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This paper is dedicated with great esteem
and admiration to Professor Leon Mikołajczyk.

REMARKS FOR ONE-DIMENSIONAL FRACTIONAL EQUATIONS

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Abstract. In this paper we study a class of one-dimensional Dirichlet boundary value problems involving the Caputo fractional derivatives. The existence of infinitely many solutions for this equations is obtained by exploiting a recent abstract result. Concrete examples of applications are presented.

Keywords: fractional differential equations, Caputo fractional derivatives, variational methods.

Mathematics Subject Classification: 34A08, 26A33, 35A15.

1. INTRODUCTION

The aim of this short note is to study nonlinear fractional boundary value problems whose general form is given by

$$\begin{cases} \Delta_{F,\alpha} u(t) + f(t, u(t)) = 0 & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where

$$\Delta_{F,\alpha} u(t) := \frac{d}{dt}\Big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big),$$

$\alpha \in (1/2, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^\alpha$ and ${}_t^cD_T^\alpha$ are the left and right Caputo fractional derivatives of order α respectively, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Following [12], denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $g \in C^\infty([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$. The fractional derivative Hilbert space E_0^α is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\| := \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\right)^{1/2}$$

for every $u \in E_0^\alpha$.

For basic facts and usual notation on the variational setting adopted here we refer the reader to [12, 18]. Let us denote

$$\kappa := \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}},$$

$$C(T,\alpha) := \int_0^{T/4} t^{2-2\alpha}\,dt + \int_{T/4}^{3T/4} \left[t^{1-\alpha} - \left(t - \frac{T}{4}\right)^{1-\alpha}\right]^2 dt$$
$$+ \int_{3T/4}^{T} \left[t^{1-\alpha} - \left(t - \frac{T}{4}\right)^{1-\alpha} - \left(t - \frac{3T}{4}\right)^{1-\alpha}\right]^2 dt,$$

and

$$B^0 := \limsup_{\xi\to 0^+} \frac{\displaystyle\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2},$$

where F is the potential of f defined by

$$F(t,\xi) := \int_0^\xi f(t,x)\,dx, \quad (t,\xi) \in [0,T] \times \mathbb{R}.$$

With the above notation, in [2, Theorem 3.2], exploiting a quoted critical point theorem established by Ricceri in [17], the following result has been shown.

Theorem 1.1. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that*

(f_1) $F(t,\xi) \geq 0$ *for every* $(t,\xi) \in ([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]) \times \mathbb{R}$.

Assume that there exist two real sequences $\{c_n\}$ *and* $\{d_n\}$ *in* $[0,+\infty)$, *with* $\lim_{n\to\infty} d_n = 0$, *satisfying the conditions:*

(h_3) *for some* $n_0 \in \mathbb{N}$ *one has*

$$c_n < \frac{T|\cos(\pi\alpha)|\Gamma(2-\alpha)}{4\kappa\sqrt{C(T,\alpha)}} d_n$$

for each $n \geq n_0$,

(h$_4$) $\mathcal{A}_0 := \lim_{n\to\infty} \varphi(c_n, d_n, \alpha, T) < \dfrac{B^0}{16\kappa^2 C(T,\alpha)}$, *where*

$$\varphi(c_n, d_n, \alpha, T) := \frac{\int_0^T \max_{|\xi|\le d_n} F(t,\xi)dt - \int_{T/4}^{3T/4} F(t, c_n)\, dt}{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)d_n^2 - 16\kappa^2 c_n^2 C(T,\alpha)}.$$

Then, for each

$$\lambda \in \left(\frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^0}, \frac{1}{\kappa^2 T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|\mathcal{A}_0} \right),$$

problem

$$\begin{cases} \Delta_{F,\alpha} u(t) + \lambda f(t, u(t)) = 0 & a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

admits a sequence of non-zero solutions which strongly converges to zero in E_0^α.

The aim of this paper is to prove the following remarkable consequence of Theorem 1.1.

Theorem 1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $f|_{(-\infty,0]} \equiv 0$ *and* $\inf_{\xi\ge 0} F(\xi) = 0$. *Further, let* $h \in C^0([0,T])$ *with*

(a$_0$) $\min_{t\in[0,T]} h(t) > 0$.

