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Dedicated to the Memory of Professor Zdzisław Kamont

GLOBAL CONVERGENCE OF SUCCESSIVE APPROXIMATIONS OF THE DARBOUX PROBLEM FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. We consider the Darboux problem for the hyperbolic partial functional differential equation with infinite delay. We deal with generalized (in the "almost everywhere" sense) solutions of this problem. We prove a theorem on the global convergence of successive approximations to a unique solution of the Darboux problem.

Keywords: successive approximations, Darboux problem, infinite delay.

Mathematics Subject Classification: 35R10.

1. INTRODUCTION

In this paper we deal with the following Darboux problem for the second order partial functional differential equation

$$D_{xy}z(x,y) = f\big(x,y,z_{(x,y)}\big), \quad (x,y) \in G, \tag{1.1}$$

$$z(x,y) = \phi(x,y), \qquad (x,y) \in E_0, \tag{1.2}$$

where

$$G = [0,a] \times [0,b],\ E_0 = E \setminus ((0,a] \times (0,b]) \quad \text{and} \quad E = (-\infty,a] \times (-\infty,b].$$

In the above problem $f : G \times \mathfrak{B} \to \mathbb{R}$, $\phi : E_0 \to \mathbb{R}$ are given functions. In the right-hand side of (1.1) the functional dependence is described by the operator $G \ni (x,y) \mapsto z_{(x,y)} \in \mathfrak{B}$, where $z_{(x,y)} : (-\infty,0]^2 \to \mathbb{R}$ is a function defined by the formula $z_{(x,y)}(s,t) = z(x+s, y+t)$, $(s,t) \in (-\infty,0]^2$. Thus $\mathfrak{B}$ is a vector space of real-valued functions defined in $(-\infty, 0]^2$. The space $\mathfrak{B}$ is equipped with a seminorm and satisfies some suitable axioms, which will be given in Section 2.

The axiomatic approach and the model of functional dependence which we use in this paper is well known for ordinary functional differential equations. Systems of axioms most often used in this case were given in [9, 11, 14] (see also [4, 10] with a rich bibliography concerning functional differential equations with infinite delay). We adapt the system of [9] to partial functional differential equations.

Convergence of successive approximations for ordinary functional differential equations as well as for integral functional equations with infinite delay has been proved by Shin [12,13]. In the case of ordinary differential equations with finite delay, convergence of successive approximations follows from the results of Chen [3]. The fact that the convergence of successive approximations is a generic property has been proved for equations with finite delay by De Blasi and Myjak [7], while for equations with infinite delay by Faina [8].

The Darboux problem for partial functional differential equtions with infinite delay has been studied in [5, 6], both papers concern classical solutions. The axiomatic approach for such equations was introduced in [5], where the existence theorem was proved by means of the measure of noncompactness technique. Some existence results for equations involving first order derivatives of an unknown function were obtained in [6] via the Banach or the Schauder fixed-point theorems. The Darboux problem for fractional order partial functional differential equations with infinite delay has been studied in [1, 2]. The existence and uniqueness results in these papers are obtained by using the Banach fixed-point theorem or some nonlinear alternatives of the Leray-Schauder theorem.

In this paper we get the global convergence of successive appoximations as well as the uniqueness of solutions for the Darboux problem (1.1), (1.2). We deal with generalized solutions, i.e. $z : (-\infty, a] \times (-\infty, b] \to \mathbb{R}$ is a solution of (1.1) if it is absolutely continuous and satisfies this equation almost everywhere on G. In Section 3 we prove a comparison theorem and with the help of it we get the main result in Section 4. The method of the proof follows the ideas of Shin [12].

2. THE PHASE SPACE $\mathfrak{B}$

Let $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, +\infty)$. Assume that $\mathfrak{B}$ is a linear space of functions mapping $\mathbb{R}^2_-$ into $\mathbb{R}$ equipped with a seminorm $|\cdot|_{\mathfrak{B}}$. If in a classical definition of continuity we replace a norm with the seminorm $|\cdot|_{\mathfrak{B}}$ then we may discuss continuity of a function with arguments or values in $\mathfrak{B}$.

For any $(\xi, \eta) \in G$ denote the rectangle $[\xi, a] \times [\eta, b]$ by $G_{\xi\eta}$ and its "Darboux--boundary", where initial values are prescribed in the classical case, by $\Gamma_{\xi\eta} = ([\xi, a] \times \{\eta\}) \cup (\{\xi\} \times [\eta, b])$.

Suppose that $\mathfrak{B}$ satisfies the following axioms:

(A_1) If $z : E \to \mathbb{R}$ and $(\xi, \eta) \in G$ are such that z is continuous on $G_{\xi\eta}$ and $z_{(s,t)} \in \mathfrak{B}$ for all $(s, t) \in \Gamma_{\xi\eta}$ then for any $(x, y) \in G_{\xi\eta}$ we have $z_{(x,y)} \in \mathfrak{B}$.

