_i(s)^{-\frac{1}{\alpha_i}} ds \in \mathrm{SV} \ \text{ and } \ P_i(t) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-\frac{1}{\alpha_i}} \in \mathrm{RV}\Big(\frac{\alpha_i - \lambda_i}{\alpha_i}\Big),$$

and an attempt will be made to detect solutions belonging to $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > 0$, $i = 1, \dots, n$, or to $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, according to whether (1.1) or (b) holds, respectively.

Let $(x_1(t), \dots, x_n(t))$ be a strongly increasing solution of (1.1) on $[T, \infty)$. Integrating (1.1) twice on $[T, t]$, we have

$$x_i(t) = c_{i0} + \int_T^t \left[\frac{1}{p_i(s)}\left(c_{i1} + \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr\right)\right]^{\frac{1}{\alpha_i}} ds, \quad t \geq T, \quad i = 1, \dots, n, \tag{3.5}$$

where $c_{i0} = x_i(T) > 0$ and $c_{i1} = p_i(T) x_i'(T)^{\alpha_i} \geq 0$. This applies to both cases (1.3) and (1.4). Note that in view of (1.7) the solution is required to satisfy

$$\int_T^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds = \infty, \quad i = 1, \dots, n. \tag{3.6}$$

Our task is to solve the system of integral equations (3.5) plus (3.6) in the class of regularly varying functions. This can be accomplished through the analysis of regularly varying functions satisfying the system of integral asymptotic relations

$$x_i(t) \sim \int_T^t \left(\frac{1}{p_i(s)} \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr\right)^{\frac{1}{\alpha_i}} ds, \quad t \to \infty, \quad i = 1, \dots, n. \tag{3.7}$$

It turns out that one can acquire thorough knowledge of all possible regularly varying solutions of positive indices of (3.7) plus (3.6), and this fact will play an essential role in constructing the strongly increasing solutions of system (1.1) by means of fixed point techniques and in determining their accurate asymptotic behavior at infinity.

We begin by considering system (3.7) with $p_i(t)$ satisfying condition (1.4). Note that $\lambda_i \geq \alpha_i$, $i = 1, \ldots, n$ (cf. (3.4)). Suppose that (3.6)–(3.7) has a positive solution $(x_1(t), \ldots, x_n(t)) \in \mathrm{RV}(\rho_1, \ldots, \rho_n)$ on $[T, \infty)$ with $\rho_i > 0$, $i = 1, \ldots, n$. Using (3.1) and (3.2), we have

$$\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds = \int_T^t s^{\mu_i + \beta_i \rho_{i+1}} m_i(s) \xi_{i+1}(s)^{\beta_i} ds, \quad t \geq T, \quad i = 1, \ldots, n. \tag{3.8}$$

The divergence of (3.6) as $t \to \infty$ implies $\mu_i + \beta_i \rho_{i+1} \geq -1$, $i = 1, \ldots, n$. It should be noted that the equality is not allowed in any of the last inequalities. In fact, if $\mu_i + \beta_i \rho_{i+1} = -1$ for some i, then from (3.8) we obtain

$$\left(\frac{1}{p_i(t)} \int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds \right)^{\frac{1}{\alpha_i}} = t^{-\frac{\lambda_i}{\alpha_i}} l_i(t)^{-\frac{1}{\alpha_i}} \left(\int_T^t s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds \right)^{\frac{1}{\alpha_i}}, \tag{3.9}$$

for $t \geq T$. If $\lambda_i > \alpha_i$, then the right-hand side of (3.9) is integrable on $[T, \infty)$, and so (3.7) implies that $\lim_{t \to \infty} x_i(t) = \mathrm{const} > 0$, which is impossible. If $\lambda_i = \alpha_i$, then integrating (3.9) gives

$$x_i(t) \sim \int_T^t s^{-1} l_i(s)^{-\frac{1}{\alpha_i}} \left(\int_T^s r^{-1} m_i(r) \xi_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds \in \mathrm{SV} = \mathrm{RV}(0),$$

which contradicts the assumption that $\rho_i > 0$. Therefore, it holds that $\mu_i + \beta_i \rho_{i+1} > -1$ for $i = 1, \ldots, n$. Then, applying Karamata's integration theorem to (3.8), we see that

$$\left(\frac{1}{p_i(t)} \int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds \right)^{\frac{1}{\alpha_i}} \sim \frac{t^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}} \xi_{i+1}(t)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{\frac{1}{\alpha_i}}}, \quad t \to \infty, \tag{3.10}$$

for $i = 1, \ldots, n$. Since the left-hand side of (3.10) is not integrable on $[T, \infty)$ we see that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$, $i = 1, \ldots, n$. Suppose that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i = -1$ for some i. Then, $\mu_i + \beta_i \rho_{i+1} + 1 = \lambda_i - \alpha_i \geq 0$, which means that the possibility $\lambda_i = \alpha_i$ is ruled out, that is, $\lambda_i > \alpha_i$. In this case, integrating (3.10) from T to t and using (3.7), we obtain

$$x_i(t) \sim (\lambda_i - \alpha_i)^{-\frac{1}{\alpha_i}} \int_T^t s^{-1} l_i(s)^{-\frac{1}{\alpha_i}} m_i(s)^{\frac{1}{\alpha_i}} \xi_{i+1}(s)^{\frac{\beta_i}{\alpha_i}} ds \in \mathrm{SV}, \quad t \to \infty,$$

a contradiction. Consequently, we must have $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i > -1$ for all i, in which case, integrating (3.10) on $[T, t]$, we conclude via Karamata's integration theorem and (3.7) that

$$x_i(t) \sim \frac{t^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}} \xi_{i+1}(t)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{\frac{1}{\alpha_i}} \left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} + 1\right)}, \quad t \to \infty, \quad i = 1, \dots, n. \tag{3.11}$$

This shows that $\rho_i = (-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i + 1$, which can be expressed as