Suppose that there exist two sequences $\{c_n\}$ *and* $\{d_n\}$ *in* $(0, +\infty)$, *with* $c_n < d_n$ *for every* $n \ge \nu$, *and* $\lim_{n\to\infty} d_n = 0$, *such that:*

(a$_1$) $\lim_{n\to\infty} \dfrac{d_n}{c_n} = +\infty$,

(a$_2$) $\max_{x\in[c_n,d_n]} f(x) \le 0$ *for every* $n \ge \nu$,

(a$_3$) $$\frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)| \displaystyle\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\, dt} < \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty.$$

Then, the following problem

$$\begin{cases} \Delta_{F,\alpha} u(t) + h(t) f(u(t)) = 0 & a.e.\ t \in [0,T], \\ u(0) = u(T) = 0 \end{cases}$$

admits a sequence of non-zero solutions which strongly converges to zero in E_0^α.

For several results on fractional differential equations, one can see, for example, the monographs of Miller and Ross [15], Samko *et al.* [18], Podlubny [16], Hilfer [11], Kilbas *et al.* [13] and the papers [1, 3–7]

We cite a recent monograph by Kristály, Rădulescu and Varga [14] as a general reference on variational methods adopted here.

2. PROOF OF THE MAIN RESULT

Our aim is to apply Theorem 1.1. First of all observe that, by (a_0), condition (f_1) holds. Further, if $\{c_n\}$ and $\{d_n\}$ are two real sequences satisfying our assumptions, we have that there exists $n_0 \geq \nu$ such that

$$\frac{c_n^2}{d_n^2} < \frac{T|\cos(\pi\alpha)|\Gamma(2-\alpha)}{4\kappa\sqrt{C(T,\alpha)}},$$

for every $n \geq n_0$. Hence the hypothesis (h_3) in Theorem 1.1 is verified. We will prove that

$$\mathcal{A}_0 := \lim_{n\to\infty} \frac{\|h\|_{L^1([0,T])} \max_{|\xi|\leq d_n} F(\xi) - \left(\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt\right) F(c_n)\,dt}{T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)d_n^2 - 16\kappa^2 c_n^2 C(T,\alpha)} = 0.$$

Set

$$h_n := \|h\|_{L^1([0,T])} \frac{\max_{|\xi|\leq d_n} F(\xi)}{c_n^2} - \left(\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt\right) \frac{F(c_n)}{c_n^2}$$

for every $n \geq n_0$ and observe that hypothesis (a_2) yields

$$\max_{|\xi|\leq d_n} F(\xi) = \max_{|\xi|\leq c_n} F(\xi). \tag{2.1}$$

Thus, since

$$\frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt}{\|h\|_{L^1([0,T])}} \leq 1 \quad \text{and} \quad F(c_n) \geq 0,$$

by (2.1), we can write

$$\frac{\max_{|\xi|\leq d_n} F(\xi)}{c_n^2} = \frac{\max_{|\xi|\leq c_n} F(\xi)}{c_n^2} \geq \frac{F(c_n)}{c_n^2} \geq \frac{\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt}{\|h\|_{L^1([0,T])}} \frac{F(c_n)}{c_n^2}.$$

for every $n \geq n_0$.

Since $h_n \geq 0$ for every $n \geq n_0$, one easily gets

$$0 \leq \limsup_{n\to\infty} h_n.$$

Further, by (a_3) we have

$$0 < \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} < +\infty, \tag{2.2}$$

and consequently (note that $c_n \searrow 0^+$ as $n \to \infty$) we obtain

$$0 \leq \limsup_{n\to\infty} \frac{F(c_n)}{c_n^2} < +\infty. \tag{2.3}$$

Now, let $\xi_n \in (0, c_n]$ be a sequence such that $F(\xi_n) := \max_{|\xi|\leq c_n} F(\xi)$ for every $n \geq n_0$. Thus

$$\limsup_{n\to\infty} \frac{\max\limits_{|\xi|\leq d_n} F(\xi)}{c_n^2} = \limsup_{n\to\infty} \frac{\max\limits_{|\xi|\leq c_n} F(\xi)}{c_n^2} = \limsup_{n\to\infty} \frac{F(\xi_n)}{c_n^2} \leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2}.$$

The above inequalities and (2.2) yield

$$0 \leq \limsup_{n\to\infty} \frac{\max\limits_{|\xi|\leq d_n} F(\xi)}{c_n^2} \leq \limsup_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2} < +\infty.$$