(A_2) If z and (ξ,η) are such as in (A_1) then the function $G_{\xi\eta} \ni (x,y) \mapsto z_{(x,y)} \in \mathfrak{B}$ is continuous.

(A_3) There is a constant $H \geq 0$, a continuous function $K : \mathbb{R}^2_+ \to \mathbb{R}_+$ and a locally bounded function $M : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying the following conditions: if z and (ξ,η) are as in (A_1) then for any $(x,y) \in G_{\xi\eta}$ we have:

(i) $|z(x,y)| \leq H|z_{(x,y)}|_{\mathfrak{B}}$,

(ii) $$|z_{(x,y)}|_{\mathfrak{B}} \leq K(x-\xi, y-\eta) \sup_{(s,t)\in[\xi,x]\times[\eta,y]} |z(s,t)| + \\ + M(x-\xi,y-\eta) \sup_{(s,t)\in([\xi,x]\times\{\eta\})\cup(\{\xi\}\times[\eta,y])} |z_{(s,t)}|_{\mathfrak{B}}.$$

Remark 2.1. Axioms (A_1)–(A_3) that we consider for partial functional differential equations are adapted from those introduced by Hale and Kato [9] for ordinary functional differential equations.

Now, we show examples of the phase space $\mathfrak{B}$ satisfying the axioms (A_1)–(A_3).

Example 2.2. Let $\mathfrak{B}$ be the set of all functions $\phi : \mathbb{R}^2_- \to \mathbb{R}$ which are continuous on $[-a_0,0]\times[-b_0,0]$, $a_0, b_0 \geq 0$, with the seminorm

$$|\phi|_{\mathfrak{B}} = \sup\{|\phi(s,t)| \colon (s,t) \in [-a_0,0]\times[-b_0,0]\}.$$

Then $H = K = M = 1$, and the quotient space $\hat{\mathfrak{B}} = \mathfrak{B}/|\cdot|_{\mathfrak{B}}$ is isometric to the space $C([-a_0,0]\times[-b_0,0],\mathbb{R})$ of all continuous functions from $[-a_0,0]\times[-b_0,0]$ into $\mathbb{R}$ with the supremum norm. This means that functional differential equations with finite delay are included in our axiomatic model.

Example 2.3. Let C_γ, $\gamma \in \mathbb{R}$, be the set of all continuous functions $\phi\colon \mathbb{R}^2_- \to \mathbb{R}$ for which a limit $\lim_{|(s,t)|\to\infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm $|\phi|_{C_\gamma} = \sup\{e^{\gamma(s+t)}|\phi(s,t)| : \ (s,t) \in \mathbb{R}^2_-\}$. Then we have $H = 1$, $K(x,y) = \max\{e^{-\gamma(x+y)}, 1\}$ and $M(x,y) = e^{-\gamma(x+y)}\max\{e^{\gamma x}, e^{\gamma y}, 1\}$.

Example 2.4. Let $a_0, b_0, \gamma \geq 0$ and let

$$|\phi|_{CL_\gamma} = \sup\{|\phi(s,t)| : \ (s,t) \in [-a_0,0]\times[-b_0,0]\} + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}|\phi(s,t)|dsdt$$

be the seminorm for the space CL_γ of all functions $\phi : \mathbb{R}^2_- \to \mathbb{R}$ which are continuous on $[-a_0,0]\times[-b_0,0]$, measurable on $(-\infty,-a_0]\times(-\infty,0]\cup(-\infty,0]\times(-\infty,-b_0]$, and such that $|\phi|_{CL_\gamma} < +\infty$. Then

$$H = 1, \quad K(x,y) = \int_{-x}^{0}\int_{-y}^{0} e^{\gamma(s+t)}dsdt,$$

$$M(x,y) = \max\{1, 2e^{-\gamma(x+y)}\max(e^{\gamma x}, e^{\gamma y})\}.$$

3. A COMPARISON THEOREM

We say that $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ satisfies the Carathéodory conditions if:

(i) $\omega(\cdot, \cdot, u)$ is measurable for all $u \in [0, 2r]$,
(ii) $\omega(x, y, \cdot)$ is continuous for almost all $(x, y) \in G_{\xi\eta}$,
(iii) there is a Lebesgue integrable function $\mu : G_{\xi\eta} \to \mathbb{R}_+$ such that

$$\omega(x, y, u) \leq \mu(x, y)$$

for all $u \in [0, 2r]$ and almost all $(x, y) \in G_{\xi\eta}$.

Having fixed $(\xi, \eta) \in G$ let us consider the integral equation

$$v(x,y) = K(x-\xi, y-\eta) \int_{\xi}^{x} \int_{\eta}^{y} \omega(s,t,v(s,t))dsdt \tag{3.1}$$

for $(x, y) \in G_{\xi\eta}$, where $K(x, y)$ is as in (A_3) and $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ is a comparison function satisfying the above Carathéodory conditions.