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = 1, \dots, n. \tag{3.12}$$

To solve the algebraic linear system (3.12) in ρ_i, $i = 1, \dots, n$, it suffices to observe that the coefficient matrix

$$A = A\Big(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n}\Big) = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{\beta_{n-1}}{\alpha_{n-1}} \\ -\frac{\beta_n}{\alpha_n} & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{3.13}$$

is invertible, because

$$|A| = \frac{A_n - B_n}{A_n} > 0, \quad \text{where} \quad A_n = \alpha_1\alpha_2 \dots \alpha_n, \quad B_n = \beta_1\beta_2 \dots \beta_n, \tag{3.14}$$

and its inverse is given explicitly by

$$A^{-1} = \frac{A_n}{A_n - B_n} \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \dots & \dots & \frac{\beta_1\beta_2\dots\beta_{n-1}}{\alpha_1\alpha_2\dots\alpha_{n-1}} \\ & 1 & \frac{\beta_2}{\alpha_2} & \frac{\beta_2\beta_3}{\alpha_2\alpha_3} & \dots & \frac{\beta_2\beta_3\dots\beta_{n-1}}{\alpha_2\alpha_3\dots\alpha_{n-1}} \\ & & 1 & \frac{\beta_3}{\alpha_3} & \dots & \frac{\beta_3\dots\beta_{n-1}}{\alpha_3\dots\alpha_{n-1}} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \frac{\beta_{n-1}}{\alpha_{n-1}} \\ * & & & & & 1 \end{pmatrix}, \tag{3.15}$$

where the lower triangular elements are omitted for economy of notation. Let (M_{ij}) denote the matrix on the right-hand side of (3.15). It is easy to see that the i-th row of (M_{ij}) is obtained by shifting the vector

$$\Big(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \dots, \frac{\beta_i\beta_{i+1}\dots\beta_{i+(n-2)}}{\alpha_i\alpha_{i+1}\dots\alpha_{i+(n-2)}}\Big) \qquad (\alpha_{n+1} = \alpha_1, \;\; \beta_{n+1} = \beta_1)$$

$(i-1)$-times to the right cyclically, so that the lower triangular elements M_{ij}, $i > j$, satisfy the relations

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2\ldots\beta_n}{\alpha_1\alpha_2\ldots\alpha_n}, \quad i > j, \quad i = 1, 2, \ldots, n. \tag{3.16}$$

Then the unique solution ρ_i, $i = 1, \ldots, n$, of (3.12) is given explicitly by

$$\rho_i = \frac{A_n}{A_n - B_n}\sum_{j=1}^{n} M_{ij}\frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = 1, \ldots, n, \tag{3.17}$$

from which it follows that all ρ_i are positive if

$$\sum_{j=1}^{n} M_{ij}\frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} > 0, \quad i = 1, \ldots, n. \tag{3.18}$$

We observe that (3.11) can be rewritten in the form

$$x_i(t) \sim \frac{t^{\frac{\alpha_i+1}{\alpha_i}} p_i(t)^{-\frac{1}{\alpha_i}} q_i(t)^{\frac{1}{\alpha_i}} x_{i+1}(t)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad t \to \infty, \tag{3.19}$$

where

$$D_i = (\lambda_i - \alpha_i + \alpha_i\rho_i)^{\frac{1}{\alpha_i}}\rho_i, \tag{3.20}$$

for $i = 1, \ldots, n$. This is a cyclic system of asymptotic relations, from which one can derive without difficulty the following independent explicit asymptotic formulas for each $x_i(t)$:

$$x_i(t) \sim \left[\prod_{j=1}^{n}\left(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-\frac{1}{\alpha_j}} q_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n-B_n}}, \quad t \to \infty, \quad i = 1, \ldots, n. \tag{3.21}$$

Notice that (3.21) is rewritten in the form

$$x_i(t) \sim t^{\rho_i}\left[\prod_{j=1}^{n}\left(\frac{l_j(t)^{-\frac{1}{\alpha_j}} m_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n-B_n}}, \quad t \to \infty, \quad i = 1, \ldots, n. \tag{3.22}$$

We now assume that (3.18) holds, define the constants ρ_i by (3.17) and consider the functions $X_i(t) \in \mathrm{RV}(\rho_i)$ on $[a, \infty)$ defined by

$$X_i(t) = \left[\prod_{j=1}^{n}\left(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-\frac{1}{\alpha_j}} q_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n-B_n}}, \quad i = 1, \ldots, n. \tag{3.23}$$

Then $X_i(t)$ satisfy the system of asymptotic relations (3.7), i.e.,

$$\int_b^t \left(\frac{1}{p_i(s)}\int_b^s q_i(r)X_{i+1}(r)^{\beta_i}dr\right)^{\frac{1}{\alpha_i}} ds \sim X_i(t), \quad t \to \infty, \quad i = 1, , , n, \tag{3.24}$$

for any $b \geq a$, where $X_{n+1}(t) = X_1(t)$. In fact, noting that $X_i(t)$ are expressed as

$$X_i(t) = t^{\rho_i}\Xi_i(t), \quad \Xi_i(t) = \left[\prod_{j=1}^{n}\left(\frac{l_j(t)^{-\frac{1}{\alpha_j}} m_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n - B_n}},$$

and using Karamata's integration theorem, we obtain

$$\left(\frac{1}{p_i(t)}\int_b^t q_i(s)X_{i+1}(s)^{\beta_i}\,ds\right)^{\frac{1}{\alpha_i}} \sim \frac{t^{\rho_i - 1} l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}} \Xi_{i+1}(t)^{\frac{\beta_i}{\alpha_i}}}{(\lambda_i - \alpha_i + \alpha_i\rho_i)^{\frac{1}{\alpha_i}}},$$

and

$$\int_b^t \left(\frac{1}{p_i(s)}\int_b^s q_i(r)X_{i+1}(r)^{\beta_i}\,dr\right)^{\frac{1}{\alpha_i}} ds \sim \frac{t^{\rho_i} l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}} \Xi_{i+1}(t)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \tag{3.25}$$

as $t \to \infty$. Since simple calculation with the help of the relations

$$M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_n}{A_n}, \qquad M_{i+1.j}\frac{\beta_i}{\alpha_i} = M_{ij}, \quad \text{for } j \neq i$$

between the i-th and $(i+1)$-th rows of the matrix A shows that

$$\begin{aligned}
&\frac{l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}}}{D_i}\Xi_{i+1}(t)^{\frac{\beta_i}{\alpha_i}} \\
&= \frac{l_i(t)^{-\frac{1}{\alpha_i}} m_i(t)^{\frac{1}{\alpha_i}}}{D_i}\left[\prod_{j=1}^{n}\left(\frac{l_j(t)^{-\frac{1}{\alpha_j}} m_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}}\right]^{\frac{A_n}{A_n - B_n}} \\
&= \left[\prod_{j=1}^{n}\left(\frac{l_j(t)^{-\frac{1}{\alpha_j}} m_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n - B_n}} = \Xi_i(t),
\end{aligned}$$

we conclude from (3.25) that $X_i(t)$ satisfy the asymptotic relations (3.24) as desired.

