

## EXISTENCE AND CONTROLLABILITY RESULTS FOR DAMPED SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY

M. Mallika Arjunan and N.Y. Nadaf

*Communicated by Alexander Domoshnitsky*

**Abstract.** In this paper, we investigate the existence and controllability of mild solutions for a damped second order impulsive functional differential equation with state-dependent delay in Banach spaces. The results are obtained by using Sadovskii's fixed point theorem combined with the theories of a strongly continuous cosine family of bounded linear operators. Finally, an example is provided to illustrate the main results.

**Keywords:** damped second order differential equations, impulsive differential equations, controllability, state-dependent delay, cosine function, mild solution, fixed point.

**Mathematics Subject Classification:** 34A37, 93B05.

### 1. INTRODUCTION

In this paper, we are interested to study the existence and controllability of mild solutions for a damped second order impulsive functional differential equation with state-dependent delay in Banach spaces. First, we consider the following class of damped second order impulsive neutral functional differential equation with state-dependent delay in the form:

$$x''(t) = Ax(t) + \mathcal{D}x'(t) + f(t, x_{\rho(t, x_t)}), \quad t \in I = [0, a], \quad (1.1)$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \quad (1.2)$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \dots, n, \quad (1.3)$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \dots, n, \quad (1.4)$$

where  $A$  is the infinitesimal generator of a strongly continuous cosine function of bounded linear operator  $(C(t))_{t \in \mathbb{R}}$  defined on a Banach space  $X$ ; the function  $x_s : (-\infty, 0] \rightarrow X$ ,  $x_s(\theta) = x(s + \theta)$ , belongs to some abstract phase space  $\mathcal{B}$

described axiomatically;  $\mathcal{D}$  is a bounded linear operator on a Banach space  $X$ ;  $0 < t_1 < \dots < t_n < a$  are prefixed numbers;  $f : I \times \mathcal{B} \rightarrow X$ ,  $\rho : I \times \mathcal{B} \rightarrow (-\infty, a]$ ,  $I_i(\cdot) : \mathcal{B} \rightarrow X$ ,  $J_i(\cdot) : \mathcal{B} \rightarrow X$  are appropriate functions and the symbol  $\Delta\xi(t)$  represents the jump of the function  $\xi(\cdot)$  at  $t$ , which is defined by  $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$ .

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These process tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more details on this theory and on its applications, we refer to the monographs of Lakshmikantham *et al.* [39], Samoilenko and Perestyuk [50], Bainov and Simeonov [5], and the papers of [9, 10, 13–17, 29–31, 41, 47, 48] and the references therein.

In control theory, one of the most important qualitative properties of dynamical systems is controllability. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where the initial and final states may vary over the entire space. Many authors have studied the controllability of nonlinear systems with and without impulses, see for instance [6, 37, 43–46, 51]. In dynamical systems damping is another important issue; it may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. Concerning first and second order differential equations with a damped term we cite [3, 4, 32, 40, 52] among some works.

On the other hand, functional differential equations with state-dependent delay appear frequently in applications as the model of equations and for this reason the study of this type of equations has received much attention in the recent years. The reader is referred to [1, 2, 7, 8, 12, 18–20, 22–28, 42] and the references therein for some examples and applications. The literature related to second order impulsive differential systems with state-dependent delay is very limited, and related to this matter we only cite [11, 34]. To the best of our knowledge, the study of the existence and controllability system described in the abstract form (1.1)–(1.4) is an untreated problem, and this fact is the main motivation of this paper.

This paper is organized as follows. In Section 2, we recall some notations, definitions and preliminary facts which will be used throughout this paper. In Section 3, we establish sufficient conditions for the existence of mild solutions for the problem (1.1)–(1.4) by using Sadovskii's fixed point theorem combined with the theories of a cosine family of bounded linear operators. In Section 4, we study controllability results for the problem (1.1)–(1.4). In Section 5, we present some examples to show the application of the results.

## 2. PRELIMINARIES

In this section, we recall briefly some notations, definitions and lemmas needed to establish our main results.

Throughout this paper  $(X, \|\cdot\|)$  is a Banach space and  $A$  is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators  $(C(t))_{t \in \mathbb{R}}$  on Banach space  $X$ .

**Definition 2.1.** A one parameter family  $(C(t))_{t \in \mathbb{R}}$  of bounded linear operators mapping the Banach space  $X$  into itself is called a strongly continuous cosine family iff:

- (i)  $C(s+t) + C(s-t) = 2C(s)C(t)$  for all  $s, t \in \mathbb{R}$ ,
- (ii)  $C(0) = I$ ,
- (iii)  $C(t)x$  is continuous in  $t$  on  $\mathbb{R}$  for each fixed  $x \in X$ .

We denote by  $(S(t))_{t \in \mathbb{R}}$  the sine function associated with  $(C(t))_{t \in \mathbb{R}}$  which is defined by  $S(t)x = \int_0^t C(s)x ds$ ,  $x \in X$ ,  $t \in \mathbb{R}$  and we always assume that  $N$  and  $\bar{N}$  are positive constants such that  $\|C(t)\| \leq N$  and  $\|S(t)\| \leq \bar{N}$ , for every  $t \in I$ . The infinitesimal generator of a strongly continuous cosine family  $(C(t))_{t \in \mathbb{R}}$  is the operator  $A : X \rightarrow X$  defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where

$$D(A) = \{x \in X : C(t)x \text{ is twice differentiable in } t\}.$$

Define

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}.$$

The following properties are well known ([53]):

- (i) If  $x \in X$  then  $S(t)x \in E$  for every  $t \in \mathbb{R}$ .
- (ii) If  $x \in E$  then  $S(t)x \in D(A)$ ,  $(\frac{d}{dt})C(t)x = AS(t)x$  and  $(\frac{d^2}{dt^2})S(t)x = S(t)x$  for every  $t \in \mathbb{R}$ .
- (iii) If  $x \in D(A)$  then  $C(t)x \in D(A)$ , and  $(\frac{d^2}{dt^2})C(t)x = AC(t)x = C(t)Ax$  for every  $t \in \mathbb{R}$ .
- (iv) If  $x \in D(A)$  then  $S(t)x \in D(A)$ , and  $(\frac{d^2}{dt^2})S(t)x = AS(t)x = S(t)Ax$  for every  $t \in \mathbb{R}$ .