For any $c \in [0, a]$ and $d \in [0, b]$ write $S(c, d) = G \setminus ((c, a] \times (d, b])$. We deal with continuous solutions of the above integral equation on $G_{\xi\eta} \cap S(c, d)$. In other words a solution of (3.1) is a function which belongs to the space $C(G_{\xi\eta} \cap S(c, d), [0, 2r])$. In the sequel we will write $\mathcal{C}_{cd}$ instead of $C(G_{\xi\eta} \cap S(c, d), [0, 2r])$ for simplicity.

Theorem 3.1. *Suppose that $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ is a comparison function satisfying the Carathéodory conditions and such that $\omega(x, y, \cdot)$ is nondecreasing for almost all $(x, y) \in G_{\xi\eta}$. Then the following conditions hold:*

(i) *There are constants $c \in (\xi, a]$, $d \in (\eta, b]$ such that for any $\varepsilon \in [0, r)$ equation*

$$v(x,y) = \varepsilon + K(x-\xi, y-\eta) \int_{\xi}^{x} \int_{\eta}^{y} \omega(s,t,v(s,t))dsdt \tag{3.2}$$

has a solution $v(x, y; \varepsilon)$ in $\mathcal{C}_{cd}$.

(ii) *If $0 \leq \varepsilon_1 < \varepsilon_2 < r$, then $v(x, y; \varepsilon_1) < v(x, y; \varepsilon_2)$ on $G_{\xi\eta} \cap S(c, d)$.*
(iii) *There exists a solution $\tilde{v}(x, y)$ of equation* (3.1) *for which we have $\lim_{\varepsilon \to 0^+} v(x, y; \varepsilon) = \tilde{v}(x, y)$ uniformly on $G_{\xi\eta} \cap S(c, d)$ and that it is a maximal solution of* (3.1).

Proof. (i) There are $c \in (\xi, a]$ and $d \in (\eta, b]$ such that if $(x, y) \in G_{\xi\eta} \cap S(c, d)$ then

$$\sup_{u \in \mathcal{C}_{cd}} \int_{\xi}^{x} \int_{\eta}^{y} \omega(s,t,u(s,t))dsdt \leq \frac{r}{K_{cd}}, \tag{3.3}$$

where

$$K_{cd} = \sup\{K(x-\xi, y-\eta) : \ (x, y) \in G_{\xi\eta} \cap S(c, d)\}.$$

Obviously $\mathcal{C}_{cd}$ is a closed bounded convex subset of the Banach space $C(G_{\xi\eta} \cap S(c,d), \mathbb{R})$. Define the operators

$$(\mathcal{T}_\varepsilon u)(x,y) = \varepsilon + (\mathcal{T}u)(x,y), \qquad (\mathcal{T}u)(x,y) = K(x-\xi, y-\eta)(\mathcal{W}u)(x,y)$$

and

$$(\mathcal{W}u)(x,y) = \int_\xi^x \int_\eta^y \omega(s,t,u(s,t))dsdt$$

for $(x,y) \in G_{\xi\eta} \cap S(c,d)$ and $u \in \mathcal{C}_{cd}$. Then from the Carathéodory condition (iii) it follows that

$$|(\mathcal{W}u)(x,y)| \le \int_\xi^c \int_\eta^d \mu(s,t)dsdt,$$

$$|(\mathcal{W}u)(x,y) - (\mathcal{W}u)(\bar{x},\bar{y})| \le \left| \int_\xi^c \int_{\bar{y}}^y \mu(s,t)dsdt \right| + \left| \int_{\bar{x}}^x \int_\eta^d \mu(s,t)dsdt \right|,$$

which means that the set $\{\mathcal{W}u : \ u \in \mathcal{C}_{cd}\}$ is uniformly bounded and equicontinuous on $G_{\xi\eta} \cap S(c,d)$. Hence, by the Arzelà-Ascoli lemma this set is relatively compact in $C(G_{\xi\eta} \cap S(c,d), \mathbb{R})$. Since $K(x,y)$ is continuous, it is clear that also $\{\mathcal{T}_\varepsilon u : \ u \in \mathcal{C}_{cd}\}$ is relatively compact in $C(G_{\xi\eta} \cap S(c,d), \mathbb{R})$. From condition (3.3) we obtain that $\mathcal{T}_\varepsilon \mathcal{C}_{cd} \subset \mathcal{C}_{cd}$ for $\varepsilon \in [0,r)$.

