

POSITIVITY PRESERVING RESULTS FOR A BIHARMONIC EQUATION UNDER DIRICHLET BOUNDARY CONDITIONS

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Abstract. We prove a dichotomy result giving the positivity preserving property for a biharmonic equation with Dirichlet boundary conditions arising in MEMS models. We adapt some ideas in [H.-Ch. Grunau, G. Sweers, *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. 307 (1997), 589–626].

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1. INTRODUCTION

In this paper, we focus on the following biharmonic equation with Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u - \tau \Delta u = f & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\tau > 0$ and Ω is a smooth bounded domain of \mathbb{R}^n ($n \geq 2$). Equation (1.1) arises for example in the micro-electromechanical systems (MEMS devices) giving the modelization of electrostatic actuation for membranes deflecting on thin plates in the field of nanotechnology detection systems [7,8]. The Dirichlet boundary conditions are also called clamped boundary conditions giving rise to zero vertical displacement and zero slope. The positive parameter τ represents the tension constant rising in the stretching energy sector in the presence of elastic deformation.

Our motivation comes from the fact that the maximum principle or the *positivity preserving property* fails often in the case of Dirichlet boundary conditions with the polyharmonic operators, on the contrary for the case of Navier boundary conditions ($u = \Delta u = 0$ on $\partial\Omega$). More precisely, we are interested in the following question: if the

source term f is positive in Ω , do we have the solution u of (1.1) positive in Ω , for any $\tau > 0$?

For the one-dimensional case, Grunau (see [3, Proposition 1]) has given a positive answer to this question. He proved that for $\lambda \leq 0$ and $a \in \mathbb{R}$, for every solution u of

$$\begin{cases} u'''' + au'' + \lambda u'' = f & \text{in } (-1, 1), \\ u(-1) = u'(-1) = u(1) = u'(1) = 0, \end{cases} \quad (1.2)$$

we have $0 \not\equiv f \geq 0$ implies $u > 0$ (the solution operator for (1.2) is said to be *strongly positively preserving*). The case $\lambda > 0$ (which corresponds to $\tau < 0$ in our situation) was also well understood in [3].

This result was recently extended by Laurençot and Walker [6] giving a general result on the sign-preserving property in radial symmetry for a ball in \mathbb{R}^n , $n \geq 2$. Nevertheless, in Section 2, we give a very simple and direct proof showing that if Ω is a ball in \mathbb{R}^n , ($n \geq 2$) and if the source term f is radially symmetric, then for any $\tau > 0$, problem (1.1) is strongly positivity preserving, i.e. $u > 0$ if $0 \not\equiv f \geq 0$ (see Proposition 2.1 below).

Our main concern here is to show a dichotomy result, adapting some ideas and techniques used by Grunau and Sweers [4] in the study of positivity for equations involving $(-\Delta)^m$ and the Dirichlet boundary conditions (see also [2] and the references therein). In fact, we consider the problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = a(x)u + f & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Let Λ_1^τ denote the first eigenvalue of the operator $L_\tau = \Delta^2 - \tau \Delta$ under the Dirichlet boundary conditions given by

$$\Lambda_1^\tau = \inf_{u \in H_0^2(\Omega) \setminus \{0\}} \frac{\|\Delta u\|_{L^2}^2 + \tau \|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} > 0.$$

Let Φ_1^τ be a corresponding eigenfunction. Define $(L_\tau - a)^{inv}$ the Green operator corresponding to the problem (1.3):

$$u = (L_\tau - a)^{inv} f.$$

It is well known that for $a \in C(\overline{\Omega})$ verifying $\max_{\overline{\Omega}} a < \Lambda_1^\tau$ and $p > 1$, the operators

$$(L_\tau - a)^{inv} : L^p(\Omega) \rightarrow W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$$

are well defined.

We say that an operator $T : L^p(\Omega) \rightarrow L^p(\Omega)$ is *positivity preserving* if $0 \not\equiv f \geq 0$ implies $Tf \geq 0$, and T is *strongly positivity preserving* if $0 \not\equiv f \geq 0$ implies $Tf > 0$.

Our main result is the following.