In this paper,  $[D(A)]$  stands for the domain of the operator  $A$  endowed with the graph norm  $\|x\|_A = \|x\| + \|Ax\|$ ,  $x \in D(A)$ . Moreover, in this work,  $E$  is the space formed by the vectors  $x \in X$  for which  $C(\cdot)x$  is of class  $C^1$  on  $\mathbb{R}$ . It was proved by Kisinsky ([38]) that  $E$  endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t)x\|, \quad x \in E, \quad (2.1)$$

is a Banach space. The operator valued function  $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$  is a strongly continuous group of bounded linear operators on the space  $E \times X$  generated by the operator  $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$  defined on  $D(A) \times E$ . It follows from this that  $AS(t) : E \rightarrow X$

is a bounded linear operator and that  $AS(t)x \rightarrow 0$ ,  $t \rightarrow 0$ , for each  $x \in E$ . Furthermore, if  $x : [0, \infty) \rightarrow X$  is a locally integrable function, then  $z(t) = \int_0^t S(t-s)x(s)ds$  defines an  $E$ -valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s) ds, & \int_0^t C(t-s)x(s) ds \end{bmatrix}^T$$

defines an  $E \times X$ -valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad 0 \leq t \leq a, \quad (2.2)$$

$$x(0) = z, \quad x'(0) = w, \quad (2.3)$$

where  $h : I \rightarrow X$  is an integrable function has been discussed in [53]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [54]. We only mention here that the function  $x(\cdot)$  given by

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad 0 \leq t \leq a, \quad (2.4)$$

is called a mild solution of (2.2)–(2.3), and that when  $z \in E$ ,  $x(\cdot)$  is continuously differentiable and

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad 0 \leq t \leq a. \quad (2.5)$$

For additional details about the cosine function theory, we refer the reader to [53, 54].

To consider the impulsive conditions (1.3)–(1.4), it is convenient to introduce some additional concepts and notations.

A function  $u : [\sigma, \tau] \rightarrow X$  is said to be a normalized piecewise continuous function on  $[\sigma, \tau]$  if  $u$  is piecewise continuous and left continuous on  $(\sigma, \tau]$ . We denote by  $\mathcal{PC}([\sigma, \tau], X)$  the space of normalized piecewise continuous functions from  $[\sigma, \tau]$  into  $X$ . In particular, we introduce the space  $\mathcal{PC}$  formed by all normalized piecewise continuous functions  $u : [0, a] \rightarrow X$  such that  $u$  is continuous at  $t \neq t_i$ ,  $u(t_i^-) = u(t_i)$  and  $u(t_i^+)$  exists, for  $i = 1, \dots, n$ . It is clear that  $\mathcal{PC}$  endowed with the norm  $\|u\|_{\mathcal{PC}} = \sup_{s \in I} \|u(s)\|$  is a Banach space.

In what follows, we set  $t_0 = 0$ ,  $t_{n+1} = a$ , and for  $u \in \mathcal{PC}$  we denote by  $\tilde{u}_i$ , for  $i = 0, 1, \dots, n-1$ , the function  $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$  given by  $\tilde{u}_i(t) = u(t)$  for  $t \in (t_i, t_{i+1}]$  and  $\tilde{u}_i(t_i) = \lim_{t \rightarrow t_i^+} u(t)$ . Moreover, for a set  $B \subseteq \mathcal{PC}$ , we denote by  $\tilde{B}_i$ , for  $i = 0, 1, \dots, n-1$ , the set  $\tilde{B}_i = \{\tilde{u}_i : u \in B\}$ .

**Lemma 2.2** ([33]). *A set  $B \subseteq \mathcal{PC}$  is relatively compact in  $\mathcal{PC}$  if and only if each set  $\tilde{B}_i$ ,  $i = 0, 1, \dots, n-1$ , is relatively compact in  $C([t_i, t_{i+1}], X)$ .*

In this work we will employ an axiomatic definition of the phase space  $\mathcal{B}$ , which has been used in [33] and suitably modified to treat retarded impulsive differential equations. Specifically,  $\mathcal{B}$  will be a linear space of functions mapping  $(-\infty, 0]$  into  $X$  endowed with a seminorm  $\|\cdot\|_{\mathcal{B}}$  and we will assume that  $\mathcal{B}$  satisfies the following axioms:

- (A) If  $x : (-\infty, \sigma + b) \rightarrow X$ ,  $b > 0$ , is such that  $x_{\sigma} \in \mathcal{B}$  and  $x|_{[\sigma, \sigma + b]} \in \mathcal{PC}([\sigma, \sigma + b], X)$ , then for every  $t \in [\sigma, \sigma + b)$  the following conditions hold:
  - (i)  $x_t$  is in  $\mathcal{B}$ ,
  - (ii)  $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}}$ ,
  - (iii)  $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma) \|x_{\sigma}\|_{\mathcal{B}}$ , where  $H > 0$  is a constant;  $K, M : [0, \infty) \rightarrow [1, \infty)$ ,  $K$  is continuous,  $M$  is locally bounded, and  $H, K, M$  are independent of  $x(\cdot)$ .
- (B) The space  $\mathcal{B}$  is complete.

For more details about phase space axioms and examples, we refer the reader to [27].

Additional terminologies and notations used in the sequel are standard in functional analysis. In particular, for Banach spaces  $(Z, \|\cdot\|_Z)$ ,  $(W, \|\cdot\|_W)$ , the notation  $\mathcal{L}(Z, W)$  stands for the Banach space of bounded linear operators from  $Z$  into  $W$  and we abbreviate to  $\mathcal{L}(Z)$  whenever  $Z = W$ . Additionally,  $B_r(x, Z)$  denotes the closed ball with center at  $x$  and radius  $r > 0$  in  $Z$ .

Our main results are based upon the following well-known result.

**Lemma 2.3** (Sadovskii's Fixed Point Theorem [49]). *Let  $G$  be a condensing operator on a Banach space  $X$ . If  $G(S) \subset S$  for a convex, closed and bounded set  $S$  of  $X$ , then  $G$  has a fixed point in  $S$ .*

### 3. EXISTENCE RESULTS

In this section, we discuss the existence of mild solutions for the abstract system (1.1)–(1.4). We also suppose that  $\rho : I \times \mathcal{B} \rightarrow (-\infty, a]$  is a continuous function. Additionally, we introduce the following conditions:

- ( $H_{\varphi}$ ) Let  $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B}, \rho(s, \psi) \leq 0\}$ . The function  $t \rightarrow \varphi_t$  is well defined from  $\mathcal{R}(\rho^-)$  into  $\mathcal{B}$  and there exists a continuous and bounded function  $J^{\varphi} : \mathcal{R}(\rho^-) \rightarrow (0, \infty)$  such that  $\|\varphi_t\|_{\mathcal{B}} \leq J^{\varphi}(t)\|\varphi\|_{\mathcal{B}}$  for every  $t \in \mathcal{R}(\rho^-)$ .
- ( $H_1$ ) The function  $f : I \times \mathcal{B} \rightarrow X$  satisfies the following conditions:
  - (i) Let  $x : (-\infty, a] \rightarrow X$  be such that  $x_0 = \varphi$  and  $x|_I \in \mathcal{PC}$ . The function  $t \rightarrow f(t, x_{\rho(t, x_t)})$  is measurable on  $I$  and the function  $t \rightarrow f(s, x_t)$  is continuous on  $\mathcal{R}(\rho^-) \cup I$  for every  $s \in I$ .
  - (ii) For each  $t \in I$ , the function  $f(t, \cdot) : \mathcal{B} \rightarrow X$  is continuous.
  - (iii) There exists an integrable function  $m : I \rightarrow [0, \infty)$  and a continuous non-decreasing function  $W : [0, \infty) \rightarrow (0, \infty)$  such that for every  $(t, \psi) \in I \times \mathcal{B}$

$$\|f(t, \psi)\| \leq m(t)W(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\xi \rightarrow \infty} \frac{W(\xi)}{\xi} = \Lambda < \infty.$$