Summarizing the above discussions, we obtain the following noteworthy result which provide complete information about the existence and asymptotic behavior of regularly varying solutions with positive indices of system (3.7).

Theorem 3.1. *Suppose that* $p_i \in \mathrm{RV}(\lambda_i)$ *and* $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \ldots, n$, *and that* $p_i(t)$ *satisfy condition* (1.4). *System of asymptotic relations* (3.7) *has regularly varying solutions* $(x_1(t), \ldots, x_n(t)) \in \mathrm{RV}(\rho_1, \ldots, \rho_n)$ *with* $\rho_i > 0$, $i = 1, \ldots, n$, *if and only if* (3.18) *holds, in which case* ρ_i *are uniquely determined by* (3.17) *and the asymptotic behavior of any such solution is governed by the formula* (3.21).

Next we consider the case where $p_i(t)$ satisfy condition (1.3) and show that for the two special cases (i) $\lambda_i = \alpha_i$ and (ii) $\lambda_i < \alpha_i$ for all $i = 1, \ldots, n$, complete analysis can be made of strongly increasing solutions of systems of asymptotic relations (3.7) in the framework of regular variation.

Theorem 3.2. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ for $i = 1, \dots, n$ and $p_i(t)$ satisfy condition* (1.3)*.*

(i) *Suppose that $\lambda_i = \alpha_i$, $i = 1, \dots, n$. System* (3.7) *has regularly varying solutions $(x_1(t), \dots, x_n(t)) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > 0$, $i = 1, \dots, n$, if and only if*

$$\sum_{j=1}^{n} M_{ij} \frac{\mu_j + 1}{\alpha_j} > 0, \quad i = 1, \dots, n, \tag{3.26}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_n}{A_n - B_n} \sum_{j=1}^{n} M_{ij} \frac{\mu_j + 1}{\alpha_j}, \quad i = 1, \dots, n, \tag{3.27}$$

and the asymptotic behavior of any such solutions is governed by the set of formulas (3.21) *with $D_j = \alpha_j^{1/\alpha_j} \rho_j^{(\alpha_j+1)/\alpha_j}$, $j = 1, \dots, n$.*

(ii) *Suppose that $\lambda_i < \alpha_i$, $i = 1, \dots, n$. System* (3.7) *has regularly varying solutions $(x_1(t), \dots, x_n(t)) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, if and only if*

$$\sum_{j=1}^{n} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) > 0, \quad i = 1, \dots, n, \tag{3.28}$$

where $\alpha_{n+1} = \alpha_1, \lambda_{n+1} = \lambda_1$, in which case ρ_i are uniquely determined by (3.17) *and the asymptotic behavior of any such solution is governed by the set of formulas* (3.21)*.*

Proof. (i) Suppose that (3.7) has a solution $(x_1(t), \dots, x_n(t)) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with all $\rho_i > 0$. Starting from (3.8) one can proceed exactly as in the proof of Theorem 3.1 to reach the conclusion that (3.18) holds, that ρ_i are given by (3.17) and that all the components $x_i(t)$ must obey the unique growth law (3.21). Note that since $\lambda_i = \alpha_i$, (3.18) and (3.17) are simplified to (3.26) and (3.27), respectively, and in (3.21) D_j reduce to $D_j = \alpha_j^{1/\alpha_j} \rho_j^{(\alpha_j+1)/\alpha_j}$. This proves the "only if" part. To prove the "if" part we need only to simply repeat the same argument as in Theorem 3.1.

(ii) Suppose that (3.7) has a solution $(x_1(t), \dots, x_n(t)) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$. We claim that $\mu_i + \beta_i \rho_{i+1} > -1$ for all i. In fact, if $\mu_i + \beta_i \rho_{i+1} = -1$ for some i, then integrating (3.9) on $[T, t]$ and using Karamata's integration theorem, we have

$$x_i(t) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-\frac{1}{\alpha_i}} \left(\int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds \right)^{\frac{1}{\alpha_i}} \in \mathrm{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i} \right), \quad t \to \infty,$$

which contradicts the hypothesis that $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$. Therefore, $\mu_i + \beta_i \rho_{i+1} > -1$ for all i and we see that (3.10) holds. The divergence of the integral of (3.10) on $[T, \infty)$ implies that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$ for all i, but all of them should hold as strict inequalities because the equality for some i would imply that $0 < \mu_i + \beta_i \rho_{i+1} + 1 = \lambda_i - \alpha_i < 0$, an impossibility. This fact allows us to apply Karamata's integration theorem to the integral of (3.10) on $[T, t]$, and as a consequence we obtain (3.11) which shows that ρ_i must satisfy (3.12) so that they are determined uniquely by the formulas (3.17). Putting $\sigma_i = \rho_i - (\alpha_i - \lambda_i)/\alpha_i > 0$, we transform (3.12) into

$$\sigma_i - \frac{\beta_i}{\alpha_i}\sigma_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i \alpha_{i+1}}, \quad i = 1, \dots, n,$$

from which (3.28) follows immediately. Finally, the asymptotic formula (3.21) follows from the system of cyclic relations (3.19) into which (3.11) is transformed. Conversely, suppose that (3.28) holds. If we define ρ_i by (3.17) and the functions $X_i(t)$ by (3.23), then it can be easily verified that $X_i(t)$ satisfy the system of integral asymptotic relations (3.24). This completes the proof. □

Remark 3.3. It is easily seen that Theorem 3.1 and (i) of Theorem 3.2 can be unified into the following theorem.

Theorem 3.4. *Suppose that* $p_i \in \mathrm{RV}(\lambda_i)$ *and* $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. *Suppose in addition that* $\lambda_i \geq \alpha_i$, $i = 1, \dots, n$. *System* (3.7) *has regularly varying solutions* $(x_1(t), \dots, x_n(t)) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ *with* $\rho_i > 0$, $i = 1, \dots, n$, *if and only if* (3.18) *holds, in which case* ρ_i *are uniquely determined by* (3.17) *and the asymptotic behavior of any such solution is governed by the set of formulas* (3.21).

Note that this result applies to those systems of the form (3.7) in which some or all of $p_i(t)$ such that $\lambda_i = \alpha_i$ satisfy the condition $\int_a^\infty p_i(t)^{-1/\alpha_i} dt = \infty$.

4. STRONGLY INCREASING SOLUTIONS OF (1.1)

This section is devoted to the study of the existence and the asymptotic behavior of strongly increasing solutions of system (1.1) which are regularly varying of positive indices. Our main results are the following two theorems in which use is made of the notation and properties of matrix (3.13) and its inverse (3.15).