Theorem 1.1. *Let $p > 1$. Suppose that there holds $\Phi_1^\tau > 0$ in Ω . Then there exists $-\infty < \lambda_c \leq \Lambda_1^\tau$ such that for all $a \in C(\bar{\Omega})$, we have, for u and $f \in L^p(\Omega)$ verifying (1.3):*

- 1) *If $\lambda_c \leq a < \Lambda_1^\tau$ in $\bar{\Omega}$, then $(L_\tau - a)^{inv}$ is positivity preserving, i.e. $f \geq 0 \implies u \geq 0$.*
- 2) *If $a < \lambda_c$ in $\bar{\Omega}$, then $(L_\tau - a)^{inv}$ is not positivity preserving. Indeed, we have in this case:*
 - i) $0 \neq f \geq 0 \implies u \not\geq 0$,
 - ii) $\exists 0 \neq f \geq 0$ with $u \geq 0$,
 - iii) $\exists 0 \neq f \geq 0$ with $u \not\leq 0$.

This result is analogous to Corollary 6.4 in [4] stated for general bounded domains Ω in \mathbb{R}^n , $n \geq 2$, with $\partial\Omega \in C^{2m,\gamma}$ for some $\gamma > 0$, $m \geq 2$:

$$\begin{cases} (-\Delta)^m u = a(x)u + f & \text{in } \Omega, \\ \mathcal{D}_m u := (D^k u)_{k \in \mathbb{N}^n, |k| \leq m-1} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

The problem (1.4) with $a(x) \equiv 0$ and Ω is a unit ball $B \subset \mathbb{R}^n$ for any n , was studied by Boggio [1] who showed the positivity preserving result by proving the positivity of the Green function. However, it is well known that the *positivity preserving property* could no longer hold true even for some convex two-dimensional domains with Δ^2 and the Dirichlet boundary conditions. On the other hand, Grunau and Sweers proved in [5] that for domains with small C^k deformation from the ball, the Green function remains positive, while the positivity preserving breaks down when the deformation becomes important. For more discussions, we refer to the book [2].

Here we consider the case $m = 2$ but with $\tau > 0$ in (1.3). That means that we consider a lower order perturbation of the biharmonic operator Δ^2 by adding the term $-\tau\Delta u$. We should remark that the coercivity of the operator L_τ cannot imply the positivity preserving property for L_τ . We know also by Theorem 5.1 in [4] that when τ and $a(x)$ are small and Ω is a ball, problem (1.3) is strongly positivity preserving. Our contribution consists then in considering $\Delta^2 - \tau\Delta$ instead of just the biharmonic operator Δ^2 (in view of Corollary 6.4) and a general bounded smooth domain Ω instead of a ball.

Following the ideas in [4], a key point to prove in Theorem 1.1 above is some kind of *continuity result* for the positivity preserving property (see Proposition 3.2) and a crucial result asserting that when $\lambda \ll 0$, the operator

$$(L_\tau - \lambda)^{inv} : L^p(\Omega) \rightarrow W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$$

is not positivity preserving (see Proposition 3.4). Such a result for $\lambda \ll 0$ is proved using the continuity result together with the construction of some changing-sign solution. We adapted techniques in [4] and used a simple but very helpful result (see Lemma 3.3) in order to overcome the difficulty introduced by the harmonic perturbation. Actually, what we will need is a pointwise estimate for the infinity norm of a solution u to problem (1.1) with a constant source term $f \equiv 1$ on a ball B_R , and this is provided by the technical Lemma 3.3.

2. POSITIVITY PRESERVING FOR RADIAL SOLUTIONS IN DIMENSION n

Consider the problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = f(x) & \text{in } B, \\ u = \nabla u = 0 & \text{on } \partial B, \end{cases} \quad (2.1)$$

where $B := B(0, 1)$ is the unit ball in \mathbb{R}^n , $n \geq 2$.

Proposition 2.1. *Suppose that the source term f is radially symmetric, then for any $\tau > 0$, problem (2.1) is strongly positivity preserving, i.e. $u > 0$ if $0 \not\equiv f \geq 0$.*

Proof. Let f be nonnegative and radially symmetric, i.e. $f(x) = f(r)$, with $0 \leq r = |x| \leq 1$. Then the solution u to problem (2.1) is radially symmetric. Let $v = -\Delta u$ in B then v is radially symmetric and satisfies

$$(-\Delta + \tau)v = f \geq 0 \quad \text{in } B. \quad (2.2)$$

If $v(1) \geq 0$, by the strong maximum principle for $L_\tau := -\Delta + \tau$, we know that $v > 0$ in B . This is just impossible, since using the Dirichlet boundary condition, there holds

$$\int_B v dx = \int_{\partial B} \frac{\partial u}{\partial \nu} d\sigma = 0.$$

Therefore, $v(1) < 0$.

