Opuscula Math. 34, no. 3 (2014), 609–620
http://dx.doi.org/10.7494/OpMath.2014.34.3.609 Opuscula Mathematica

SIGNED STAR (k,k)-DOMATIC NUMBER OF A GRAPH

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Communicated by Hao Li

Abstract. Let G be a simple graph without isolated vertices with vertex set $V(G)$ and edge set $E(G)$ and let k be a positive integer. A function $f : E(G) \longrightarrow \{-1,1\}$ is said to be a signed star k-dominating function on G if $\sum_{e\in E(v)} f(e) \geq k$ for every vertex v of G, where $E(v) = \{uv \in E(G) \mid u \in N(v)\}$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed star k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a signed star (k,k)-dominating family (of functions) on G. The maximum number of functions in a signed star (k,k)-dominating family on G is the signed star (k,k)-domatic number of G, denoted by $d_{SS}^{(k,k)}(G)$. In this paper we study properties of the signed star (k,k)-domatic number $d_{SS}^{(k,k)}(G)$. In particular, we present bounds on $d_{SS}^{(k,k)}(G)$, and we determine the signed (k,k)-domatic number of some regular graphs. Some of our results extend these given by Atapour, Sheikholeslami, Ghameslou and Volkmann [*Signed star domatic number of a graph*, Discrete Appl. Math. 158 (2010), 213–218] for the signed star domatic number.

Keywords: signed star (k,k)-domatic number, signed star domatic number, signed star k-dominating function, signed star dominating function, signed star k-domination number, signed star domination number, regular graphs.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use [8] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every nonempty subset E' of $E(G)$, the subgraph $G[E']$ induced by E' is the graph whose vertex set consists of those vertices of G incident with at least one edge of E' and whose edge set is E'.

Two edges e_1, e_2 of G are called *adjacent* if they are distinct and have a common vertex. The *open neighborhood* $N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N_G[e] = N_G(e) \cup \{e\}$. For a function $f : E(G) \longrightarrow \{-1,1\}$ and a subset S of $E(G)$ we define $f(S) = \sum_{e\in S} f(e)$. The *edge-neighborhood* $E_G(v) = E(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex v. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$.

Let k be a positive integer. A function $f : E(G) \longrightarrow \{-1, 1\}$ is called a *signed star k-dominating function* (SSkDF) on G, if $f(v) \geq k$ for every vertex v of G. The *signed star k-domination number* of a graph G is

$$\gamma_{kSS}(G) = \min\Big\{ \sum_{e \in E(G)} f(e) \mid f \text{ is a SSkDF on } G \Big\}.$$

The signed star k-dominating function f on G with $f(E(G)) = \gamma_{kSS}(G)$ is called a *$\gamma_{kSS}(G)$-function.* As the assumption $\delta(G) \geq k$ is clearly necessary, we will always assume that when we discuss $\gamma_{kSS}(G)$ all graphs involved satisfy $\delta(G) \geq k$. The signed star k-domination number was introduced by Xu and Li in [11] and has been studied by several authors (see for instance [4, 5]). The signed star 1-domination number is the usual signed star domination number which has been introduced by Xu in [9] and has been studied by several authors (see for instance [4, 6, 10]).

A set $\{f_1, f_2, \dots, f_d\}$ of signed star k-dominating functions on G with $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a *signed star (k, k)-dominating family* (of functions) on G. The maximum number of functions in a signed star (k, k)-dominating family on G is the *signed star (k, k)-domatic number* of G, denoted by $d_{SS}^{(k,k)}(G)$. The signed star (k, k)-domatic number is well-defined and

$$d_{SS}^{(k,k)}(G) \geq 1 \tag{1.1}$$

for all graphs G with $\delta(G) \geq k$, since the set consisting of any one SSkD function forms a SS(k,k)D family on G. A *$d_{SS}^{(k,k)}$-family* of a graph G is a SS(k,k)D family containing $d_{SS}^{(k,k)}(D)$ SSkD functions. The signed star (1,1)-domatic number $d_{SS}^{(1,1)}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour, Sheikholeslami, Ghameslou and Volkmann in [1].