Theorem 4.1. *Let* $p_i \in \mathrm{RV}(\lambda_i)$ *and* $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. *Suppose that* $\lambda_i \geq \alpha_i$, $i = 1, \dots, n$. *System* (1.1) *possesses strongly increasing solutions in* $\mathrm{RV}(\rho_1, \dots, \rho_n)$ *with* $\rho_i > 0$, $i = 1, \dots, n$, *if and only if* (3.18) *holds, in which case* ρ_i *are given by* (3.17) *and the asymptotic behavior of any such solution* $(x_1(t), \dots, x_n(t))$ *is governed by the set of formulas* (3.21).

Theorem 4.2. *Let* $p_i \in \mathrm{RV}(\lambda_i)$ *and* $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. *Suppose that* $\lambda_i < \alpha_i$, $i = 1, \dots, n$. *System* (1.1) *possesses strongly increasing solutions in* $\mathrm{RV}(\rho_1, \dots, \rho_n)$ *with* $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, *if and only if* (3.28) *holds, in which case* ρ_i *are given by* (3.17) *and the asymptotic behavior of any such solution* $(x_1(t), \dots, x_n(t))$ *is governed by the set of formulas* (3.21).

We note that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 3.3 and (ii) of Theorem 3.2. The "if" parts are proved by way of the following results ensuring the existence of strongly increasing solutions for systems of the form (1.1) with nearly regularly varying coefficients $p_i(t)$ and $q_i(t)$ in the sense defined below.

Definition 4.3. Let $f(t)$ be a regularly varying function of index σ and suppose that $g(t)$ satisfies $kf(t) \leq g(t) \leq Kf(t)$ for some positive constants k, K and for all large t. Then $g(t)$ is said to be a *nearly regularly varying function of index* σ. Such a relation between $f(t)$ and $g(t)$ is denoted by $g(t) \asymp f(t)$ as $t \to \infty$.

Theorem 4.4. *Let $p_i(t)$ and $q_i(t)$ be nearly regularly varying indices λ_i and μ_i, respectively, that is, there exist $\tilde{p}_i \in \mathrm{RV}(\lambda_i)$ and $\tilde{q}_i \in \mathrm{RV}(\mu_i)$ such that*

$$p_i(t) \asymp \tilde{p}_i(t), \quad q_i(t) \asymp \tilde{q}_i(t), \quad t \to \infty, \quad i = 1, \dots, n. \tag{4.1}$$

Suppose in addition that $\lambda_i \geq \alpha_i$, $i = 1, \dots, n$, and that (3.18) *holds. Then, system* (1.1) *possesses strongly increasing solutions $(x_1(t), \dots, x_n(t))$ which are nearly regularly varying of positive index $(\rho_1, \dots, \rho_n)$ in the sense that*

$$x_i(t) \asymp \left[\prod_{j=1}^{n}\left(\frac{t^{\frac{\alpha_j+1}{\alpha_j}}\tilde{p}_j(t)^{-\frac{1}{\alpha_j}}\tilde{q}_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n-B_n}}, \quad t \to \infty, \quad i = 1, \dots, n, \tag{4.2}$$

where ρ_i and D_j are defined by (3.17) *and* (3.20)*, respectively.*

Theorem 4.5. *Let $p_i(t)$ and $q_i(t)$ be nearly regularly varying of indices λ_i and μ_i, respectively, $i = 1, \dots, n$. Suppose that $\lambda_i < \alpha_i$, $i = 1, \dots, n$, and that* (3.28) *holds. Then, system* (1.1) *possesses strongly increasing solutions $(x_1(t), \dots, x_n(t))$ which are nearly regularly varying of positive index $(\rho_1, \dots, \rho_n)$ such that $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = 1, \dots, n$, and satisfy* (4.2)*, where ρ_i and D_j are defined by* (3.17) *and* (3.20)*, respectively.*

Proof of Theorem 4.4. We assume that the regularly varying functions $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ in (4.1) are expressed as

$$\tilde{p}_i(t) = t^{\lambda_i} l_i(t), \quad \tilde{q}_i(t) = t^{\mu_i} m_i(t), \quad l_i, m_i \in \mathrm{SV}. \tag{4.3}$$

By hypothesis there exist positive constants h_i, H_i, k_i and K_i such that

$$h_i\tilde{p}_i(t) \leq p_i(t) \leq H_i\tilde{p}_i(t), \quad k_i\tilde{q}_i(t) \leq q_i(t) \leq K_i\tilde{q}_i(t), \quad t \geq a, \quad i = 1, \dots, n. \tag{4.4}$$

Let the functions $X_i(t) \in \mathrm{RV}(\rho_i)$ be defined by

$$X_i(t) = t^{\rho_i}\left[\prod_{j=1}^{n}\left(\frac{l_j(t)^{-\frac{1}{\alpha_j}}m_j(t)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_n}{A_n-B_n}}, \quad i = 1, \dots, n. \tag{4.5}$$

It is known that

$$\int_b^t\left(\frac{1}{\tilde{p}_i(t)}\int_b^s \tilde{q}_i(r)X_{i+1}(r)^{\beta_i}dr\right)^{\frac{1}{\alpha_i}}ds \sim X_i(t), \quad t \to \infty, \quad i = 1, \dots, n, \tag{4.6}$$

for any $b \geq a$. By (4.6), there exists $T > a$ such that

$$\int_T^t \left(\frac{1}{\tilde{p}_i(t)} \int_T^s \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds \leq 2X_i(t), \quad t \geq T, \quad i = 1, \dots, n. \tag{4.7}$$

We may assume that each $X_i(t)$ is increasing on $[T, \infty)$ because it is known ([2, Theorem 1.5.3]) that any regularly varying function of positive index is asymptotic to an increasing function. Since (4.6) holds with $b = T$, one can choose $T_1 > T$ so large that

$$\int_T^t \left(\frac{1}{\tilde{p}_i(t)} \int_T^s \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds \geq \frac{1}{2} X_i(t) \quad \text{for} \quad t \geq T_1, \quad i = 1, \dots, n. \tag{4.8}$$

Define the positive constants l_i and L_i ($l_i \leq L_i$) by

$$l_i = \left[\prod_{j=1}^n \frac{1}{2} \left(\frac{k_j}{H_j} \right)^{\frac{M_{ij}}{\alpha_j}} \right]^{\frac{A_n}{A_n - B_n}}, \qquad L_i = \left[\prod_{j=1}^n 4 \left(\frac{K_j}{h_j} \right)^{\frac{M_{ij}}{\alpha_j}} \right]^{\frac{A_n}{A_n - B_n}}, \quad i = 1, \dots, n. \tag{4.9}$$