(H<sub>2</sub>) There are positive constants  $L_{I_i}, L_{J_i}$  such that

$$\begin{aligned} \|I_i(\psi_1) - I_i(\psi_2)\| &\leq L_{I_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \dots, n, \\ \|J_i(\psi_1) - J_i(\psi_2)\| &\leq L_{J_i} \|\psi_1 - \psi_2\|_{\mathcal{B}}, \quad \psi_j \in \mathcal{B}, \quad i = 1, 2, \dots, n. \end{aligned}$$

**Remark 3.1.** The condition (H<sub>φ</sub>) is frequently satisfied by functions that are continuous and bounded. In fact, assume that the space of continuous and bounded functions  $C_b((-\infty, 0], X)$  is continuously included in  $\mathcal{B}$ . Then, there exists  $L > 0$  such that

$$\|\varphi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \|\varphi(\theta)\|}{\|\varphi\|_{\mathcal{B}}} \|\varphi\|_{\mathcal{B}}, \quad t \leq 0, \varphi \neq 0, \varphi \in C_b((-\infty, 0], X).$$

It is easy to see that the space  $C_b((-\infty, 0], X)$  is continuously included in  $\mathcal{PC}_g(X)$  and  $\mathcal{PC}_g^0(X)$ . Moreover, if  $g(\cdot)$  verifies (g-5)–(g-6) in [36] and  $g(\cdot)$  is integrable on  $(-\infty, -r]$ , then the space  $C_b((-\infty, 0], X)$  is also continuously included in  $\mathcal{PC}_r \times L^p(g; X)$ . For complementary details related to this matter, see Proposition 7.1.1 and Theorems 1.3.2 and 1.3.8 in [36].

Motivated by (2.4) we define the following concept of mild solutions for the system (1.1)–(1.4). If  $x(\cdot)$  is a solution of (1.1)–(1.4), then from (2.4), we adopt the following concept of mild solution,

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)\eta + \int_0^t S(t-s)[\mathcal{D}x'(s) + f(s, x_{\rho(s, x_s)})]ds + \\ &+ \sum_{0 < t_i < t} C(t-t_i)I_i(x_{t_i}) + \sum_{0 < t_i < t} S(t-t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

Motivated from the above expression, we present the following definition.

**Definition 3.2.** A function  $x : (-\infty, a] \rightarrow X$  is called a mild solution of the abstract Cauchy problem (1.1)–(1.4) if  $x_0 = \varphi, x_{\rho(s, x_s)} \in \mathcal{B}$  for every  $s \in I, x(\cdot)|_I \in \mathcal{PC}$  and

$$\begin{aligned} x(t) &= C(t)\varphi(0) + S(t)\eta + \sum_{i=0}^{j-1} [S(t-t_{i+1})\mathcal{D}x(t_{i+1}^-) - S(t-t_i)\mathcal{D}x(t_i^+)] - \\ &- S(t-t_j)\mathcal{D}x(t_j^+) + \int_0^t C(t-s)\mathcal{D}x(s)ds + \int_0^t S(t-s)f(s, x_{\rho(s, x_s)})ds + \\ &+ \sum_{0 < t_i < t} C(t-t_i)I_i(x_{t_i}) + \sum_{0 < t_i < t} S(t-t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

**Remark 3.3.** In the rest of this paper,  $y : (-\infty, a] \rightarrow X$  is the function defined by  $y(t) = \varphi(t)$  on  $(-\infty, 0]$  and  $y(t) = C(t)\varphi(0) + S(t)\zeta$  for  $t \in I$ . In addition,  $\|y\|_a, M_a, K_a,$  and  $J_0^\varphi$  are the constants defined by  $\|y\|_a = \sup_{s \in [0, a]} \|y(s)\|, M_a = \sup_{s \in [0, a]} M(s), K_a = \sup_{s \in [0, a]} K(s)$  and  $J_0^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t)$ .

**Lemma 3.4** ([35, Lemma 2.1]). *Let  $x : (-\infty, a] \rightarrow X$  be a function such that  $x_0 = \varphi$  and  $x(\cdot)|_I \in \mathcal{PC}$ . Then*

$$\|x_s\|_{\mathcal{B}} \leq (M_a + J_0^\varphi)\|\varphi\|_{\mathcal{B}} + K_a \sup\{\|x(\theta)\|, \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup I.$$

**Theorem 3.5.** *Let conditions  $(H_\varphi)$ ,  $(H_1)$ ,  $(H_2)$  hold and assume that  $S(t)$  is compact for every  $t \in \mathbb{R}$ . If*

$$K_a \left[ \frac{1}{K_a} (3\bar{N} + aN)\|\mathcal{D}\| + \bar{N} \liminf_{\xi \rightarrow \infty} \frac{W(\xi)}{\xi} \int_0^a m(s)ds + \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \right] < 1,$$

then there exists a mild solution of (1.1)–(1.4).

*Proof.* On the space  $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$  endowed with the uniform convergence topology, we define the operator  $\Gamma : Y \rightarrow Y$  by

$$\begin{aligned} \Gamma x(t) &= C(t)\varphi(0) + S(t)\eta + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \int_0^t S(t-s)f(s, \bar{x}_{\rho(s, \bar{x}_s)})ds + \\ &+ \sum_{i=0}^{j-1} [S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - S(t-t_j^+)\mathcal{D}\bar{x}(t_j^+) + \\ &+ \sum_{0 < t_i < t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0 < t_i < t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I, \end{aligned}$$

where  $\bar{x} : (-\infty, a] \rightarrow X$  is such that  $\bar{x}_0 = \varphi$  and  $\bar{x} = x$  on  $I$ . From the axiom (A) and our assumptions on  $\varphi$ , we infer that  $\Gamma x \in \mathcal{PC}$ .

Next, we prove that there exists  $r > 0$  such that  $\Gamma(B_r(y|_I, Y)) \subseteq B_r(y|_I, Y)$ . If we assume this property is false, then for every  $r > 0$  there exist  $x^r \in B_r(y|_I, Y)$  and  $t^r \in I$  such that  $r < \|\Gamma x^r(t^r) - y(t^r)\|$ . Then, from Lemma 3.4, we get

$$\begin{aligned} r &< \|\Gamma x^r(t^r) - y(t^r)\| \leq \\ &\leq 3\bar{N}\|\mathcal{D}\|r + aN\|\mathcal{D}\|r + \bar{N} \int_0^{t^r} m(s)W(\|\bar{x}_{\rho(s, (\bar{x}^r)_s)}\|_{\mathcal{B}})ds + \\ &+ \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \bar{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \leq \\ &\leq 3\bar{N}\|\mathcal{D}\|r + aN\|\mathcal{D}\|r + \\ &+ \bar{N}W((M_a + J_0^\varphi)\|\varphi\|_{\mathcal{B}} + K_a r + K_a\|y\|_a) \int_0^a m(s)ds + \\ &+ \sum_{i=1}^n N(L_{I_i}K_a r + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \bar{N}(L_{J_i}K_a r + \|J_i(y_{t_i})\|), \end{aligned}$$

and hence

$$1 \leq K_a \left[ \frac{1}{K_a} (3\bar{N} + aN) \|\mathcal{D}\| + \bar{N} \liminf_{\xi \rightarrow \infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \right],$$

which is contrary to our assumption.