Using again the maximum principle for L_τ , we cannot have a subdomain $\Omega \subset B$ such that $v < 0$ in Ω and $v = 0$ on $\partial\Omega$. So there exists $r_0 \in (0, 1)$ such that $v \geq 0$ in $(0, r_0)$ and $v < 0$ in $(r_0, 1]$. Then the function

$$r \mapsto h(r) := \int_{B_r} v dx = -u'(r)r^{n-1}|\partial B|$$

is nondecreasing in $(0, r_0)$ and decreasing in $(r_0, 1]$, which means $h(r) \geq 0$ for $0 < r < 1$ as $h(0) = h(1) = 0$. So u is nonincreasing w.r.t. r , which yields finally $u > 0$ in B since $u|_{\partial B} = 0$ and u is decreasing near 1. \square

3. A DICHOTOMY RESULT FOR POSITIVITY PRESERVING PROPERTY

In what follows, Ω is a general smooth domain and we prove some results inspired by [4].

Proposition 3.1. *Let $p > 1$. Let $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ and $f \in L^p(\Omega)$ satisfy (1.3). Suppose that we have $\Phi_1^\tau > 0$ in Ω . If $0 \not\equiv a - \Lambda_1^\tau \geq 0$, then*

$$0 \not\equiv f \geq 0 \implies u \not\geq 0 \quad (\text{positivity killing}).$$

If $a \equiv \Lambda_1^\tau$, then for $0 \not\equiv f \geq 0$ there is no solution.

Proof. Suppose that we have $0 \neq f \geq 0$ and $0 \neq u \geq 0$. Then

$$0 < \int_{\Omega} f \Phi_1^\tau dx = \int_{\Omega} (L_\tau u - au) \Phi_1^\tau dx = \int_{\Omega} u (\Lambda_1^\tau - a) \Phi_1^\tau dx \leq 0, \quad (3.1)$$

which is a contradiction if $0 \neq a - \Lambda_1^\tau \geq 0$.

If $a \equiv \Lambda_1^\tau$, again from (3.1), we see that no solution could exist. \square

Proposition 3.2. *Let $\tau > 0$, $a \in C(\bar{\Omega})$ with $a < \Lambda_1^\tau$. Suppose that*

$$(L_\tau - a)^{inv} : L^p(\Omega) \rightarrow W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$$

is positivity preserving (resp. strongly positivity preserving) for $p > 1$. Then for all $b \in C(\bar{\Omega})$ with $a \leq b < \Lambda_1^\tau$ in $\bar{\Omega}$, we have $(L_\tau - b)^{inv}$ is also positivity preserving (resp. strongly positivity preserving).

Proof. We will prove only for the positivity preserving case, since the other one is completely similar.

First assume that $p \geq 2$. Suppose that $u \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$ satisfies $L_\tau u - bu = f$ in Ω with $b - a \geq 0$. Then u solves $L_\tau u - au = f + (b - a)u$ in Ω .

Consider now the nonlinear problem

$$\begin{cases} (L_\tau - a)v = f + (b - a)|v| & \text{in } \Omega, \\ v = \nabla v = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

and the corresponding energy functional F on $H_0^2(\Omega)$ defined by

$$F(v) = \int_{\Omega} [(\Delta v)^2 + \tau |\nabla v|^2 - av^2 - 2fv - (b - a)|v|v] dx.$$

Set

$$\varepsilon = \min \left\{ 2\Lambda_1^\tau, \min_{x \in \bar{\Omega}} (\Lambda_1^\tau - b(x)) \right\} > 0.$$