It is easy to see that l_i and L_i in (4.9) satisfy the cyclic systems of equations

$$l_i = \frac{1}{2} \left(\frac{k_i}{H_i} \right)^{\frac{1}{\alpha_i}} l_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad L_i = 4 \left(\frac{K_i}{h_i} \right)^{\frac{1}{\alpha_i}} L_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = 1, \dots, n, \quad (l_{n+1} = l_1, \ L_{n+1} = L_1).$$

Since

$$\frac{L_i}{l_i} = \left[\prod_{j=1}^n 8 \left(\frac{H_j K_j}{h_j k_j} \right)^{\frac{M_{ij}}{\alpha_j}} \right]^{\frac{A_n}{A_n - B_n}},$$

one can choose the constants h_i, H_i, k_i and K_i so that $L_i/l_i \geq 2X_i(T_1)/X_i(T)$, that is,

$$2l_i X_i(T_1) \leq L_i X_i(T), \quad i = 1, \dots, n, \tag{4.10}$$

because these constants are independent of $X_i(t)$ and the choice of T and T_1.

Let us now define $\mathcal{X}$ to be the set of continuous vector functions $(x_1(t), \dots, x_n(t))$ on $[T, \infty)$ satisfying

$$l_i X_i(t) \leq x_i(t) \leq L_i X_i(t), \quad t \geq T, \quad i = 1, \dots, n. \tag{4.11}$$

Clearly, $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T, \infty)^n$. Let $\mathcal{F}_i$ denote the integral operators

$$\mathcal{F}_i x(t) = c_i + \int_T^t \left(\frac{1}{p_i(s)} \int_T^s q_i(r) x(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds, \quad t \geq T, \quad i = 1, \dots, n, \tag{4.12}$$

where c_i are positive constants such that

$$ {}_i X_i(T_1) \le c_i \le \frac{1}{2} L_i X_i(T), \quad i = 1, \dots, n, \tag{4.13} $$

and define the mapping $\Phi : \mathcal{X} \to C[T, \infty)^n$ by

$$ \begin{aligned} &\Phi(x_1, x_2, \dots, x_n)(t) \\ &= (\mathcal{F}_1 x_2(t), \mathcal{F}_2 x_3(t), \dots, \mathcal{F}_n x_{n+1}(t)), \quad t \ge T, \quad (x_{n+1}(t) = x_1(t)). \end{aligned} \tag{4.14} $$

It can be shown that Φ is a self-map on $\mathcal{X}$ and sends $\mathcal{X}$ into a relatively compact subset of $C[T, \infty)^n$, so that Φ has a fixed point in $\mathcal{X}$ by the Schauder-Tychonoff fixed point theorem.

(i) $\Phi(\mathcal{X}) \in \mathcal{X}$. Let $(x_1, \dots, x_n) \in \mathcal{X}$. Then, using (4.7)–(4.14) we see that

$$ \mathcal{F}_i x_{i+1}(t) \ge c_i \ge l_i X_i(T_1) \ge l_i X_i(t) \quad \text{for} \quad T \le t \le T_1, $$

$$ \begin{aligned} \mathcal{F}_i x_{i+1}(t) &\ge \left(\frac{k_i l_{i+1}^{\beta_i}}{H_i} \right)^{\frac{1}{\alpha_i}} \int_T^t \left(\frac{1}{\tilde{p}_i(s)} \int_T^s \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds \\ &\ge \frac{1}{2} \left(\frac{k_i l_{i+1}^{\beta_i}}{H_i} \right)^{\frac{1}{\alpha_i}} X_i(t) \ge l_i X_i(t) \quad \text{for} \quad t \ge T_1, \end{aligned} $$

and

$$ \begin{aligned} \mathcal{F}_i x_{i+1}(t) &\le \frac{1}{2} L_i X_i(T) + \left(\frac{K_i L_{i+1}^{\beta_i}}{h_i} \right)^{\frac{1}{\alpha_i}} \int_T^t \left(\frac{1}{\tilde{p}_i(s)} \int_T^s \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds \\ &\le \frac{1}{2} L_i X_i(T) + 2 \left(\frac{K_i L_{i+1}^{\beta_i}}{h_i} \right)^{\frac{1}{\alpha_i}} X_{i+1}(t) \le \frac{1}{2} L_i X_i(t) + \frac{1}{2} L_i X_i(t) \\ &= L_i X_i(t) \quad \text{for} \quad t \ge T. \end{aligned} $$

This shows that $\Phi(x_1, \dots, x_n) \in \mathcal{X}$, that is, Φ maps $\mathcal{X}$ into itself.

(ii) $\Phi(\mathcal{X})$ is relatively compact. The inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ implies that $\Phi(\mathcal{X})$ is locally uniformly bounded on $[T, \infty)$. From the inequalities

$$ 0 \le (\mathcal{F}_i x_{i+1})'(t) \le L_{i+1}^{\frac{\beta_i}{\alpha_i}} \left(\frac{1}{p_i(t)} \int_T^t q_i(s) X_{i+1}(s)^{\beta_i} ds \right)^{\frac{1}{\alpha_i}}, \quad t \ge T, \quad i = 1, \dots, n, $$

holding for all $(x_1, \dots, x_n) \in \mathcal{X}$ it follows that $\Phi(\mathcal{X})$ is locally equicontinuous on $[T, \infty)$. The relative compactness of $\Phi(\mathcal{X})$ then follows from the Arzela-Ascoli lemma.