Let  $r > 0$  be such that  $\Gamma(B_r(y|_I, Y)) \subset B_r(y|_I, Y)$ . In order to prove that  $\Gamma$  is a condensing map on  $B_r(y|_I, Y)$  into  $B_r(y|_I, Y)$ . For this, we decompose  $\Gamma$  as  $\Gamma_1 + \Gamma_2$ , where

$$\begin{aligned} \Gamma_1 x(t) &= C(t)\varphi(0) + S(t)\eta + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \\ &\quad + \sum_{i=0}^{j-1} [S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - \\ &\quad - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \sum_{0 < t_i < t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0 < t_i < t} S(t-t_i)J_i(\bar{x}_{t_i}), \\ \Gamma_2 x(t) &= \int_0^t S(t-s)f(s, \bar{x}_{\rho(s, \bar{x}_s)})ds, \end{aligned}$$

From the proof of [26, Theorem 3.4], we conclude that  $\Gamma_2$  is completely continuous. Moreover, from the estimate

$$\begin{aligned} \|\Gamma_1 x - \Gamma_1 z\|_{\mathcal{PC}} &\leq \\ &\leq 3\bar{N}\|\mathcal{D}\| \|x - z\|_{\mathcal{PC}} + aN\|\mathcal{D}\| \|x - z\|_{\mathcal{PC}} + K_a \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \|x - z\|_{\mathcal{PC}} \leq \\ &\leq K_a \left[ \frac{1}{K_a} (3\bar{N} + aN) \|\mathcal{D}\| + \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \right] \|x - z\|_{\mathcal{PC}}. \end{aligned}$$

It follows that  $\Gamma_1$  is contraction on  $B_r(y|_I, Y)$ , which implies that  $\Gamma$  is a condensing operator on  $B_r(y|_I, Y)$ .

Finally, from Lemma 2.3 we infer that there exists a mild solution of (1.1)–(1.4). This completes the proof.  $\square$

#### 4. CONTROLLABILITY RESULTS

In this section, we shall establish sufficient conditions for the controllability of mild solutions for a damped second order impulsive functional differential equation with



state-dependent delay. More precisely, we consider the following abstract control system in the form:

$$x''(t) = Ax(t) + \mathcal{D}x'(t) + Bu(t) + f(t, x_{\rho(t, x_t)}), \quad t \in I = [0, a], \quad (4.1)$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = \eta \in X, \quad (4.2)$$

$$\Delta x(t_i) = I_i(x_{t_i}), \quad i = 1, 2, \dots, n, \quad (4.3)$$

$$\Delta x'(t_i) = J_i(x_{t_i}), \quad i = 1, 2, \dots, n, \quad (4.4)$$

where  $A, \mathcal{D}, f, I_i$  and  $J_i$  are defined as in equations (1.1)–(1.4), the control function  $u(\cdot)$  given in  $L^2(I, U)$ , a Banach space of admissible control functions with  $U$  as a Banach space and  $B : U \rightarrow X$  is a bounded linear operator on a Banach space  $X$  with  $D(\mathcal{D}) \subset D(A)$ .

Furthermore, we assume the following conditions:

( $H_1$ )' The function  $f : I \times \mathcal{B} \rightarrow X$  satisfies the following conditions:

- (i) The function  $f : I \times \mathcal{B} \rightarrow X$  is completely continuous.
- (ii) For every positive constant  $r$ , there exists an  $\alpha_r \in L^1(I)$  such that

$$\sup_{\|\psi\| \leq r} \|f(t, \psi)\| \leq \alpha_r(t).$$

( $H_3$ )  $B$  is continuous operator from  $U$  to  $X$  and the linear operator  $W : L^2(I, U) \rightarrow X$ , defined by

$$Wu = \int_0^a S(a-s)Bu(s)ds,$$

has a bounded invertible operator,  $W^{-1}$  which takes the values in  $L^2(I, U)/\text{Ker}W$  such that  $\|B\| \leq M_1$  and  $\|W^{-1}\| \leq M_2$  for some positive integers  $M_1, M_2$ .

( $H_4$ ) The maps  $I_i, J_i : \mathcal{B} \rightarrow X, i = 1, 2, \dots, n$ , are completely continuous and there exist continuous nondecreasing functions  $\lambda_i, \mu_i : [0, \infty) \rightarrow (0, \infty), i = 1, 2, \dots, n$ , such that

$$\|I_i(\psi)\| \leq \lambda_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\xi \rightarrow \infty} \frac{\lambda_i(\xi)}{\xi} = \xi_i < \infty,$$

$$\|J_i(\psi)\| \leq \mu_i(\|\psi\|_{\mathcal{B}}), \quad \liminf_{\xi \rightarrow \infty} \frac{\mu_i(\xi)}{\xi} = \eta_i < \infty.$$

**Definition 4.1.** The system (4.1)–(4.4) is said to be controllable on the interval  $[0, a]$  if for every  $x_0 = \varphi \in \mathcal{B}, x'(0) = \eta \in X$  and  $x_1 \in X$ , there exists a control  $u \in L^2(I, U)$  such that the mild solution  $x(t)$  of (4.1)–(4.4) satisfies  $x(a) = x_1$ .

**Definition 4.2.** A functions  $x : (-\infty, a] \rightarrow X$  is called a mild solution of the abstract Cauchy problem (4.1)–(4.4) if  $x_0 = \varphi, x_{\rho(s, x_s)} \in \mathcal{B}$  for every  $s \in I; x(\cdot)|_I \in \mathcal{PC}$  and

$$\begin{aligned} x(t) = & C(t)\varphi(0) + S(t)\eta + \sum_{i=0}^{j-1} [S(t - t_{i+1})\mathcal{D}x(t_{i+1}^-) - S(t - t_i)\mathcal{D}x(t_i^+)] - \\ & - S(t - t_j)\mathcal{D}x(t_j^+) + \int_0^t C(t - s)\mathcal{D}x(s)ds + \\ & + \int_0^t S(t - s)[Bu(s) + f(s, x_{\rho(s, x_s)})]ds + \sum_{0 < t_i < t} C(t - t_i)I_i(x_{t_i}) + \\ & + \sum_{0 < t_i < t} S(t - t_i)J_i(x_{t_i}), \quad t \in I. \end{aligned}$$

**Theorem 4.3.** Let conditions  $(H_\varphi), (H_1) - (H_3)$  and  $(H_1)'$  hold. Then the system (4.1)–(4.4) is controllable on  $(-\infty, a]$  provided that

$$\begin{aligned} (1 + a\bar{N}M_1M_2) \left[ K_a \left( \frac{1}{K_a} (3\bar{N} + aN) \|\mathcal{D}\| + \bar{N}\Lambda \int_0^a m(s)ds + \right. \right. \\ \left. \left. + \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \right) \right] < 1. \end{aligned}$$