We show that the operator $\mathcal{T}_\varepsilon : \mathcal{C}_{cd} \to \mathcal{C}_{cd}$ is continuous. Indeed, if $\{u^{(n)}\}$ is any sequence in $\mathcal{C}_{cd}$ such that $u^{(n)} \to u_0$ then, by the Lebesgue dominated convergence theorem we have $(\mathcal{W}u^{(n)})(x,y) \to (\mathcal{W}u_0)(x,y)$ for each $(x,y) \in G_{\xi\eta} \cap S(c,d)$. Since the sequence $\{\mathcal{W}u^{(n)}\}_{n=1}^\infty$ is equicontinuous, this convergence is uniform on $G_{\xi\eta} \cap S(c,d)$. Analogously, the equicontinuity of $\{\mathcal{T}_\varepsilon u^{(n)}\}$ yields that $(\mathcal{T}_\varepsilon u^{(n)})(x,y) \to (\mathcal{T}_\varepsilon u_0)(x,y)$ uniformly on $G_{\xi\eta} \cap S(c,d)$. Therefore, by the Schauder fixed point theorem, there exists a fixed point of $\mathcal{T}_\varepsilon$ in $\mathcal{C}_{cd}$, which obviously is a solution of equation (3.2).

(ii) Since $v(\xi, y; \varepsilon_1) = v(x, \eta; \varepsilon_1) = \varepsilon_1 < \varepsilon_2 = v(\xi, y; \varepsilon_2) = v(x, \eta; \varepsilon_2)$ our claim follows from the monotonicity of ω with respect to the last variable.

(iii) Let

$$w(x,y;\varepsilon) = \int_\xi^x \int_\eta^y \omega(s,t,v(s,t;\varepsilon))dsdt.$$

Analogously as in (i) we may prove that the sets $\{w(x,y;\varepsilon) : \ \varepsilon \in [0,r)\}$ and consequently $\{v(x,y;\varepsilon) : \ \varepsilon \in [0,r)\}$ are equicontinuous. From (ii) it follows that the limit $\lim_{\varepsilon \to 0^+} v(x,y;\varepsilon) = \tilde{v}(x,y)$ exists and by the equicontinuity of $\{v(x,y;\varepsilon) : \ \varepsilon \in [0,r)\}$ it is uniform on $G_{\xi\eta} \cap S(c,d)$. Obviously $\tilde{v}(x,y)$ is a maximal solution of (3.1). □

Having fixed $(\xi,\eta) \in G$ consider the following integral inequality:

$$z(x,y) \le K(x-\xi, y-\eta) \int_\xi^x \int_\eta^y \omega(s,t,z(s,t))dsdt \tag{3.4}$$

for $(x,y) \in G_{\xi\eta}$.

Theorem 3.2. *Suppose that* $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ *is a comparison function satisfying the Carathéodory conditions such that* $\omega(x, y, \cdot)$ *is nondecreasing for almost all* $(x, y) \in G_{\xi\eta}$. *Then for any* $c \in (\xi, a]$ *and* $d \in (\eta, b]$, $v(x, y) \equiv 0$ *is the only solution of equation* (3.1) *in* $\mathcal{C}_{cd}$ *if and only if* $z(x, y) \equiv 0$ *is the only solution of inequality* (3.4) *in* $\mathcal{C}_{cd}$.

We omit the proof of the above theorem which follows from Theorem 3.1 and from the monotonicity of ω.

4. SUCCESSIVE APPROXIMATIONS

Let $\mathfrak{X}$ denote the set of all functions $\phi : E_0 \to \mathbb{R}$ such that $\phi_{(s,t)} \in \mathfrak{B}$ for $(s, t) \in E_0 \cap G$, and ϕ is continuous on $E_0 \cap G$. We say that $f : G \times \mathfrak{B} \to \mathbb{R}$ satisfies the Carathéodory conditions if:

(i) $f(\cdot, \cdot, w)$ is measurable for all $w \in \mathfrak{B}$,
(ii) $f(x, y, \cdot)$ is continuous for almost all $(x, y) \in G$,
(iii) there is a Lebesgue integrable function $m : G \to \mathbb{R}_+$ such that

$$|f(x, y, w)| \le m(x, y) \quad \text{for all } w \in \mathfrak{B} \text{ and almost all } (x, y) \in G.$$

Observe that z is a solution of the Darboux problem (1.1), (1.2), with $f : G \times \mathfrak{B} \to \mathbb{R}$ satisfying the Carathéodory conditions and $\phi \in \mathfrak{X}$, if and only if it satisfies the integral equation

$$z(x, y) = \begin{cases} \phi(x, y) & \text{for } (x, y) \in E_0, \\ \phi(x, 0) + \phi(0, y) - \phi(0, 0) + & \\ \quad + \int_0^x \int_0^y f\big(s, t, z_{(s,t)}\big) ds dt & \text{for } (x, y) \in G. \end{cases} \tag{4.1}$$

Therefore, we define the successive approximations of problem (1.1), (1.2) as follows:

$$z^{(0)}(x, y) = \begin{cases} \phi(x, y) & \text{for } (x, y) \in E_0, \\ \phi(x, 0) + \phi(0, y) - \phi(0, 0) & \text{for } (x, y) \in G, \end{cases}$$

$$z^{(n+1)}(x, y) = \begin{cases} \phi(x, y) & \text{for } (x, y) \in E_0, \\ \phi(x, 0) + \phi(0, y) - \phi(0, 0) & \\ \quad + \int_0^x \int_0^y f\big(s, t, z^{(n)}_{(s,t)}\big) ds dt & \text{for } (x, y) \in G. \end{cases}$$

For any $\phi \in \mathfrak{X}$ we denote by $\mathfrak{S}_\phi$ the space of all functions $u : E \to \mathbb{R}$ such that $u(x, y) = \phi(x, y)$ for all $(x, y) \in E_0$ and u is continuous on G.