(iii) Φ is a continuous map. Let $\{(x_1^\nu(t), \dots, x_n^\nu(t))\}$ be a sequence in $\mathcal{X}$ converging as $\nu \to \infty$ to $(x_1(t), \dots, x_n(t)) \in \mathcal{X}$ uniformly on compact subintervals of $[T, \infty)$. Using (4.12) we obtain

$$ |\mathcal{F}_i x_{i+1}^\nu(t) - \mathcal{F}_i x_{i+1}(t)| \le \int_T^t p_i(s)^{-\frac{1}{\alpha_i}} F_i^\nu(s) ds, \quad t \ge T, \tag{4.15} $$

where

$$F_i^\nu(t) = \left| \Bigg(\int_T^t q_i(s) x_{i+1}^\nu(s)^{\beta_i} ds \Bigg)^{\frac{1}{\alpha_i}} - \Bigg(\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds \Bigg)^{\frac{1}{\alpha_i}} \right|.$$

It is easy to see that

$$F_i^\nu(t) \le \Bigg(\int_T^t q_i(s) \Big| x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i} \Big| ds \Bigg)^{\frac{1}{\alpha_i}}, \tag{4.16}$$

if $\alpha_i \ge 1$ and

$$F_i^\nu(t) \le \frac{1}{\alpha_i} \Bigg(L_{i+1} \int_T^t q_i(s) X_{i+1}(s)^{\beta_i} ds \Bigg)^{\frac{1}{\alpha_i}-1} \int_T^t q_i(s) \Big| x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i} \Big| ds, \tag{4.17}$$

if $\alpha_i < 1$. Combining (4.15) with (4.16) or (4.17) and applying the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{\nu\to\infty} \mathcal{F}_i x_{i+1}^\nu(t) = \mathcal{F}_i x_{i+1}(t) \text{ uniformly on any compact subset of } [T,\infty),\ i = 1,\dots,n,$$

which proves the continuity of Φ.

Therefore, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $(x_1,\dots,x_n) \in \mathcal{X}$ of Φ, which satisfies

$$x_i(t) = \mathcal{F}_i x_{i+1}(t) = c_i + \int_T^t \Bigg(\frac{1}{p_i(s)} \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr \Bigg)^{\frac{1}{\alpha_i}} ds, \quad t \ge T, \quad i = 1,\dots,n. \tag{4.18}$$

This shows that $(x_1(t),\dots,x_n(t))$ is a solution of system (1.1) on $[T,\infty)$. Since the solution obtained is a member of $\mathcal{X}$, it is nearly regularly varying of positive index $(\rho_1,\dots,\rho_n)$ and hence is a strongly increasing solution of (1.1). This completes the proof. □

The proof of Theorem 4.5 is essentially the same as above, and so it may be omitted.

To complete the proof of the "if" parts of Theorems 4.1 and 4.2 it suffices to verify the regularity of the nearly regularly varying solutions $(x_1,\dots,x_n)$ obtained if $p_i(t)$ and $q_i(t)$ are assumed to be regularly varying functions. To this end the following generalized L'Hospital's rule is utilized. See, for example, Haupt and Aumann [5].

Lemma 4.6. *Let $f(t), g(t) \in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0 \ \text{ for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0 \ \text{ for all large } t.$$

Then,

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \qquad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

Proof of the "if" parts of Theorem 4.1. Assume that $p_i \in \mathrm{RV}(\lambda_i)$, $\lambda \ge \alpha_i$, and $q_i \in \mathrm{RV}(\mu_i)$. Define the positive constants ρ_i by (3.17) and let $X_i \in \mathrm{RV}(\rho_i)$ denote the functions on the right-hand side of (4.2) with $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ replaced by $p_i(t)$ and $q_i(t)$, respectively. Then, Theorem 4.4 ensures the existence of a nearly regularly varying solution $(x_1(t),\dots,x_n(t))$ of (1.1) such that $x_i(t) \asymp X_i(t)$ as $t\to\infty$, $i=1,\dots,n$. Note that $x_i(t)$ satisfy the system of integral equations (4.18).

It remains to verify that $x_i(t)$ are regularly varying functions of index ρ_i, $i = 1,\dots,n$. We define

$$u_i(t) = \int_T^t \left(\frac{1}{p_i(s)} \int_T^s q_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{\frac{1}{\alpha_i}} ds, \quad i=1,\dots,n, \tag{4.19}$$

and put

$$l_i = \liminf_{t\to\infty} \frac{x_i(t)}{u_i(t)}, \quad L_i = \limsup_{t\to\infty} \frac{x_i(t)}{u_i(t)}.$$

Since $x_i(t) \asymp X_i(t)$ and

$$u_i(t) \sim X_i(t), \quad t\to\infty, \quad i=1,\dots,n, \tag{4.20}$$

it follows that $0 < l_i \le L_i < \infty, i=1,\dots,n$. Using Lemma 4.6 we obtain

$$\begin{aligned}
l_i &\ge \liminf_{t\to\infty} \frac{x_i'(t)}{u_i'(t)} = \liminf_{t\to\infty} \frac{\left(\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds\right)^{\frac{1}{\alpha_i}}}{\left(\int_T^t q_i(s) X_{i+1}(s)^{\beta_i} ds\right)^{\frac{1}{\alpha_i}}} \\
&= \liminf_{t\to\infty} \left(\frac{\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds}{\int_T^t q_i(s) X_{i+1}(s)^{\beta_i} ds} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{t\to\infty} \frac{\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds}{\int_T^t q_i(s) X_{i+1}(s)^{\beta_i} ds} \right)^{\frac{1}{\alpha_i}} \\
&\ge \left(\liminf_{t\to\infty} \frac{q_i(t) x_{i+1}(t)^{\beta_i}}{q_i(t) X_{i+1}(t)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{t\to\infty} \left(\frac{x_{i+1}(t)}{X_{i+1}(t)} \right)^{\frac{\beta_i}{\alpha_i}} = l_{i+1}^{\frac{\beta_i}{\alpha_i}},
\end{aligned}$$

where (4.20) has been used in the last step. Thus, l_i satisfy the cyclic system of inequalities

$$l_i \ge l_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i=1,\dots,n, \quad (l_{n+1} = l_1). \tag{4.21}$$

Likewise, by taking the upper limits instead of the lower limits we are led to the cyclic inequalities

$$L_i \le L_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i=1,\dots,n, \quad (L_{n+1} = L_1), \tag{4.22}$$

satisfied by L_i. From (4.21) and (4.22) we see that

$$l_i \ge l_i^{\frac{\beta_1\cdots\beta_n}{\alpha_1\cdots\alpha_n}}, \qquad L_i \le L_i^{\frac{\beta_1\cdots\beta_n}{\alpha_1\cdots\alpha_n}},$$

whence, because of $\beta_1 \ldots \beta_n/\alpha_1 \ldots \alpha_n < 1$, it follows that $l_i \geq 1$ and $L_i \leq 1$, and hence that $l_i = L_i = 1$ or $\lim_{t\to\infty} x_i(t)/u_i(t) = 1$ for $i = 1, \ldots, n$. This combined with (4.20) shows that $x_i(t) \sim u_i(t) \sim X_i(t)$ as $t \to \infty$, which implies that each $x_i(t)$ is a regularly varying function of index ρ_i. This proves the "if" part of Theorem 4.1. Essentially the same proof applies to the "if" part of Theorem 4.2. □

5. APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

The aim of the final section is to demonstrate that our main results on systems of ordinary differential equations (1.1) can be applied to some classes of partial differential equations to shed light on the problem of existence and asymptotic behavior of their radial positive solutions. Throughout this section $x = (x_1, \ldots, x_N)$ represents the space variable in $\mathbb{R}^N$, $N \geq 2$, and $|x|$ denotes the Euclidean length of x. All partial differential equations will be considered in an exterior domain $\Omega_R = \{x \in \mathbb{R}^N : |x| \geq R\}$, $R > 0$.