*Proof.* Consider the space  $Y = \{x \in \mathcal{PC} : u(0) = \varphi(0)\}$  endowed with the uniform convergence topology. Using the assumption  $(H_3)$ , for an arbitrary function  $x(\cdot)$ , we define the control

$$\begin{aligned} u(t) = & W^{-1} \left[ x_1 - C(a)\varphi(0) - S(t)\eta - \sum_{i=0}^{j-1} [S(a - t_{i+1})\mathcal{D}x(t_{i+1}^-) - S(a - t_i)\mathcal{D}x(t_i^+)] + \right. \\ & + S(a - t_j)\mathcal{D}x(t_j^+) - \int_0^a C(a - s)\mathcal{D}x(s)ds - \int_0^a S(a - s)f(s, x_{\rho(s, x_s)})ds - \\ & \left. - \sum_{0 < t_i < a} C(a - t_i)I_i(x_{t_i}) - \sum_{0 < t_i < a} S(a - t_i)J_i(x_{t_i}) \right] (t). \end{aligned}$$

Using this control, we shall show that the operator  $\Gamma : Y \rightarrow Y$  defined by

$$\begin{aligned}
\Gamma x(t) = & C(t)\varphi(0) + S(t)\eta + \sum_{i=0}^{j-1} [S(t-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t-t_i)\mathcal{D}\bar{x}(t_i^+)] - \\
& - S(t-t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t-s)\mathcal{D}\bar{x}(s)ds + \\
& + \int_0^t S(t-s)f(s, \bar{x}_{\rho(s, \bar{x}_s)})ds + \\
& + \int_0^t S(t-\xi)BW^{-1} \left[ x_1 - C(a)\varphi(0) - S(a)\eta - \right. \\
& - \sum_{i=0}^{j-1} [S(a-t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - \\
& - S(a-t_i)\mathcal{D}\bar{x}(t_i^+)] + S(a-t_j)\mathcal{D}\bar{x}(t_j^+) - \int_0^a C(a-s)\mathcal{D}\bar{x}(s)ds - \\
& - \int_0^a S(a-s)f(s, \bar{x}_{\rho(s, \bar{x}_s)})ds - \sum_{0 < t_i < a} C(a-t_i)I_i(\bar{x}_{t_i}) - \\
& \left. - \sum_{0 < t_i < a} S(a-t_i)J_i(\bar{x}_{t_i}) \right] (\xi) d\xi + \\
& + \sum_{0 < t_i < t} C(t-t_i)I_i(\bar{x}_{t_i}) + \sum_{0 < t_i < t} S(t-t_i)J_i(\bar{x}_{t_i}), \quad t \in I,
\end{aligned}$$

has a fixed point  $x(\cdot)$ . This fixed point  $x(\cdot)$  is then a mild solution of the system (4.1)–(4.4). Clearly,  $(\Gamma x)(a) = x_1$ , which means that the control  $u$  steers the system from the initial state  $\varphi$  to  $x_1$  in time  $a$ , provided we can obtain a fixed point of the operator  $\Gamma$  which implies that the system is controllable. Here  $\bar{x} : (-\infty, a] \rightarrow X$  is such that  $\bar{x}_0 = \varphi$  and  $\bar{x} = x$  on  $I$ . From axiom (A) and our assumptions on  $\varphi$ , we infer that  $\Gamma x \in \mathcal{PC}$ .

Next, we claim that there exists  $r > 0$  such that  $\Gamma(B_r(y|_I, Y)) \subseteq B_r(y|_I, Y)$ . If we assume this property is false, then for every  $r > 0$  there exist  $x^r \in B_r(y|_I, Y)$

and  $t^r \in I$  such that  $r < \|\Gamma x^r(t^r) - y(t^r)\|$ . Then, from Lemma 3.1, we get

$$\begin{aligned}
& r < \|\Gamma x^r(t^r) - y(t^r)\| \leq \\
& \leq 3\bar{N}\|\mathcal{D}\|r + aN\|\mathcal{D}\|r + \bar{N} \int_0^{t^r} m(s)W(\|\bar{x}^r_{\rho(s,(\bar{x}^r)_s)}\|_{\mathcal{B}})ds + \\
& + \bar{N}M_1M_2 \int_0^{t^r} \left[ \|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \bar{N}\|\eta\| + \right. \\
& + 3\bar{N}\|\mathcal{D}\|r + aN\|\mathcal{D}\|r + \bar{N} \int_0^a m(s)W(\|\bar{x}^r_{\rho(s,(\bar{x}^r)_s)}\|_{\mathcal{B}})ds + \\
& + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) + \\
& + \left. \sum_{i=1}^n \bar{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \right] d\xi + \\
& + \sum_{i=1}^n N(L_{I_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|I_i(y_{t_i})\|) + \\
& + \sum_{i=1}^n \bar{N}(L_{J_i}\|\bar{x}_{t_i} - y_{t_i}\|_{\mathcal{B}} + \|J_i(y_{t_i})\|) \leq \\
& \leq 3\bar{N}\|\mathcal{D}\|r + aN\|\mathcal{D}\|r + \\
& + \bar{N}W\left((M_a + J_0^\varphi)\|\varphi\|_{\mathcal{B}} + K_a r + K_a\|y\|_a\right) \int_0^a m(s)ds + \\
& + a\bar{N}M_1M_2 \left[ \|x_1\| + NH\|\varphi\|_{\mathcal{B}} + \bar{N}\|\eta\| + 3\bar{N}\|\mathcal{D}\|r + \right. \\
& + aN\|\mathcal{D}\|r + \bar{N}W\left((M_a + J_0^\varphi)\|\varphi\|_{\mathcal{B}} + K_a r + K_a\|y\|_a\right) \int_0^a m(s)ds + \\
& + \left. \sum_{i=1}^n N(L_{I_i}K_a r + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \bar{N}(L_{J_i}K_a r + \|J_i(y_{t_i})\|) \right] + \\
& + \sum_{i=1}^n N(L_{I_i}K_a r + \|I_i(y_{t_i})\|) + \sum_{i=1}^n \bar{N}(L_{J_i}K_a r + \|J_i(y_{t_i})\|),
\end{aligned}$$

and hence

$$1 \leq (1 + a\bar{N}M_1M_2) \left[ K_a \left( \frac{1}{K_a} (3\bar{N} + aN) \|\mathcal{D}\| + \bar{N}\Lambda \int_0^a m(s) ds + \sum_{i=1}^n (NL_{I_i} + \bar{N}L_{J_i}) \right) \right],$$

which contradicts to our assumption.