Assumption 4.1. Let $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$, where $(\xi, \eta) \in G$, be a comparison function satisfying the Carathéodory conditions such that $\omega(x, y, \cdot)$ is nondecreasing for almost all $(x, y) \in G_{\xi\eta}$. Furthermore suppose that $v(s, t) \equiv 0$ is the only function in $C(G_{\xi\eta} \cap S(c, d), [0, 2r])$ satisfying the integral equation (3.1), with any $c \in (\xi, a]$, $d \in (\eta, b]$.

Theorem 4.2. *Let $f : G \times \mathfrak{B} \to \mathbb{R}$ satisfy the Carathéodory conditions, $\mathfrak{B}$ satisfy axioms (A_1)–(A_3) and $\phi \in \mathfrak{X}$. Furthermore, suppose that for any $(\xi, \eta) \in G$ there exist a constant $r > 0$ and a comparison function $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ satisfying Assumption 4.1 such that the inequality*

$$|f(x, y, u_{(x,y)}) - f(x, y, v_{(x,y)})| \le \omega(x, y, |u_{(x,y)} - v_{(x,y)}|_{\mathfrak{B}}) \tag{4.2}$$

holds for all $(x, y) \in G_{\xi\eta}$ and $u, v \in \mathfrak{S}_\phi$ such that $|u_{(x,y)} - v_{(x,y)}|_{\mathfrak{B}} \le 2r$. Then the successive approximations $z^{(n)}$ are well defined and converge to a unique solution of problem (1.1), (1.2) uniformly on G.

Proof. Since f satisfies the Carathéodory conditions the successive approximations are well defined. Furthermore, the sequences $\{z^{(n)}_{(x,y)}\}$ and $\{z^{(n)}(x, y)\}$ are equicontinuous on G. Let

$$\tau = \sup\Big\{\theta \in [0, 1] : \ \{z^{(n)}(x, y)\} \text{ converges uniformly on } S(\theta a, \theta b)\Big\}.$$

Since the sequence $\{z^{(n)}(x, y)\}$ is constant on $E_0 = S(0, 0)$, the above supremum is well defined. If $\tau = 1$, then we have the global convergence of successive approximations. Suppose that $\tau < 1$ and put $\xi = \tau a$ and $\eta = \tau b$. This means that the sequence $\{z^{(n)}(x, y)\}$ converges uniformly on $G \setminus G_{\xi\eta}$ Since this sequence is equicontinuous it converges uniformly on $S(\xi, \eta)$ to a continuous function $\tilde{z}(x, y)$. If we prove that there are $c \in (\xi, a]$ and $d \in (\eta, b]$ such that $\{z^{(n)}(x, y)\}$ converges uniformly on $S(c, d)$, this will yield a contradiction.

Put

$$z(x, y) = \begin{cases} \phi(x, y) & \text{for } (x, y) \in E_0, \\ \tilde{z}(x, y) & \text{for } (x, y) \in S(\xi, \eta). \end{cases}$$

By force of condition (A$_3$)-(ii) we see that $\lim_{n\to\infty} z^{(n)}_{(s,t)} = z_{(s,t)}$ for each $(s, t) \in \varGamma_{\xi\eta}$ and the convergence is uniform with respect to $(s, t) \in \varGamma_{\xi\eta}$. There exists a constant $r > 0$ and a comparison function $\omega : G_{\xi\eta} \times [0, 2r] \to \mathbb{R}_+$ satisfying inequality (4.2). There exist $c \in (\xi, a]$, $d \in (\eta, b]$ and $n_0 \in \mathbb{N}$ such that $\big|z^{(n)}_{(x,y)} - z^{(m)}_{(x,y)}\big|_{\mathfrak{B}} \le 2r$ for all $(x, y) \in G_{\xi\eta} \cap S(c, d)$ and $n, m \ge n_0$. To simplfy the notation we assume that $n_0 = 1$.