Our purpose in this paper is to initiate the study of signed star (k, k)-domatic numbers in graphs. We first study basic properties and bounds for the signed star (k, k)-domatic number of a graph where some of them are analogous to those of the signed star domatic number $d_{SS}(G)$ in [1]. In addition, we determine the signed star (k, k)-domatic number of some regular graphs.

We start with a simple known observation which is important for our investigations.

Observation 1.1 ([5])**.** *Let G be a graph of size m with $\delta(G) \geq k$. Then $\gamma_{kSS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint u such that* $\deg(u) = k$ *or* $\deg(u) = k + 1$.

2. BASIC PROPERTIES OF THE SIGNED STAR (k, k)-DOMATIC NUMBER

In this section we study basic properties of $d_{SS}^{(k,k)}(G)$.

Proposition 2.1. *If $k \geq 1$ is an integer and G is a graph of minimum degree $\delta(G) \geq k$, then*

$$d_{SS}^{(k,k)}(G) \leq \delta(G).$$

Moreover, if $d_{SS}^{(k,k)}(G) = \delta(G)$, then for each function of any signed star (k, k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ with $d = d_{SS}^{(k,k)}(G)$, and for all vertices v of degree $\delta(G)$, $\sum_{e \in E(v)} f_i(e) = k$ and $\sum_{i=1}^{d} f_i(e) = k$ for every $e \in E(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star (k, k)-dominating family on G such that $d = d_{SS}^{(k,k)}(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} d \cdot k = \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) = \\ = \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) = \\ \leq \sum_{e \in E(v)} k = k \cdot \delta(G), \end{aligned}$$

and this implies the desired upper bound on the signed star (k, k)-domatic number.

If $d_{SS}^{(k,k)}(G) = \delta(G)$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. □

The special case $k = 1$ in Proposition 2.1 can be found in [1]. As an application of Proposition 2.1, we will prove the following Nordhaus-Gaddum type result.

Corollary 2.2. *If $k \geq 1$ is an integer and G is a graph of order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, then*

$$d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 1.$$

If $d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) = n - 1$, then G is regular.

Proof. Since $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$, it follows from Proposition 2.1 that

$$\begin{aligned} d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) &\leq \delta(G) + \delta(\overline{G}) = \\ &= \delta(G) + (n - \Delta(G) - 1) \leq \\ &\leq n - 1, \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geq 1$, and the above inequality chain leads to the better bound $d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 2$. This completes the proof. □

Theorem 2.3. *If v is a vertex of a graph G such that $d(v)$ is odd and k is even or $d(v)$ is even and k is odd, then*

$$d_{SS}^{(k,k)}(G) \leq \frac{k}{k+1} \cdot d(v).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star (k,k)-dominating family on G such that $d = d_{SS}^{(k,k)}(G)$. Assume first that $d(v)$ is odd and k is even. The definition yields $\sum_{e\in E(v)} f_i(e) \geq k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{e\in E(v)} f_i(e) \geq k+1$ for each $i \in \{1,2,\ldots,d\}$. It follows that

$$\begin{aligned} k \cdot d(v) &= \sum_{e\in E(v)} k \geq \sum_{e\in E(v)} \sum_{i=1}^{d} f_i(e) = \\ &= \sum_{i=1}^{d} \sum_{e\in E(v)} f_i(e) \geq \\ &\geq \sum_{i=1}^{d} (k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound. Assume next that $d(v)$ is even and k is odd. Note that $\sum_{e\in E(v)} f_i(e) \geq k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{e\in E(v)} f_i(e) \geq k+1$ for each $i \in \{1,2,\ldots,d\}$. Now the desired bound follows as above, and the proof is complete. □

The next result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *If G is a graph such that $\delta(G)$ is odd and k is even or $\delta(G)$ is even and k is odd, then*

$$d_{SS}^{(k,k)}(G) \leq \frac{k}{k+1} \cdot \delta(G).$$

The bound is sharp for cycles when $k = 1$.

As an application of Corollary 2.4, we will improve the Nordhaus-Gaddum bound in Corollary 2.2 for many cases.