Let  $r > 0$  be such that  $\Gamma(B_r(y|_I, Y)) \subset B_r(y|_I, Y)$ . In order to prove that  $\Gamma$  is a condensing map on  $B_r(y|_I, Y)$  into  $B_r(y|_I, Y)$ . We introduce the decomposition  $\Gamma = \Gamma_1 + \Gamma_2$ , where

$$\begin{aligned} \Gamma_1 x(t) &= C(t)\varphi(0) + S(t)\eta + \sum_{i=0}^{j-1} [S(t - t_{i+1})\mathcal{D}\bar{x}(t_{i+1}^-) - S(t - t_i)\mathcal{D}\bar{x}(t_i^+)] - \\ &\quad - S(t - t_j)\mathcal{D}\bar{x}(t_j^+) + \int_0^t C(t - s)\mathcal{D}\bar{x}(s)ds + \sum_{0 < t_i < t} C(t - t_i)I_i(\bar{x}_{t_i}) + \\ &\quad + \sum_{0 < t_i < t} S(t - t_i)J_i(\bar{x}_{t_i}), \\ \Gamma_2 x(t) &= \int_0^t S(t - s) [f(s, \bar{x}_{\rho(s, \bar{x}_s)}) + Bu(s)] ds. \end{aligned}$$

Now

$$\begin{aligned} \|Bu(s)\| &\leq \|B\| \|W^{-1}\| \left[ \|x_1\| + \|C(a)\| \|\varphi(0)\| + \|S(a)\| \|\eta\| + \right. \\ &\quad + \sum_{i=0}^{j-1} [\|S(a - t_{i+1})\| \|\mathcal{D}\| \|\bar{x}(t_{i+1}^-)\| + \|S(a - t_i)\| \|\mathcal{D}\| \|\bar{x}(t_i^+)\|] + \\ &\quad \left. + \|S(a - t_j)\| \|\mathcal{D}\| \|\bar{x}(t_j^+)\| + \int_0^a \|C(a - s)\| \|\mathcal{D}\| \|\bar{x}(s)\| ds + \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^a \|S(a-s)\| \|f(s, \bar{x}_{\rho(s, \bar{x}_s)})\| ds + \sum_{0 < t_i < a} \|C(a-t_i)\| \|I_i(\bar{x}_{t_i})\| + \\
& + \sum_{0 < t_i < a} \|S(a-t_i)\| \|J_i(\bar{x}_{t_i})\| \leq \\
& \leq M_1 M_2 \left[ \|x_1\| + NH \|\varphi\|_{\mathcal{B}} + \bar{N} \|\eta\| + 3\bar{N} \|D\| r + aN \|\mathcal{D}\| r + \right. \\
& \left. + \bar{N} \int_0^a \alpha_r(s) ds + N \sum_{i=1}^n \lambda_i \|\bar{x}_{t_i}\| + \bar{N} \sum_{i=1}^n \mu_i \|\bar{x}_{t_i}\| \right] \leq \\
& \leq M_1 M_2 \left[ \|x_1\| + NH \|\varphi\|_{\mathcal{B}} + \bar{N} \|\eta\| + 3\bar{N} \|D\| r + aN \|\mathcal{D}\| r + \right. \\
& \left. + \bar{N} \int_0^a \alpha_r(s) ds + \sum_{i=1}^n r(N\lambda_i + \bar{N}\mu_i) \right] = A_0.
\end{aligned}$$

Here by applying the same technique that is used in the proof of [43, Lemma 3.1], we arrived that  $\Gamma_2$  is completely continuous.

Next, we show that  $\Gamma_1$  is contraction on  $B_r(y|_I, Y)$ . Indeed, for  $x, z \in B_r(y|_I, Y)$ , we have

$$\begin{aligned}
\|\Gamma_1 x - \Gamma_1 z\|_{\mathcal{P}\mathcal{C}} & \leq aN \|\mathcal{D}\| \|x - z\|_{\mathcal{P}\mathcal{C}} + 3\bar{N} \|\mathcal{D}\| \|x - z\|_{\mathcal{P}\mathcal{C}} + \sum_{i=1}^n NL_{I_i} K_a \|x - z\|_{\mathcal{P}\mathcal{C}} + \\
& + \sum_{i=1}^n \bar{N} L_{J_i} K_a \|x - z\|_{\mathcal{P}\mathcal{C}} \leq \\
& \leq K_a \left[ \frac{1}{K_a} (3\bar{N} + aN) \|\mathcal{D}\| + \sum_{i=1}^n (NL_{I_i} + \bar{N} L_{J_i}) \right] \|x - z\|_{\mathcal{P}\mathcal{C}}.
\end{aligned}$$

It follows that  $\Gamma_1$  is a contraction on  $B_r(y|_I, Y)$  which implies that  $\Gamma$  is a condensing operator on  $B_r(y|_I, Y)$ .

Finally, from the Sadovskii's fixed point theorem,  $\Gamma$  has a fixed point on  $Y$ . This means that any fixed point of  $\Gamma$  is a mild solution of the problem (4.1)–(4.4). The proof is now complete.  $\square$

## 5. EXAMPLES

In this section, we consider an application of our abstract results. We choose the space  $X = L^2([0, \pi])$ ,  $\mathcal{B} = \mathcal{P}\mathcal{C}_0 \times L^2(g, X)$  is the space introduced in [36] and  $A : D(A) \subset X \rightarrow X$  is the operator defined by  $Au = u''$  with domain  $D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\}$ . It is well-known that  $A$  is the infinitesimal generator of a strongly continuous cosine family  $(C(t))_{t \in \mathbb{R}}$  on  $X$ . Furthermore,  $A$  has a discrete spectrum, the eigenvalues are  $-n^2$ , for  $n \in \mathbb{N}$ , with corresponding eigenvectors  $z_n(\tau) = \left(\frac{2}{\pi}\right)^{1/2} \sin(n\tau)$ , and the following properties hold.

- (a) The set of functions  $\{z_n : n \in \mathbb{N}\}$  forms an orthonormal basis of  $X$ .
- (b) If  $x \in D(A)$ , then  $Ax = -\sum_{n=1}^{\infty} n^2 \langle x, z_n \rangle z_n \cdot \cos nt$ .
- (c) For  $x \in X$ ,  $C(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, z_n \rangle z_n$  and the associated sine family is

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, z_n \rangle z_n,$$

which implies that the operator  $S(t)$  is compact, for all  $t \in \mathbb{R}$  and that  $\|C(t)\| = \|S(t)\| = 1$ , for all  $t \in \mathbb{R}$ .

- (d) If  $G$  is the group of translations on  $X$  defined by  $G(t)x(\zeta) = \tilde{x}(\zeta + t)$ , where  $\tilde{x}(\cdot)$  is the extension of  $x(\cdot)$  with period  $2\pi$ , then  $C(t) = \frac{1}{2}[\Phi(t) + \Phi(-t)]$  and  $A = B^2$ , where  $B$  is the infinitesimal generator of  $\Phi$  and  $E = \{x \in H^1(0, \pi) : x(0) = x(\pi) = 0\}$  (see [21] for details).