For any $(x, y) \in G_{\xi\eta} \cap S(c, d)$ put

$$v^{(m,n)}(x, y) = \Big|z^{(m)}_{(x,y)} - z^{(n)}_{(x,y)}\Big|_{\mathfrak{B}} \quad \text{and} \quad v^{(k)}(x, y) = \sup_{n,m\ge k} v^{(m,n)}(x, y).$$

Since the sequence $\{v^{(k)}(x, y)\}$ is nonincreasing, it is convergent to a function $v(x, y)$ for each $(x, y) \in G_{\xi\eta} \cap S(c, d)$. From the equicontinuity of $\{v^{(k)}(x, y)\}$, it follows that $\lim_{k\to\infty} v^{(k)}(x,y) = v(x,y)$ uniformly on $G_{\xi\eta}\cap S(c,d)$. Furthermore, for $(x,y)\in G_{\xi\eta}\cap S(c,d)$ and $m,n\geq k$ we have

$$\begin{aligned} v^{(m,n)}(x,y) = \big|z^{(m)}_{(x,y)} - z^{(n)}_{(x,y)}\big|_{\mathfrak{B}} \leq \\ \leq K(x-\xi,y-\eta) \sup_{(s,t)\in[\xi,x]\times[\eta,y]} |z^{(m)}(s,t) - z^{(n)}(s,t)| + \\ + M(x-\xi,y-\eta) \sup_{(s,t)\in([\xi,x]\times\{\eta\})\cup(\{\xi\}\times[\eta,y])} \big|z^{(m)}_{(s,t)} - z^{(n)}_{(s,t)}\big|_{\mathfrak{B}}, \end{aligned} \tag{4.3}$$

by condition (A_3)-(ii). We also have the estimate

$$\sup_{(s,t)\in[\xi,x]\times[\eta,y]} |z^{(m)}(s,t) - z^{(n)}(s,t)| \leq \int_0^x\int_0^y \big|f\big(s,t,z^{(m-1)}_{(s,t)}\big) - f\big(s,t,z^{(n-1)}_{(s,t)}\big)\big| dsdt. \tag{4.4}$$

The integral on $[0,x]\times[0,y]$ can be divided into two integrals on $[\xi,x]\times[\eta,y]$ and $([0,x]\times[0,y])\cap S(\xi,\eta)$. The latter can be estimated by

$$\delta^{(k-1)} = \sup_{m,n\geq k} \iint_{S(\xi,\eta)} \big|f\big(s,t,z^{(m-1)}_{(s,t)}\big) - f\big(s,t,z^{(n-1)}_{(s,t)}\big)\big| dsdt.$$

If we futhermore set

$$\begin{aligned} \varepsilon^{(k)} &= \sup_{m,n\geq k}\sup_{(s,t)\in\Gamma_{\xi\eta}} \big|z^{(m)}_{(s,t)} - z^{(n)}_{(s,t)}\big|_{\mathfrak{B}}, \\ K_{cd} &= \sup\{K(x-\xi,y-\eta):\ (x,y)\in G_{\xi\eta}\cap S(c,d)\}, \\ M_{cd} &= \sup\{M(x-\xi,y-\eta):\ (x,y)\in G_{\xi\eta}\cap S(c,d)\}, \end{aligned}$$

then by (4.2), (4.3) and (4.4) we obtain for $(x,y)\in G_{\xi\eta}\cap S(c,d)$ the estimate

$$\begin{aligned} v^{(m,n)}(x,y) = \big|z^{(m)}_{(x,y)} - z^{(n)}_{(x,y)}\big|_{\mathfrak{B}} \leq \\ \leq K(x-\xi,y-\eta)\int_\xi^x\int_\eta^y \big|f\big(s,t,z^{(m-1)}_{(s,t)}\big) - f\big(s,t,z^{(n-1)}_{(s,t)}\big)\big| dsdt + \\ + K_{cd}\delta^{(k-1)} + M_{cd}\sup_{(s,t)\in\Gamma_{\xi\eta}} \big|z^{(m)}_{(s,t)} - z^{(n)}_{(s,t)}\big|_{\mathfrak{B}} \leq \\ \leq K(x-\xi,y-\eta)\int_\xi^x\int_\eta^y \omega\big(s,t,v^{(m,n)}(s,t)\big) dsdt + \\ + K_{cd}\delta^{(k-1)} + M_{cd}\varepsilon^{(k)}, \end{aligned}$$

from which it follows that

$$v^{(k)}(x,y) \leq K(x-\xi,y-\eta)\int_\xi^x\int_\eta^y \omega\big(s,t,v^{(k-1)}(s,t)\big) dsdt + K_{cd}\delta^{(k-1)} + M_{cd}\varepsilon^{(k)}.$$

Since $\lim_{k\to\infty}\delta^{(k)} = \lim_{k\to\infty}\varepsilon^{(k)} = 0$, by the Lebesgue dominated convergence theorem we get

$$v(x,y) \le K(x-\xi, y-\eta)\int_\xi^x\int_\eta^y \omega(s,t,v(s,t))dsdt.$$

By Theorem 3.2 and condition (iii) in Assumption 4.1, we have $v(x,y) \equiv 0$ on $G_{\xi\eta}\cap S(c,d)$, which yields that $\lim_{k\to\infty} v^{(k)}(x,y) = 0$ uniformly on $G_{\xi\eta}\cap S(c,d)$. Thus $\{z^{(k)}_{(x,y)}\}_{k=1}^{\infty}$ is a Cauchy sequence on $G_{\xi\eta}\cap S(c,d)$. By condition (A_3)-(i), we have $|z^{(m)}(x,y) - z^{(n)}(x,y)| \le H\big|z^{(m)}_{(x,y)} - z^{(n)}_{(x,y)}\big|_{\mathfrak{B}}$ and consequently $\{z^{(k)}(x,y)\}_{k=0}^{\infty}$ is uniformly convergent on $G_{\xi\eta}\cap S(c,d)$ which yields the contradiction.