Theorem 2.5. *Let $k \geq 1$ be an integer, and let G be a graph of order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$. If $\Delta(G) - \delta(G) \geq 1$ or k is odd or k is even and $\delta(G)$ is odd or k, $\delta(G)$ and n are even, then*

$$d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 2.$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Corollary 2.2 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1. Assume that k is odd. If $\delta(G)$ is even, then it follows from Proposition 2.1 and Corollary 2.4 that

$$\begin{aligned} d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) &\leq \frac{k}{k+1}\delta(G) + \delta(\overline{G}) = \\ &= \frac{k}{k+1}\delta(G) + (n - \delta(G) - 1) < \\ &< n - 1, \end{aligned}$$

and we obtain the desired bound. If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Proposition 2.1 and Corollary 2.4, we find that

$$\begin{aligned} d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) &\le \delta(G) + \frac{k}{k+1}\delta(\overline{G}) = \\ &= (n - \delta(\overline{G}) - 1) + \frac{k}{k+1}\delta(\overline{G}) < \\ &< n - 1, \end{aligned}$$

and this completes the proof of Case 1.

Case 2. Assume that k is even. If $\delta(G)$ is odd, then it follows from Proposition 2.1 and Corollary 2.4 that

$$d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \le \frac{k}{k+1}\delta(G) + (n - \delta(G) - 1) < n - 1.$$

If $\delta(G)$ is even and n is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Theorem 2.6. *If G is a graph such that k is odd and $d_{SS}^{(k,k)}(G)$ is even or k is even and $d_{SS}^{(k,k)}(G)$ is odd, then*

$$d_{SS}^{(k,k)}(G) \le \frac{k-1}{k}\delta(G).$$

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a signed star (k,k)-dominating family on G such that $d = d_{SS}^{(k,k)}(G)$. Assume first that k is odd and d is even. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \le k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{i=1}^{d} f_i(e) \le k - 1$ for each $e \in E(G)$. If v is a vertex of minimum degree, then it follows that

$$\begin{aligned} d \cdot k = \sum_{i=1}^{d} k &\le \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) = \\ &= \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \le \\ &\le \sum_{e \in E(v)} (k-1) = \delta(G)(k-1), \end{aligned}$$

and this yields to the desired bound. Assume second that k is even and d is odd. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \le k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{i=1}^{d} f_i(e) \le k - 1$ for each $e \in E(G)$. Now the desired bound follows as above, and the proof is complete. □

According to (1.1), $d_{SS}^{(k,k)}(G)$ is a positive integer. If we suppose in the case $k = 1$ that $d_{SS}(G) = d_{SS}^{(1,1)}(G)$ is an even integer, then Theorem 2.6 leads to the contradiction $d_{SS}(G) \leq 0$. Consequently, we obtain the next known result.

Corollary 2.7 ([1])**.** *The signed star domatic number $d_{SS}(G)$ is an odd integer.*

Proposition 2.8. *Let $k \geq 2$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. Then $d_{SS}^{(k,k)}(G) = 1$ if and only if each edge $e \in E(G)$ has an endpoint u such that $\deg(u) = k$ or $\deg(u) = k + 1$.*

Proof. Assume that each edge $e \in E(G)$ has an endpoint u such that $\deg(u) = k$ or $\deg(u) = k + 1$. It follows from Observation 1.1 that $\gamma_{kSS}(G) = m$ and thus $d_{SS}^{(k,k)}(G) = 1$.

Conversely, assume that $d_{SS}^{(k,k)}(G) = 1$. If G contains an edge $e = uv$ such that $d(u) \geq k + 2$ and $d(v) \geq k + 2$, then the functions $f_1, f_2 : E(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in E(G)$ and $f_2(e) = -1$ and $f_2(x) = 1$ for each edge $x \in E(G) \setminus \{e\}$ are signed star k-dominating functions on G such that $f_1(x) + f_2(x) \leq 2 \leq k$ for each edge $x \in E(G)$. Thus $\{f_1, f_2\}$ is a signed star (k, k)-dominating family on G, a contradiction to $d_{SS}^{(k,k)}(G) = 1$. □

The next result is an immediate consequence of Observation 1.1 and Proposition 2.8.