5.1. SYSTEMS OF p-LAPLACIAN EQUATIONS

We first consider the system of nonlinear p-Laplacian equations

$$\operatorname{div}\Big(|\nabla u_i|^{p-2}\nabla u_i\Big) = f_i(|x|)|u_{i+1}|^{\gamma_i-1}u_{i+1}, \quad i = 1, \ldots, n, \quad (u_{n+1} = u_1) \tag{5.1}$$

where $p > 1$ and $\gamma_i > 0$ are constants, and $f_i(t)$ are positive continuous functions on $[a, \infty)$ which are regularly varying of indices ν_i, $i = 1, \ldots, n$, respectively. Our attention will be focused on radial solutions $(u_1(|x|), \ldots, u_n(|x|))$ of (5.1) defined in Ω_R. A radial vector function $(u_1(|x|), \ldots, u_n(|x|))$ is a solution of (5.1) in Ω_a if and only if $(u_1(t), \ldots, u_n(t))$ is a solution of the system of ordinary differential equations

$$(t^{N-1}|u_i'|^{p-2}u_i')' = t^{N-1}f_i(t)|u_{i+1}|^{\gamma_i-1}u_{i+1}, \quad t \geq a, \quad i = 1, \ldots, n, \quad (u_{n+1} = u_1) \tag{5.2}$$

which is a special case of system (1.1) with

$$\alpha_1 = \ldots = \alpha_n = p - 1, \qquad \beta_i = \gamma_i, \quad i = 1, \ldots, n,$$

$$\lambda_1 = \ldots = \lambda_n = N - 1, \qquad \mu_i = N - 1 + \nu_i, \quad i = 1, \ldots, n.$$

It is assumed that

$$\gamma_1 \ldots \gamma_n < (p - 1)^n. \tag{5.3}$$

Using the inverse of the matrix $A\Big(\frac{\gamma_1}{p-1}, \ldots, \frac{\gamma_n}{p-1}\Big)$ associated with (5.2) (cf. (3.13)) we define

$$(M_{ij}) = \frac{(p-1)^n - \gamma_1 \ldots \gamma_n}{(p-1)^n} A\Big(\frac{\gamma_1}{p-1}, \ldots, \frac{\gamma_n}{p-1}\Big)^{-1}. \tag{5.4}$$

To analyze (5.2) we need to distinguish the two cases $p \geq N$ and $p < N$ under which conditions (1.3) and (1.4) are satisfied, respectively, for system (5.2).

(i) Suppose that $p \le N$, i.e. either $p < N$ (so that (1.4) is satisfied) or $p = N$ in which case (1.3) holds. In this case applying Theorem 4.1 to (5.2), we conclude that system (5.1) possesses increasing radial solutions $(u_1(|x|), \dots, u_n(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i > 0$, $i = 1, \dots, n$, if and only if

$$\sum_{j=1}^{n} M_{ij}(p + \nu_j) > 0, \quad i = 1, \dots, n. \tag{5.5}$$

In this case ρ_i are uniquely determined by

$$\rho_i = \frac{(p-1)^{n-1}}{(p-1)^n - \gamma_1 \cdots \gamma_n} \sum_{j=1}^{n} M_{ij}(p + \nu_j), \quad i = 1, \dots, n, \tag{5.6}$$

and moreover the asymptotic behavior of any such solution as $|x| \to \infty$ is governed by the growth law

$$u_i(|x|) \sim |x|^{\rho_i} \left[\prod_{j=1}^{n} \left(\frac{\varphi_i(|x|)^{\frac{1}{p-1}}}{(N - p + (p-1)\rho_j)^{\frac{1}{p-1}} \rho_j} \right)^{M_{ij}} \right]^{\frac{(p-1)^n}{(p-1)^n - \gamma_1 \cdots \gamma_n}}, \quad |x| \to \infty, \tag{5.7}$$

where $\varphi_i \in \mathrm{SV}$ are the regularly varying parts of $f_i(t)$: $f_i(t) = t^{\nu_i}\varphi_i(t)$, $i = 1, \dots, n$.

(ii) Suppose that $p > N$. In this case from Theorem 4.2 applied to (5.2) it follows that system (5.1) possesses increasing radial solutions $(u_1(|x|), \dots, u_n(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\rho_i > (p-N)/(p-1)$, $i = 1, \dots, n$, if and only if

$$\sum_{j=1}^{n} M_{ij} \Big(N + \nu_j + \frac{p-N}{p-1} \gamma_j \Big) > 0, \quad i = 1, \dots, n. \tag{5.8}$$

In this case ρ_i are uniquely determined by (5.6) and the asymptotic behavior of any such solution as $|x| \to \infty$ is governed by the formulas (5.7).

We remark that the particular case of (5.1) in which $f_i(t) \equiv c_i > 0$, i.e.,

$$\mathrm{div}\Big(|\nabla u_i|^{p-2} \nabla u_i \Big) = c_i |u_{i+1}|^{\gamma_i - 1} u_{i+1}, \quad i = 1, \dots, n, \quad (u_{n+1} = u_1) \tag{5.9}$$

always possesses strongly increasing radial solutions $(u_1(|x|), \dots, u_n(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, where ρ_i satisfy

$$\rho_i > 0, \quad i = 1, \dots, n, \quad \text{if } p \le N, \qquad \rho_i > \frac{p-N}{p-1}, \quad i = 1, \dots, n, \quad \text{if } p > N.$$