Then we have

$$\begin{aligned} F(v) &\geq \int_{\Omega} [(\Delta v)^2 + \tau |\nabla v|^2 - 2fv - bv^2] dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau} \int_{\Omega} [(\Delta v)^2 + \tau |\nabla v|^2] dx + \int_{\Omega} \left[\left(\Lambda_1^\tau - \frac{1}{2}\varepsilon - b \right) v^2 - 2fv \right] dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau} \int_{\Omega} (\Delta v)^2 dx + \int_{\Omega} \left(\frac{1}{2}\varepsilon v^2 - 2fv \right) dx \geq \\ &\geq \frac{\varepsilon}{2\Lambda_1^\tau} \int_{\Omega} (\Delta v)^2 dx - \frac{1}{\varepsilon} \int_{\Omega} f^2 dx. \end{aligned}$$

Since $F(\cdot)$ is weakly lower semicontinuous and coercive in $H_0^2(\Omega)$, there exists a minimizer $u_* \in H_0^2(\Omega)$ which verifies the Euler-Lagrange equation $(L_\tau - a)u_* = f + (b - a)|u_*|$. Using regularity theory and appropriate embeddings, a boot-strapping argument leads to $u_* \in W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega)$.

Coming back to the positivity preserving hypothesis of the operator $(L_\tau - a)^{inv}$, as $f + (b - a)|u_*| \geq 0$, we have then the solution u_* of (3.2) is nonnegative:

$$u_* = (L_\tau - a)^{inv}(f + (b - a)|u_*|) \geq 0.$$

Hence u_* solves (1.3) with b instead of a , that is $u = u_* \geq 0$.

Finally, for $p \in (1, 2)$, the claim follows by approximating $f \in L^p(\Omega)$ by a sequence $f_k \in L^2(\Omega)$. □

Lemma 3.3. *Let $\tau > 0$ and u be the solution to*

$$\begin{cases} \Delta^2 u - \tau \Delta u = 1 & \text{in } B_R := B(0, R) \subset \mathbb{R}^n, \\ u = \Delta u = 0 & \text{on } \partial B_R. \end{cases} \tag{3.3}$$

Then $\|u\|_\infty < \frac{R^2}{2\tau n}$.

Proof. Let $v = -\Delta u$. Then $-\Delta v + \tau v = 1$ and $v = 0$ on ∂B_R . So immediately $0 \leq v < \tau^{-1}$ by the maximum principle, or equivalently, as u is radial,

$$0 \leq -\frac{1}{r^{n-1}}(r^{n-1}u')' < \frac{1}{\tau}, \quad 0 < r \leq R.$$

Integrating from 0 to $r > 0$, we obtain $0 \leq -r^{n-1}u' < \frac{r^n}{\tau n}$, that is $0 \leq -u' < \frac{r}{\tau n}$ for $r \in (0, R]$. Then integrating from 0 to R , as u is decreasing and $u = 0$ on ∂B_R , we obtain $u(0) = \|u\|_\infty < \frac{R^2}{2\tau n}$. □

Applying Proposition 3.2 and Lemma 3.3, we get the following crucial result.

Proposition 3.4. *For $\lambda \ll 0$, the operator*

$$(L_\tau - \lambda)^{inv} : L^p(\Omega) \rightarrow W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega),$$

$p > 1$, is not positivity preserving for any $\tau > 0$.

Proof. Using first Proposition 3.2, if $(L_\tau)^{inv}$ is not positivity preserving, we are done as $\Lambda_1^\tau > 0$.

Hence we may assume that $(L_\tau)^{inv}$ preserves the positivity. We will construct as in [4] a nonpositive nontrivial test function $a \in C(\bar{\Omega})$ such that $(L_\tau - a)^{inv}$ is not positivity preserving. Therefore we would have $(L_\tau - \lambda)^{inv}$ is not positivity preserving for $\lambda \leq \min a < 0$, since otherwise we get a contradiction, again by Proposition 3.2.

The construction of such a function a is related to the construction of a sign-changing function \tilde{u} satisfying $(L_\tau - a)^{inv}\tilde{u} \geq 0$ in Ω and $\tilde{u} = \nabla\tilde{u} = 0$ on $\partial\Omega$. Set $v_1 = (L_\tau)^{inv}1$, which is then positive by our assumption. By scaling, we may

assume that Ω contains the unit ball B and let u_0 be the solution to (3.3) on B . Consider $\varepsilon > 0$ small and the function

$$u_\varepsilon(x) = 16r^2 - \varepsilon - 32\tau nu_0(0) + 32\tau nu_0(x),$$

where $r = |x|$. As $u_0(0) < \frac{1}{2\tau n}$ by Lemma 3.3, we can choose ε small enough such that $u_\varepsilon(1) = 16 - \varepsilon - 32\tau nu_0(0) > 0$. Hence there exists an interval $[r_1, r_2] \subset (0, 1)$ verifying $u_\varepsilon > 0$ in $[r_1, r_2]$.