Corollary 2.9. *Let $k \geq 2$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. Then $d_{SS}^{(k,k)}(G) = 1$ if and only if $\gamma_{kSS}(G) = m$.*

Next we present a lower bound on the signed star (k, k)-domatic number.

Proposition 2.10. *Let $k \geq 1$ be an integer, and let G be a graph with minimum degree $\delta(G) \geq k$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $k + 2$, then $d_{SS}^{(k,k)}(G) \geq k$.*

Proof. Let $\{u_1, u_2, \ldots, u_k\} \subset N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $k+2$ implies that the functions $f_i : E(G) \to \{-1, 1\}$ such that $f_i(vu_i) = -1$ and $f_i(x) = 1$ for each edge $x \in E(G) \setminus \{vu_i\}$ are signed star k-dominating functions on G for $i \in \{1, 2, \ldots, k\}$. Since $f_1(x) + f_2(x) + \ldots + f_k(x) \leq k$ for each edge $x \in E(G)$, we observe that $\{f_1, f_2, \ldots, f_k\}$ is a signed star (k, k)-dominating family on G, and Proposition 2.10 is proved. □

Corollary 2.11. *If G is a graph of minimum degree $\delta(G) \geq k+2$, then $d_{SS}^{(k,k)}(G) \geq k$.*

Theorem 2.12. *Let G be a graph of size m with $\delta(G) \geq k$, signed star k-domination number $\gamma_{kSS}(G)$ and signed star (k, k)-domatic number $d_{SS}^{(k,k)}(G)$. Then*

$$\gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G) \leq mk.$$

Moreover, if $\gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G) = mk$, then for each $d_{SS}^{(k,k)}$-family $\{f_1, f_2, \ldots, f_d\}$ of G, each function f_i is a γ_{kSS}-function and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$.

Proof. If $\{f_1, f_2, \dots, f_d\}$ is a signed star (k,k)-dominating family on G such that $d = d_{SS}^{(k,k)}(G)$, then the definitions imply

$$d \cdot \gamma_{kSS}(G) = \sum_{i=1}^{d} \gamma_{kSS}(G) \le \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) =$$

$$= \sum_{e \in E(G)} \sum_{i=1}^{d} f_i(e) \le \sum_{e \in E(G)} k = mk$$

as desired.

If $\gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G) = mk$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{SS}^{(k,k)}$-family $\{f_1, f_2, \dots, f_d\}$ of G and for each i, $\sum_{e \in E(G)} f_i(e) = \gamma_{kSS}(G)$, thus each function f_i is a γ_{kSS}-function, and $\sum_{i=1}^{d} f_i(e) = k$ for all $e \in E(G)$. □

The upper bound on the product $\gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G)$ leads to an upper bound on the sum of these two parameters.

Theorem 2.13. *If $k \ge 1$ is an integer and G is a graph of size m and minimum degree $\delta(G) \ge k$, then*

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) \le m + k.$$

Proof. If $\delta(G) = k$, then it follows from Proposition 2.1 that

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) \le \delta(G) + m = m + k.$$

Assume next that $\delta(G) = k + 1$. If $\gamma_{kSS}(G) = m$, then $d_{SS}^{(k,k)}(G) = 1$ and so

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) = m + 1 \le m + k.$$

In the case that $\gamma_{kSS}(G) \le m - 1$, Proposition 2.1 implies that

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) \le \delta(G) + m - 1 = m + k.$$

Assume now that $\delta(G) \ge k + 2$. According to Theorem 2.12, we have

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) \le d_{SS}^{(k,k)}(G) + \frac{km}{d_{SS}^{(k,k)}(G)}.$$

In view of Corollary 2.11, $d_{SS}^{(k,k)}(G) \ge k$, and Proposition 2.1 implies that $d_{SS}^{(k,k)}(G) \le n - 1 \le m$. Using these inequalities, and the fact that the function $g(x) = x + (km)/x$ is decreasing for $k \le x \le \sqrt{km}$ and increasing for $\sqrt{km} \le x \le n$, we obtain

$$d_{SS}^{(k,k)}(G) + \gamma_{kSS}(G) \le \max\left\{k + \frac{km}{k}, m + \frac{km}{m}\right\} = m + k.$$