5.2. NONLINEAR METAHARMONIC EQUATIONS

Next, the nonlinear metaharmonic equation

$$\Delta^m u = g(|x|)|u|^{\gamma-1}u, \quad x \in \Omega_R, \tag{5.10}$$

is under consideration, where $m \geq 2$ and $\gamma > 0$ are constants, and $g(t)$ is a positive continuous function on $[R, \infty)$ which is regularly varying of index ν. We are interested in radial positive solutions u of (5.10) such that u and $\Delta^k u$, $k = 1, \ldots, m-1$, are regularly varying of positive indices. It is clear that seeking such solutions of (5.10) is equivalent to seeking radial regularly varying solutions of positive indices of the system

$$\Delta u_i = u_{i+1}, \quad i = 1, \ldots, m-1, \qquad \Delta u_m = g(|x|)|u_{m+1}|^{\gamma-1}u_{m+1}, \quad x \in \Omega_R, \tag{5.11}$$

where $u_{m+1} = u_1$. This system is equivalent to the system of ordinary differential equations

$$\begin{aligned} (t^{N-1}u_i')' &= t^{N-1}u_{i+1}, \quad i = 1, \ldots, m-1, \\ (t^{N-1}u_m')' &= t^{N-1}g(t)|u_{m+1}|^{\gamma-1}u_{m+1}, \quad t \geq R, \end{aligned} \tag{5.12}$$

which is a special case of (1.1) with

$$\alpha_1 = \ldots = \alpha_m = 1, \qquad \beta_i = \ldots = \beta_{m-1} = 1, \quad \beta_m = \gamma,$$

$$\lambda_1 = \ldots = \lambda_m = N-1, \qquad \mu_1 = \ldots = \mu_{m-1} = N-1, \quad \mu_m = N-1+\nu.$$

We assume that $\gamma < 1$. The $m \times m$-matrix (3.13) associated with (5.12) reads $A(1, \ldots, 1, \gamma)$. Define the matrix (M_{ij}) by

$$(M_{ij}) = (1-\gamma)A(1, \ldots, 1, \gamma)^{-1}. \tag{5.13}$$

As is easily checked, $M_{ij} = 1$ for $1 \leq i \leq j \leq m$ and $M_{ij} = \gamma$ for $1 \leq j < i \leq m$.

Observe that conditions (1.3) and (1.4) for (5.12) reduce, respectively, to $N = 2$ and $N \geq 3$. However, since $\lambda_i \geq \alpha_i$ for all i in this case, only Theorem 4.1 can be used to determine the structure of increasing regularly varying solutions $(u_1(t), \ldots, u_m(t)) \in \mathrm{RV}(\rho_1, \ldots, \rho_m)$, $\rho_i > 0$, of the cyclic system (5.12). The regularity indices ρ_i should be given by (3.17) which in the present situation reduce to

$$\rho_i = \frac{2m+\nu}{1-\gamma} - 2(i-1), \quad i = 1, \ldots, m, \tag{5.14}$$

from which we see that all ρ_i are positive if and only if $\rho_m > 0$, that is,

$$\frac{2m+\nu}{1-\gamma} > 2(m-1). \tag{5.15}$$

Taking this fact into account, we conclude from Theorem 4.1 that equation (5.10) possesses radial increasing positive solutions $u(|x|)$ in $\mathrm{RV}(\rho)$ with $\rho > 0$ if and only if (5.15) holds, in which case ρ is given by $\rho = (2m+\nu)/(1-\gamma)$ and the asymptotic behavior of $u(|x|)$ as $|x| \to \infty$ is governed by the growth formula

$$u(|x|) \sim |x|^\rho \left[\frac{\psi(|x|)}{\prod_{i=1}^n \Big((N-2+\rho_i)\rho_i\Big)}\right]^{\frac{1}{1-\gamma}}, \quad |x| \to \infty, \tag{5.16}$$

where ρ_i are as in (5.14) and $\psi(t)$ denotes the slowly varying part of $g(t) : g(t) = t^{\nu}\psi(t)$.

Since (5.15) holds if $\nu = 0$, one can assert that the particular case of (5.10)

$$\Delta^m u = c|u|^{\gamma-1}u, \quad x \in \Omega_R,$$

where $c > 0$ is a constant, always possesses radial solutions $u(|x|) \in \mathrm{RV}(2m/(1-\gamma))$, and that any such solution has one and the same asymptotic behavior (5.16) with $\rho = 2m/(1-\gamma)$.

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REFERENCES

[1] J.A.D. Appleby, D.D. Patterson, *Classification of convergence rates of solutions of perturbed ordinary differential equations with regularly varying nonlinearity*, preprint, 2013, arXiv:1303.3345v3.

[2] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge Universiy Press, 1987.

[3] C. Cîrstea, V. Radulescu, *Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach*, Asymptot. Anal. **46** (2006), 275–298.

[4] V.M. Evtukhov, E.S. Vladova, *Asymptotic representations of solutions of essentially nonlinear cyclic systems of ordinary differential equations*, Differ. Equ. **48** (2012), 630–646.

[5] O. Haupt, G. Aumann, *Differential- Und Integralrechnung*, Walter de Gruyter, Berlin, 1938.

[6] J. Jaroš, T. Kusano, *Existence and precise asymptotic behavior of strongly monotone solutions of systems of nonlinear differential equations*, Differ. Equ. Appl. **5** (2013), 185–204.

[7] J. Jaroš, T. Kusano, *Asymptotic behavior of positive solutions of a class of systems of second order nonlinear differential equations*, Electron. J. Qual. Theory Differ. Equ. **23** (2013), 1–23.

[8] J. Jaroš, T. Kusano, *On strongly decreasing solutions of cyclic systems of second order nonlinear differential equations* Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[9] J. Jaroš, T. Kusano, T. Tanigawa, *Asymptotic analysis of positive solutions of a class of third order nonlinear differential equations in the framework of regular variation*, Math. Nachr. **286** (2013), 205–223.

[10] T. Kusano, J. Manojlović, *Asymptotic behavior of positive solutions of sublinear differential equations of Emden-Fowler type*, Comput. Math. Appl. **62** (2011), 551–565.

[11] T. Kusano, J. Manojlović, *Positive solutions of fourth order Thomas-Fermi type differential equations in the framework of regular variation*, Acta Appl. Math. **121** (2012), 81–103.

[12] T. Kusano, J. Manojlović, *Asymptotic behavior of positive solutions of odd order Emden-Fowler type differential equations in the framework of regular variation*, Electron. J. Qual. Theory Differ. Equ. **45** (2012), 1–23.

[13] T. Kusano, V. Marić, T. Tanigawa, *An asymptotic analysis of positive solutions of generalized Thomas-Fermi differential equations – the sub-half-linear case*, Nonlinear Anal. **75** (2012), 2474–2485.

[14] V. Marić, *Regular Variation and Differential Equations*, Lecture Notes in Mathematics, Vol. 1726, Springer Verlag, Berlin-Heidelberg, 2000.

[15] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, Vol. 508, Springer Verlag, Berlin-Heidelberg, 1976.

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