Thus $\{z^{(k)}(x,y)\}$ converges uniformly on G to a continuous function $z^*(x,y)$. By the Carathéodory condition (iii) and the Lebesgue dominated convergence theorem we get

$$\lim_{k\to\infty}\int_0^x\int_0^y f\big(s,t,z^{(k)}_{(x,y)}\big)dsdt = \int_0^x\int_0^y f\big(s,t,z^{*}_{(x,y)}\big)dsdt,$$

for each $(x,y)\in G$. This yields that z^* is a solution of equation (4.1) which means a solution of problem (1.1), (1.2).

Finally, we show the uniqueness of solutions of problem (1.1), (1.2). Let v and w be two solutions of (4.1). As above put

$$\tau = \sup\big\{\theta\in[0,1]:\ v(x,y) = w(x,y) \quad \text{for} \quad (x,y)\in G\setminus S(\theta a,\theta b)\big\},$$

and suppose that $\tau < 1$. If we set $\xi = \tau a$ and $\eta = \tau b$ then there exist a constant $r > 0$ and a comparison function $\omega : G_{\xi\eta}\times[0,2r]\to\mathbb{R}_+$ satisfying inequality (4.2). We choose $c\in(\xi,a]$ and $d\in(\eta,b]$ such that

$$|v_{(x,y)} - w_{(x,y)}|_{\mathfrak{B}} \le 2r \quad \text{for} \quad (x,y)\in G_{\xi\eta}\cap S(c,d).$$

Then for all $(x,y)\in G_{\xi\eta}\cap S(c,d)$ we obtain

$$\begin{aligned}|v_{(x,y)} - w_{(x,y)}|_{\mathfrak{B}} &\le K(x-\xi,y-\eta)\sup_{(s,t)\in[\xi,x]\times[\eta,y]}|v(s,t)-w(s,t)| \le\\ &\le K(x-\xi,y-\eta)\int_\xi^x\int_\eta^y\big|f\big(s,t,v_{(s,t)}\big) - f\big(s,t,w_{(s,t)}\big)\big|dsdt \le\\ &\le K(x-\xi,y-\eta)\int_\xi^x\int_\eta^y \omega\big(s,t,|v_{(s,t)}-w_{(s,t)}|_{\mathfrak{B}}\big)dsdt.\end{aligned}$$

By Theorem 3.2, we have $|v_{(x,y)} - w_{(x,y)}|_{\mathfrak{B}} \equiv 0$ on $G_{\xi\eta}\cap S(c,d)$ and by condition (A_3)-(i) we get $v(x,y)\equiv w(x,y)$ on $G_{\xi\eta}\cap S(c,d)$, which yields a contradiction. Thus $\tau = 1$ and the solution of (4.1) is unique on G. This completes the proof of Theorem 4.2. □

5. EXAMPLES

Finally we present two simple equations that are examples of (1.1).

Example 5.1. Let $\mathfrak{B}$ be the space C_γ defined in Example 2.3 with $\gamma < 0$. In this case we have $K(x,y) = e^{-\gamma(x+y)}$. Suppose that $\phi : E_0 \to \mathbb{R}$ is any continuous function such that $\phi_{(s,t)} \in C_\gamma$ for $(s,t) \in E_0 \cap G$. Consider the functional differential equation

$$D_{xy}z(x,y) = \frac{(xy)^{-\frac{1}{2}}}{\sqrt{|z(\frac{x}{2},\frac{y}{2})|+1}}, \quad (x,y) \in G. \tag{5.1}$$

If we define $f : G \times C_\gamma \to \mathbb{R}$ by the formula $f(x,y,w) = \frac{(xy)^{-\frac{1}{2}}}{\sqrt{|w(-\frac{x}{2},-\frac{y}{2})|+1}}$ then f satisfies the Carathéodory conditions with $m : G \to \mathbb{R}_+$ defined by $m(x,y) = (xy)^{-\frac{1}{2}}$. Furthermore, for any $(x,y) \in G$ and $w, \bar{w} \in C_\gamma$ we have

$$|f(x,y,w) - f(x,y,\bar{w})| \leq \Big| \frac{(xy)^{-\frac{1}{2}}}{\sqrt{|w(-\frac{x}{2},-\frac{y}{2})|+1}} - \frac{(xy)^{-\frac{1}{2}}}{\sqrt{|\bar{w}(-\frac{x}{2},-\frac{y}{2})|+1}} \Big| \leq$$
$$\leq \tfrac{1}{2}(xy)^{-\frac{1}{2}} \big| \bar{w}(-\tfrac{x}{2},-\tfrac{y}{2}) - w(-\tfrac{x}{2},-\tfrac{y}{2}) \big| \leq \tfrac{1}{2}(xy)^{-\frac{1}{2}} e^{\gamma(\frac{x}{2}+\frac{y}{2})} |\bar{w} - w|_{C_\gamma}.$$