Since we have discussed all possible cases for the minimum degree $\delta(G)$, the proof of Theorem 2.13 is complete. □

3. REGULAR GRAPHS

Theorem 3.1. *Let $k \geq 1$ be an integer, and let G be an r-regular graph with $r \geq k$.*

(1) *If $k \leq r \leq k+1$, then $d_{SS}^{(k,k)}(G) = 1$.*
(2) *If $r = k + 2p + 1$ with $p \geq 1$, then $k \leq d_{SS}^{(k,k)}(G) \leq r - 3$.*
(3) *If $r = k + 2p$ with $p \geq 1$, then $d_{SS}^{(k,k)}(G) \neq r - 1$, and if $d_{SS}^{(k,k)}(G) = r$, then G contains a p-regular factor.*

Proof. (1) Assume that $k \leq r \leq k+1$. According to Observation 1.1, we have $\gamma_{kSS}(G) = m$ and thus $d_{SS}^{(k,k)}(G) = 1$.

(2) Assume that $r = k + 2p + 1$ with $p \geq 1$. In view of Proposition 2.1 and Corollary 2.11, we obtain $k \leq d_{SS}^{(k,k)}(G) \leq r$.

If we suppose that $d_{SS}^{(k,k)}(G) = r$, then Theorem 2.6 yields to the contradiction $r \leq (k-1)r/k$.

Next, we suppose that $d_{SS}^{(k,k)}(G) = r - 1 = k + 2p$. In that case Theorem 2.3 leads to the contradiction $r - 1 \leq kr/(k+1)$.

Now suppose that $d_{SS}^{(k,k)}(G) = r - 2 = k + 2p - 1$, and let $\{f_1, f_2, \ldots, f_{k+2p-1}\}$ be a signed star (k,k)-dominating family of G. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{k+2p-1} f_i(e) \leq k$. If k is odd, then on the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, it follows that $\sum_{i=1}^{k+2p-1} f_i(e) \leq k-1$. If k is even, then we obtain analogously the same bound $\sum_{i=1}^{k+2p-1} f_i(e) \leq k-1$. If $v \in V(G)$ is an arbitrary vertex, then $\sum_{e \in E(v)} f_i(e) \geq k$ for each $1 \leq i \leq k+2p-1$. Therefore $f_i(e) = -1$ for at most p edges $e \in E(v)$ and thus $\sum_{e \in E(v)} f_i(e) \geq k+1$ for each $1 \leq i \leq k+2p-1$. Using the identity $2|E(G)| = |V(G)|(k+2p+1)$, we deduce that

$$\begin{aligned} |V(G)|(k+2p+1)(k-1) = 2|E(G)|(k-1) \geq 2 \sum_{e \in E(G)} \sum_{i=1}^{r-2} f_i(e) = \\ = \sum_{v \in V(G)} \sum_{i=1}^{r-2} \sum_{e \in E(v)} f_i(e) \geq \sum_{v \in V(G)} \sum_{i=1}^{r-2} (k+1) = \\ = |V(G)|(k+2p-1)(k+1). \end{aligned}$$

It follows that $(k+2p+1)(k-1) \geq (k+2p-1)(k+1)$, and we obtain the contradiction $-2p \geq 2p$. Altogether, we have shown that $k \leq d_{SS}^{(k,k)}(G) \leq r-3$ in that case.

(3) Assume that $r = k+2p$ with $p \geq 1$. Proposition 2.1 and Corollary 2.11 imply $k \leq d_{SS}^{(k,k)}(G) \leq r$. If we suppose that $d_{SS}^{(k,k)}(G) = r - 1 = k + 2p - 1$, then it follows from Theorem 2.6 that

$$d_{SS}^{(k,k)}(G) = k + 2p - 1 \leq \frac{k-1}{k}(k+2p),$$

and we obtain the contradiction $2p \leq 0$. Hence $d_{SS}^{(k,k)}(G) \neq r - 1$.