Hence, there exists a constant β such that

$$0 \leq \limsup_{n\to\infty} h_n = \beta. \tag{2.4}$$

Then, by (a_1) and (2.4), one has

$$\mathcal{A}_0 = \limsup_{n\to\infty} \frac{h_n}{\left(T^2|\cos(\pi\alpha)|^2\Gamma^2(2-\alpha)\dfrac{d_n^2}{c_n^2} - 16\kappa^2 C(T,\alpha)\right)} = 0.$$

Concluding, hypothesis (h_4) holds. Finally, bearing in mind condition (a_3), one has

$$1 \in \left(\frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)|B^0}, +\infty\right).$$

Thanks to Theorem 1.1, the thesis is achieved. The next result is a direct consequence of Theorem 1.2.

Proposition 2.1. *Let $h \in C^0([0,T])$ satisfying condition* (a_0)*. Also let $\{c_n\}$ and $\{d_n\}$ be two sequences in $(0,+\infty)$ such that $d_{n+1} < c_n < d_n$ for every $n \geq \nu$, $\lim_{n\to\infty} d_n = 0$, and $\lim_{n\to\infty} \frac{d_n}{c_n} = +\infty$. Moreover, let $\varphi \in C^1([0,1])$ be a nonnegative function such that $\varphi(0) = \varphi(1) = \varphi'(0) = \varphi'(1) = 0$ and*

$$\max_{s\in[0,1]} \varphi(s) > \frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)| \displaystyle\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt}.$$

Further, let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(t) := \begin{cases} \varphi\left(\dfrac{t - d_{n+1}}{c_n - d_{n+1}}\right) & \text{if } t \in \bigcup_{n\geq\nu}[d_{n+1}, c_n], \\ 0 & \text{otherwise.} \end{cases}$$

Then, problem

$$\begin{cases} \Delta_{F,\alpha} u(t) + h(t) y(u(t)) = 0 \quad a.e.\ t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where

$$y(u(t)) := |u(t)|(2g(u(t)) + ug'(u(t))),$$

admits a sequence of non-zero solutions which strongly converge to zero in E_0^α.

Proof. Let $\{c_n\}$ and $\{d_n\}$ be two positive sequences satisfying our assumptions. We claim that all the hypotheses of Theorem 1.2 are verified. Indeed, one has

$$F(\xi) := \int_0^\xi y(t)dt = \xi^2 g(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^+.$$

Moreover, direct computations ensure that

$$\max_{x \in [c_{n+1}, d_{n+1}]} y(x) = 0$$

for every $n \geq \nu$, and

$$\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = \max_{s \in [0,1]} \varphi(s) > \frac{16C(T,\alpha)}{T^2 \Gamma^2(2-\alpha) |\cos(\pi\alpha)| \int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\, dt}.$$

The assertion follows by Theorem 1.2. □

In conclusion we present a concrete example of the application of Proposition 2.1.

Example 2.2. Let $h \in C^0([0,T])$ satisfying condition (a_0). Take the positive real sequences

$$a_n := \frac{1}{n!n} \quad \text{and} \quad b_n := \frac{1}{n!}$$

for every $n \geq 2$. Now, define $\varphi \in C^1([0,1])$ as follows

$$\varphi(s) := \zeta e^{\frac{1}{s(s-1)} + 4} \quad (\text{for all}\, s \in [0,1]),$$

and set

$$\widehat{g}(t) := \begin{cases} \varphi\Big(\dfrac{t - 1/(n+1)!}{1/(n!n) - 1/(n+1)!}\Big) & \text{if } t \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A := \bigcup_{n \geq 2} \Big[\frac{1}{(n+1)!}, \frac{1}{(n!n)}\Big].$$

If

$$\zeta > \frac{16C(T,\alpha)}{T^2\Gamma^2(2-\alpha)|\cos(\pi\alpha)| \displaystyle\int_{\frac{T}{4}}^{\frac{3T}{4}} h(t)\,dt},$$

then problem

$$\begin{cases} \Delta_{F,\alpha}u(t) + h(t)y(u(t)) = 0 \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where

$$y(u(t)) := |u(t)|(2\widehat{g}(u(t)) + u\widehat{g}'(u(t))),$$

admits a sequence of non-zero solutions which strongly converges to zero in E_0^α.

We just mention, for completeness, that related variational arguments have been used recently in [8] proving the existence of at least one non-zero solution for one dimensional fractional equations. See also [9, 10] for related topics.

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