### 5.1. SECOND ORDER NEUTRAL SYSTEM

Consider the following second order damped impulsive differential system with state-dependent delay

$$\begin{aligned} \frac{\partial^2}{\partial t^2} w(t, \zeta) &= \frac{\partial^2}{\partial \zeta^2} w(t, \zeta) + \alpha \frac{\partial}{\partial t} w(t, \zeta) + \int_0^\pi \beta(s) \frac{\partial}{\partial t} w(t, s) ds + \\ &+ \int_{-\infty}^t k(s-t) w(s - \rho_1(t) \rho_2(\|w(t)\|), \zeta) ds, \quad t \in I, \zeta \in [0, \pi], \end{aligned} \tag{5.1}$$

$$w(t, 0) = w(t, \pi) = 0, \quad t \in I, \tag{5.2}$$

$$\frac{\partial}{\partial t} w(0, \zeta) = \zeta(\pi), \tag{5.3}$$

$$w(\tau, \zeta) = \varphi(\tau, \zeta), \quad \tau \leq 0, \quad 0 \leq \zeta \leq \pi, \tag{5.4}$$

$$\Delta w(t_i)(\zeta) = \int_{-\infty}^{t_i} b_i(t_i - s) w(s, \zeta) ds, \quad i = 1, 2, \dots, n, \tag{5.5}$$

$$\Delta w'(t_i)(\zeta) = \int_{-\infty}^{t_i} \tilde{b}_i(t_i - s) w(s, \zeta) ds, \quad i = 1, 2, \dots, n, \tag{5.6}$$

where we assume that  $\varphi \in \mathcal{B}$  with the identity  $\varphi(s)(\zeta) = \varphi(s, \zeta)$ ,  $\varphi(0, \cdot) \in H^1([0, \pi])$  and  $0 < t_1 < t_2 < \dots < a$ . Here  $\alpha$  is a prefixed real number and  $\beta \in L^2([0, \pi])$ .

Let the functions  $\rho_i : [0, \infty) \rightarrow [0, \infty)$ ,  $i = 1, 2$ ,  $k : \mathbb{R} \rightarrow \mathbb{R}$  be continuous,  $L_f = \left( \int_{-\infty}^0 \frac{(k^2(s))}{g(s)} ds \right)^{\frac{1}{2}} < \infty$ , and the following conditions hold:

- (a)  $b_i, \tilde{b}_i \in C(\mathbb{R}, \mathbb{R})$  and  $L_{I_i} := \left( \int_{-\infty}^0 \frac{b_i^2(s)}{g(s)} ds \right)^{\frac{1}{2}}$ ,  $L_{J_i} := \left( \int_{-\infty}^0 \frac{\tilde{b}_i^2(s)}{g(s)} ds \right)^{\frac{1}{2}}$ ,  $i = 1, \dots, n$ , are finite.

Define the functions  $\mathcal{D} : X \rightarrow X$ ,  $f : I \times \mathcal{B} \rightarrow X$ ,  $\rho : I \times \mathcal{B} \rightarrow X$ ,  $I_i : \mathcal{B} \rightarrow X$  and  $J_i : \mathcal{B} \rightarrow X$  by

$$\begin{aligned}\mathcal{D}\psi(\zeta) &= \alpha\psi(t, \zeta) + \int_0^\pi \beta(s)\psi(t, s)ds, \\ f(\psi)(\zeta) &= \int_{-\infty}^0 k(s)\psi(s, \zeta)ds, \\ \rho(s, \psi) &= s - \rho_1(s)\rho_2(\|\psi(0)\|), \\ I_i(\psi)(\zeta) &= \int_{-\infty}^0 b_i(-s)\psi(s, \zeta)ds, \quad i = 1, 2, \dots, n, \\ J_i(\psi)(\zeta) &= \int_{-\infty}^0 \tilde{b}_i(-s)\psi(s, \zeta)ds, \quad i = 1, 2, \dots, n.\end{aligned}$$

With the choice of  $A, \mathcal{D}, f, \rho, I_i$  and  $J_i$ , the system (1.1)–(1.4) is the abstract formulation of (5.1)–(5.6). Moreover, the maps  $\mathcal{D}, g, f, I_i, J_i, i = 1, 2, \dots, n$ , are bounded linear operators with

$$\|\mathcal{D}\|_{\mathcal{L}(X)} \leq |\alpha| + \|\beta\|_{L^2(0,a)}, \quad \|f(t, \cdot)\|_{\mathcal{L}(\mathcal{B}, X)} \leq L_f, \quad \|I_i\|_{\mathcal{L}(\mathcal{B}, X)} \leq L_{I_i}, \quad \|J_i\|_{\mathcal{L}(\mathcal{B}, X)} \leq L_{J_i}.$$

## REFERENCES

- [1] W.G. Aiello, H.I. Freedman, J. Wu, *Analysis of a model representing stage-structured population growth with state-dependent time delay*, SIAM J. Appl. Math. **52** (1992) 3, 855–869.
- [2] A. Anguraj, M. Mallika Arjunan, E. Hernandez, *Existence results for an impulsive neutral functional differential equation with state-dependent delay*, Appl. Anal. **86** (2007), 861–872.
- [3] G. Arthi, K. Balachandran, *Controllability of damped second-order impulsive neutral functional differential systems with infinite delay*, J. Optim. Theory Appl. **152** (2012), 799–813.
- [4] G. Arthi, K. Balachandran, *Controllability of damped second-order neutral functional differential systems with impulses*, Taiwan. J. Math. **16** (2012), 89–106.
- [5] D. Bainov, P. Simeonov, *Impulsive Differential Equations*, John Wiley and Sons, New York, 1993.
- [6] K. Balachandran, S.M. Anthoni, *Controllability of second-order semilinear neutral functional differential systems in Banach spaces*, Comput. Math. Appl. **41** (2001), 1223–1235.



- [7] M. Bartha, *Periodic solutions for differential equations with state-dependent delay and positive feedback*, *Nonlinear Anal. TMA* **53** (2003) 6, 839–857.
- [8] Y. Cao, J. Fan, T.C. Gard, *The effects of state-dependent time delay on a stage-structured population growth model*, *Nonlinear Anal. TMA* **19** (1992) 2, 95–105.
- [9] Y.-K. Chang, A. Anguraj, M. Mallika Arjunan, *Existence results for impulsive neutral functional differential equations with infinite delay*, *Nonlinear Analysis: Hybrid Systems* **2** (2008), 209–218.
- [10] Y.-K. Chang, A. Anguraj, M. Mallika Arjunan, *Existence results for non-densely defined neutral impulsive differential inclusions with nonlocal conditions*, *J. Appl. Math. Comp.* **28** (2008), 79–91.
- [11] Y.-K. Chang, M. Mallika Arjunan, V. Kavitha, *Existence results for a second order impulsive functional differential equation with state-dependent delay*, *Differ. Equ. Appl.* **1** (3) (2009), 325–339.
- [12] F. Chen, D. Sun, J. Shi, *Periodicity in a food-limited population model with toxicants and state-dependent delays*, *J. Math. Anal. Appl.* **288** (2003), 136–146.
- [13] A. Domoshnitsky, M. Drakhlin, *On boundary value problems for first order impulse functional differential equations* [in:] J. Henderson (Ed.), *Boundary Value Problems for Functional Differential Equations*, World Scientific, Singapore, 1995, 107–117.
- [14] A. Domoshnitsky, M. Drakhlin, *Nonoscillation of first order impulse differential equations with delay*, *J. Math. Anal. Appl.* **206** (1997), 254–269.
- [15] A. Domoshnitsky, M. Drakhlin, E. Litsyn, *On  $n$ -th order functional differential equations with impulses*, *Mem. Differential Equations Math. Phys.* **12** (1997), 50–56.
- [16] A. Domoshnitsky, M. Drakhlin, E. Litsyn, *On boundary value problems for  $n$ -th order functional differential equations with impulses*, *Adv. Math. Sci. Appl.* **8** (1998) 2, 987–996.
- [17] A. Domoshnitsky, *On periodic boundary value problem for first order impulsive functional differential nonlinear equations*, *Funct. Differ. Equ.* **4** (1997), 37–45.
- [18] A. Domoshnitsky, M. Drakhlin, E. Litsyn, *On equations with delay depending on solution*, *Nonlinear Anal. TMA* **49** (2002) 5, 689–701.
- [19] A. Domoshnitsky, M. Drakhlin, *Periodic solutions of differential equations with delay depending on solution*, *Nonlinear Analysis, TMA* **30** (5) (1997), 2665–2672.
- [20] A. Domoshnitsky, M. Drakhlin, E. Litsyn, *Nonoscillation and positivity of solutions to first order state dependent differential equations with impulses in variable moments*, *Journal of Differential Equations* **228** (2006), 39–48.
- [21] H.O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.
- [22] F. Hartung, *Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: A numerical study*, *Proceedings of the Third World Congress of Nonlinear Analysts, Part 7* (Catania, 2000), *Nonlinear Anal. TMA* **47** (2001) 7, 4557–4566.