This means that condition (4.2) in Theorem 4.2 holds with any $(\xi,\eta) \in G$, $r > 0$ and a comparison function $\omega : G_{\xi\eta} \times [0,2r] \to \mathbb{R}_+$ given by the formula $\omega(x,y,v) = \frac{1}{2}(xy)^{-\frac{1}{2}} e^{\gamma(\frac{x}{2}+\frac{y}{2})} v$. We see that ω satisfies the Carathéodory conditions with $\mu : G_{\xi\eta} \to \mathbb{R}_+$ given by $\mu(x,y) = (xy)^{-\frac{1}{2}} e^{\gamma(\frac{x}{2}+\frac{y}{2})} r$. Comparison integral equation (3.1) in our case takes the form

$$v(x,y) = e^{-\gamma(x-\xi+y-\eta)} \int_\xi^x \int_\eta^y \tfrac{1}{2}(st)^{-\frac{1}{2}} e^{\gamma(\frac{s}{2}+\frac{t}{2})} v(s,t)\,ds\,dt. \tag{5.2}$$

Since ω is nondecreasing with respect to v, equation (5.2) has only a zero solution by Theorem 3.2 and consequently the conclusion of Theorem 4.2 holds for problem (5.1), (1.2).

Example 5.2. Let $\mathfrak{B}$ be the space CL_γ defined in Example 2.4 with $a_0 = b_0 = 0$ and $\gamma > 0$. Then we have

$$K(x,y) = \int_{-x}^0 \int_{-y}^0 e^{\gamma(s+t)}\,ds\,dt = \frac{1}{\gamma^2}(1-e^{-\gamma x})(1-e^{-\gamma y}).$$

Let $\phi : E_0 \to \mathbb{R}$ be an initial function such that $\phi_{(s,t)} \in CL_\gamma$ for $(s,t) \in E_0 \cap G$, and ϕ is continuous on $E_0 \cap G$. Consider the differential integral equation

$$D_{xy}z(x,y) = xy \sin \frac{z(x,y)}{xy} + (1 + e^{-\gamma(x+y)}) \int_{-\infty}^x \int_{-\infty}^y e^{-\gamma(x-s+y-t)} \sin z(s,t)\,ds\,dt, \tag{5.3}$$

where $(x, y) \in G$. We define $f : G \times CL_\gamma \to \mathbb{R}$ by the formula

$$f(x,y,w) \quad = \quad xy \sin \frac{w(0,0)}{xy} \; + \; (1 \; + \; e^{-\gamma(x+y)}) \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} e^{\gamma(s+y)} \sin w(s,t) ds dt.$$

We see that f satisfies the Carathéodory conditions with $m : G \to \mathbb{R}_+$ defined by $m(x,y) = xy + \frac{1}{\gamma^2}(1 + e^{-\gamma(x+y)})$. For any $(x,y) \in G$ and $w, \bar{w} \in CL_\gamma$ we also have

$$\begin{aligned}
&|f(x,y,w) - f(x,y,\bar{w})| \leq \\
&\leq xy \Big| \sin \frac{w(0,0)}{xy} - \sin \frac{\bar{w}(0,0)}{xy} \Big| + \\
&\quad + (1 + e^{-\gamma(x+y)}) \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} e^{\gamma(s+t)} |\sin w(s,t) - \sin \bar{w}(s,t)| ds dt \leq \\
&\leq |w(0,0) - \bar{w}(0,0)| + (1 + e^{-\gamma(x+y)}) \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} e^{\gamma(s+t)} |w(s,t) - \bar{w}(s,t)| ds dt \leq \\
&\leq (1 + e^{-\gamma(x+y)}) |w - \bar{w}|_{CL_\gamma}.
\end{aligned}$$

From the above estimate we get that condition (4.2) in Theorem 4.2 holds for any $(\xi, \eta) \in G$, $r > 0$ and a comparison function $\omega(x,y,v) = (1 + e^{-\gamma(x+y)})v$, which satisfies the Carathéodory conditions with $\mu(x,y) = 2(1 + e^{-\gamma(x+y)})r$. Comparison integral equation (3.1) now takes the form

$$v(x,y) = \frac{1}{\gamma^2}(1 - e^{-\gamma(x-\xi)})(1 - e^{-\gamma(y-\eta)}) \int\limits_{\xi}^{x} \int\limits_{\eta}^{y} (1 + e^{-\gamma(s+t)}) v(s,t) ds dt.$$

Since ω is nondecreasing with respect to v, the above equation has only a zero solution by Theorem 3.2. Thus conclusion of Theorem 4.2 holds for problem (5.3), (1.2).

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