Now fix $r_3 \in (r_1, r_2)$ and consider the cut-off function $\chi \in C_c^\infty(\Omega)$ such that $\text{supp}(\chi) \subset \{|x| \leq r_3\}$ and $\chi \equiv 1$ on $\{|x| \leq r_0\}$ for $r_1 < r_0 < r_3$. Define finally

$$\tilde{u}(x) = \chi(r)u_\varepsilon(x) + [1 - \chi(r)]v_1(x).$$

For $|x| \geq r_3$, $\chi \equiv 0$, we have then $\tilde{u}(x) = v_1(x) \geq 0$ and $L_\tau \tilde{u} = L_\tau v_1 = 1$.

We have also $\tilde{u}(x) = u_\varepsilon(x)$ in B_{r_0} , hence $L_\tau \tilde{u} = L_\tau u_\varepsilon = 0$ on B_{r_0} and $\tilde{u}(0) = u_\varepsilon(0) = -\varepsilon < 0$. Moreover, in $B_{r_3} \setminus B_{r_0}$, $\tilde{u} = \chi(r)u_\varepsilon(x) + [1 - \chi(r)]v_1(x) > C_\varepsilon > 0$ and $L_\tau \tilde{u} \geq -C$ for some positive constants C_ε and C . We can conclude then

$$L_\tau \tilde{u} = \begin{cases} 0 & \text{in } B_{r_0}, \\ 1 & \text{in } \Omega \setminus B_{r_3}, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{u}(0) < 0, \\ \tilde{u}(x) > 0 & \text{in } \Omega \setminus B_{r_1}, \\ \tilde{u} = \nabla \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet boundary condition comes from the fact $\tilde{u} \equiv v_1$ for $|x| \geq r_3$.

Finally, set $a(x) = -M\chi_1(r) \leq 0$ with χ_1 a cut-off function $\chi_1(r) \equiv 1$ on $\{r_0 \leq |x| \leq r_3\}$ and $\text{supp}(\chi_1) \subset \{r_1 < |x| < r_2\}$. By choosing the constant $M > 0$ sufficiently large, we can claim that the sign-changing function \tilde{u} verifies $(L_\tau - a)\tilde{u} \geq 0$ in Ω . Hence Proposition 3.2 yields that $(L_\tau - \lambda)^{inv}$ is not positivity preserving for $\lambda \leq \min a < 0$. \square

Using Propositions 3.1, 3.2 and 3.4, we are now able to prove the dichotomy result.

Proof of Theorem 1.1. From Propositions 3.1 and 3.2, one finds that the set of $\lambda \in \mathbb{R}$ for which the operator $(L_\tau - a)^{inv} : L^p(\Omega) \rightarrow W^{4,p}(\Omega) \cap W_0^{2,p}(\Omega) \subset L^p(\Omega)$ is positivity preserving is an interval $[\lambda_c, \Lambda_1^\tau]$ or $(\lambda_c, \Lambda_1^\tau)$. Proposition 3.4 yields that $\lambda_c > -\infty$. Since $\lambda \mapsto (L_\tau - \lambda)^{inv}$ is continuous except at the eigenvalues, we find that the interval is left closed.

The previous argument by Proposition 3.2 implies clearly claim 2 iii). Moreover, for claim 2 ii), we can just take $u = \Phi_1^\tau$ which verifies $\Phi_1^\tau > 0$ and $(L_\tau - a)\Phi_1^\tau = (\Lambda_1^\tau - a)\Phi_1^\tau > 0$.

Let $a < \lambda_c \leq \Lambda_1^\tau$ in $\bar{\Omega}$ and suppose that for some $0 \neq f \geq 0$, $u = (L_\tau - a)^{inv} f \leq 0$. Then

$$0 < \int_\Omega f \Phi_1^\tau dx = \int_\Omega (L_\tau u - au)\Phi_1^\tau dx = \int_\Omega u(\Lambda_1^\tau - a)\Phi_1^\tau dx \leq 0,$$

which is impossible, hence we get claim 2 i). \square

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