- [23] F. Hartung, J. Turi, *Identification of parameters in delay equations with state-dependent delays*, Nonlinear Anal. TMA **29** (1997) 11, 1303–1318.
- [24] F. Hartung, T.L. Herdman, J. Turin, *Parameter identification in classes of neutral differential equations with state-dependent delays*, Nonlinear Anal. TMA **39** (3)(2000), 305–325.
- [25] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, *Functional Differential Equations with State-Dependent Delays*, Theory and Applications, [in:] A. Canada, P. Drabek, A. Fonda (Eds.), Handbook of Differential Equations: Ordinary Differential Equations, Elsevier B.V., 2006.
- [26] E. Hernandez, *Existence of solutions for a second order abstract functional differential equation with state-dependent delay*, Electron. J. Differential Equations **21** (2007), 1–10.
- [27] E. Hernandez, R. Sakthivel, S. Tanaka, *Existence results for impulsive evolution differential equations with state-dependent delay*, Electron. J. Differential Equations **28** (2008), 1–11.
- [28] E. Hernandez, M.A. McKibben, H. Henriquez, *Existence results for partial neutral functional differential equations with state-dependent delay*, Math. Comput. Modelling **49** (2009), 1260–1267.
- [29] E. Hernandez, *A second order impulsive Cauchy problem*, Int. J. Math. Math. Sci. **8** (31) (2002), 451–461.
- [30] E. Hernandez, H. Henriquez, *Impulsive partial neutral differential equations*, Appl. Math. Lett. **19** (2006) 3, 215–222.
- [31] E. Hernandez, M. Rabello, H. Henriquez, *Existence of solutions for impulsive partial neutral functional differential equations*, J. Math. Anal. Appl. **331** (2007), 1135–1158.
- [32] E. Hernandez, K. Balachandran, N. Annapoorani, *Existence results for a damped second order abstract functional differential equation with impulses*, Math. Comput. Model. **50** (2009), 1583–1594.
- [33] E. Hernandez, M. Pierri, G. Uniao, *Existence results for a impulsive abstract partial differential equation with state-dependent delay*, Comput. Math. Appl. **52** (2006), 411–420.
- [34] E. Hernandez, A. Anguraj, M. Mallika Arjunan, *Existence results for an impulsive second order differential equation with state-dependent delay*, Dynam. Contin. Discrete Impuls. Systems **17** (2010), 287–301.
- [35] E. Hernandez, A. Prokopczyk, L.A. Ladeira, *A note on partial functional differential equations with state-dependent delay*, Nonlinear Anal. RWA **7** (2006), 510–519.
- [36] Y. Hino, S. Murakami, T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.
- [37] J.M. Jeong, H.G. Kim, *Controllability for semilinear functional integrodifferential equations*, Bull. Korean Math. Soc. **46** (2009), 463–475.
- [38] J. Kisynski, *On cosine operator functions and one parameter group of operators*, Studia Math. **49** (1972), 93–105.

- [39] V. Lakshmikantham, D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [40] Y. Lin, N. Tanaka, *Nonlinear abstract wave equations with strong damping*, J. Math. Anal. Appl. **225** (1998), 46–61.
- [41] J.K. Liu, *Nonlinear impulsive evolution equations*, Dynam. Contin. Discrete Impuls. Systems **6** (1999) 1, 77–85.
- [42] M. Mallika Arjunan, V. Kavitha, *Existence results for impulsive neutral functional differential equations with state-dependent delay*, Electron. J. Qual. Theory Differ. Equ. **26** (2009), 1–13.
- [43] S.K. Ntouyas, D.O'Regan, *Some remarks on controllability of evolution equations in Banach spaces*, Electron. J. Differential Equations **79** (2009), 1–6.
- [44] J.Y. Park, H.K. Han, *Controllability for some second order differential equations*, Bull. Korean Math. Soc. **34** (1997), 411–419.
- [45] J.Y. Park, S.H. Park, Y.H. Kang, *Controllability of second-order impulsive neutral functional differential inclusions in Banach spaces*, Math. Methods Appl. Sci. **33** (2010) 3, 249–262.
- [46] B. Radhakrishnan, K. Balachandran, *Controllability of neutral evolution integrodifferential systems with state-dependent delay*, J. Optim. Theory Appl. **153** (2012) 1, 85–97.
- [47] Y. Rogovchenko, *Impulsive evolution systems: main results and new trends*, Dynam. Contin. Discrete Impuls. Systems **3** (1997) 1, 57–88.
- [48] Y. Rogovchenko, *Nonlinear impulse evolution systems and applications to population Models*, J. Math. Anal. Appl. **207** (1997) 2, 300–315.
- [49] B.N. Sadovskii, *On a fixed point principle*, Funct. Anal. Appl. **1** (1967), 74–76.
- [50] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
- [51] R. Sakthivel, N.I. Mahmudov, J.H. Kim, *On controllability of second-order nonlinear impulsive differential systems*, Nonlinear Anal. **71** (2009), 45–52.
- [52] G.F. Webb, *Existence and asymptotic behaviour for a strongly damped nonlinear wave equations*, Can. J. Math. **32** (1980), 631–643.
- [53] C.C. Travis, G.F. Webb, *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families*, Houston J. Math. **3** (4) (1977), 555–567.
- [54] C.C. Travis, G.F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungaricae **32** (1978), 76–96.

M. Mallika Arjunan  
arjunphd07@yahoo.co.in

C.B.M. College  
Department of Mathematics  
Kovaipudur, Coimbatore-641 042, Tamil Nadu, India

N.Y. Nadaf  
nadaf\_nabisab@yahoo.com

Government PU College  
Department of Mathematics  
Vidyanagar, Hungund-587118, Bagalkot, Karnataka, India

*Received: September 3, 2012.*

*Revised: May 28, 2013.*

*Accepted: November 15, 2013.*