Now assume that $d_{SS}^{(k,k)}(G) = r = k + 2p$, and let $\{f_1, f_2, \ldots, f_{k+2p}\}$ be a signed star (k, k)-dominating family of G. Applying Proposition 2.1, we deduce that $\sum_{e \in E(v)} f_i(e) = k$ for each $v \in V(G)$ and each $1 \leq i \leq k + 2p$. Then for each $1 \leq i \leq k+2p$, each vertex $v \in V(G)$ is adjacent to exactly p edges $e_1^i, e_2^i, \ldots, e_p^i$ such that $f_i(e_1^i) = f_i(e_2^i) = \ldots f_i(e_p^i) = -1$. However, this is only possible if G contains a p-regular factor, and the proof is complete. □

Theorem 3.1 (2) implies the next result immediately.

Corollary 3.2. *If $k \geq 1$ is an integer and G is a $(k + 3)$-regular graph, then $d_{SS}^{(k,k)}(G) = k$.*

Corollary 3.3. *If $k \geq 1$ is an integer and G is a $(k+2p)$-regular graph of odd order n with $p \geq 1$ odd, then $k \leq d_{SS}^{(k,k)}(G) = k + 2p - 2$.*

Proof. Using Theorem 3.1 (3), we see that $d_{SS}^{(k,k)}(G) = k+2p$ or $d_{SS}^{(k,k)}(G) \leq k+2p-2$. If $d_{SS}^{(k,k)}(G) = k + 2p$, then Theorem 3.1 (3) implies that G contains a p-regular factor. Since n and p are odd, this is impossible, and thus Theorem 3.1 (3) yields to $k \leq d_{SS}^{(k,k)}(G) \leq k + 2p - 2$. □

Corollary 3.3 leads to the follwing supplement to Theorem 2.5.

Corollary 3.4. *Let $k \geq 2$ be an even integer, and let G be a $\delta(G)$-regular graph of odd order n such that $\delta(G) \geq k$ and $\delta(\overline{G}) \geq k$. If $\delta(G) = k + 2p$ with an odd integer $p \geq 1$, then*

$$d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 2.$$

Proof. In view of Corollary 2.2, we see that $d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 1$. Suppose to the contrary that $d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) = n - 1$. Then Proposition 2.1 implies that $d_{SS}^{(k,k)}(G) = \delta(G) = k + 2p$. However, Corollary 3.3 leads to the contradiction $d_{SS}^{(k,k)}(G) \leq k + 2p - 2$, and the proof is complete. □

Corollary 3.5. *If $k \geq 1$ is an integer and G a $(k + 2)$-regular graph of odd order n, then $d_{SS}^{(k,k)}(G) = k$.*

Let H be a $(k + 2)$-regular bipartite graph. By a well-known result of König [3], there exists a decomposition of $E(H)$ in perfect matchings $M_1, M_2, \ldots, M_{k+2}$. Now define $f_i : E(H) \longrightarrow \{-1, 1\}$ by $f_i(e) = -1$ when $e \in M_i$ and $f_i(e) = 1$ when $e \in E(H) - M_i$ for $1 \leq i \leq k + 2$. Then $f_i(v) = \sum_{e \in E(v)} f_i(e) = k$ for each $v \in V(H)$ and each $1 \leq i \leq k + 2$ and $\sum_{i=1}^{k+2} f_i(e) = k$ for every $e \in E(H)$. Therefore $\{f_1, f_2, \ldots, f_{k+2}\}$ is a signed star (k, k)-dominating family on H, and consequently $d_{SS}^{(k,k)}(H) = k + 2$. This family of examples demonstrates that $d_{SS}^{(k,k)}(G) = k + 2$ in Corollary 3.5 is possible when the order of G is even.

Theorem 3.6. *Let $k \geq 1$ and $p \geq 2$ be integers, and let G be an r-regular graph with $r = k + 2p + 1$. If $p < k + 1$, then $d_{SS}^{(k,k)}(G) \leq r - 4$.*

Proof. According to Theorem 3.1 (2), we have $d_{SS}^{(k,k)}(G) \leq r-3$. We suppose to the contrary that $d_{SS}^{(k,k)}(G) = r-3 = k+2p-2$. Let $\{f_1, f_2, \ldots, f_{k+2p-2}\}$ be a signed star (k,k)-dominating family of G. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{k+2p-2} f_i(e) \leq k$. If $v \in V(G)$ is an arbitrary vertex, then $\sum_{e\in E(v)} f_i(e) \geq k$ for each $1 \leq i \leq k+2p-2$. As in the proof of Theorem 3.1 (2), we see that $\sum_{e\in E(v)} f_i(e) \geq k+1$ for each $1 \leq i \leq k+2p-2$. Using again the identity $2|E(G)| = |V(G)|(k+2p+1)$, we deduce that

$$\begin{aligned} |V(G)|(k+2p+1)k = 2|E(G)|k &\geq 2\sum_{e\in E(G)}\sum_{i=1}^{r-3} f_i(e) = \\ &= \sum_{v\in V(G)}\sum_{i=1}^{r-3}\sum_{e\in E(v)} f_i(e) \geq \sum_{v\in V(G)}\sum_{i=1}^{r-3}(k+1) = \\ &= |V(G)|(k+2p-2)(k+1). \end{aligned}$$

It follows that $(k+2p+1)k \geq (k+2p-2)(k+1)$. This yields $k+1 \geq p$, a contradiction to the hypothesis $p < k+1$. □

Theorem 3.7. *Let $k \geq 1$ and $p \geq 2$ be integers, and let G be an r-regular graph with $r = k+2p+1$. If $k+1 < 2p$, then $d_{SS}^{(k,k)}(G) \neq r-4$.*

Proof. Suppose to the contrary that $d_{SS}^{(k,k)}(G) = r-4 = k+2p-3$. Let $\{f_1, f_2, \ldots, f_{k+2p-3}\}$ be a signed star (k,k)-dominating family of G. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{k+2p-3} f_i(e) \leq k$. If k is oddd, then on the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, it follows that $\sum_{i=1}^{k+2p-3} f_i(e) \leq k-1$. If k is even, then we obtain analogously the same bound $\sum_{i=1}^{k+2p-3} f_i(e) \leq k-1$. If $v \in V(G)$ is an arbitrary vertex, then we obtain as above that $\sum_{e\in E(v)} f_i(e) \geq k+1$ for each $1 \leq i \leq k+2p-3$. Using the identity $2|E(G)| = |V(G)|(k+2p+1)$, we deduce that

$$\begin{aligned} |V(G)|(k+2p+1)(k-1) = 2|E(G)|(k-1) &\geq 2\sum_{e\in E(G)}\sum_{i=1}^{r-4} f_i(e) = \\ &= \sum_{v\in V(G)}\sum_{i=1}^{r-4}\sum_{e\in E(v)} f_i(e) \geq \sum_{v\in V(G)}\sum_{i=1}^{r-4}(k+1) = \\ &= |V(G)|(k+2p-3)(k+1). \end{aligned}$$

It follows that $(k+2p+1)(k-1) \geq (k+2p-3)(k+1)$ and hence $k+1 \geq 2p$. This is a contradiction to the hypothesis $k+1 < 2p$, and the proof is complete. □

Combining Theorems 3.1, 3.6 and 3.7, we obtain the next bounds on $d_{SS}^{(k,k)}(G)$ immediately.

Corollary 3.8. *Let $k \geq 1$ and $p \geq 2$ be integers, and let G be an r-regular graph with $r = k+2p+1$. If $k+1 < 2p < 2k+2$, then $k \leq d_{SS}^{(k,k)}(G) \leq r-5$.*

The special case $k = p = 2$ in Corollary 3.8 leads to the following result.

Corollary 3.9. *If G is a 7-regular graph, then $d_{SS}^{(2,2)}(G) = 2$.*

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Received: February 17, 2012.
Revised: February 28, 2013.
Accepted: January 23, 2014.