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Dedicated to the Memory of Professor Zdzisław Kamont

DIFFERENCE PROBLEMS GENERATED BY INFINITE SYSTEMS OF NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH THE ROBIN CONDITIONS

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Abstract. We consider the classical solutions of mixed problems for infinite, countable systems of parabolic functional differential equations. Difference methods of two types are constructed and convergence theorems are proved. In the first type, we approximate the exact solutions by solutions of infinite difference systems. Methods of second type are truncation of the infinite difference system, so that the resulting difference problem is finite and practically solvable. The proof of stability is based on a comparison technique with nonlinear estimates of the Perron type for the given functions. The comparison system is infinite. Parabolic problems with deviated variables and integro-differential problems can be obtained from the general model by specifying the given operators.

Keywords: nonlinear parabolic equations, functional difference equations, infinite systems, Volterra type operators, nonlinear estimates of the Perron type, truncation methods.

Mathematics Subject Classification: 35R10, 35K51, 65M10.

1. INTRODUCTION

During this time numerous papers concerned problems for infinite systems of parabolic functional differential equations were published. The exposition of existence results for such problems can be found in the monograph [1], see also [9] and [4]. The papers [2,11, 12] contain uniqueness criteria for infinite parabolic problems. Various applications of infinite systems of parabolic integral differential equations, such as the discrete coagulation fragmentation model [13], are listed in [1].

We are interested in establishing numerical discretization methods for solving infinite systems of parabolic functional differential equations with initial boundary conditions of the Robin type.

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. Denote by l^∞ the class of all real sequences $p = \{p_\mu\}_{\mu\in\mathbb{N}}$ such that $\|p\|_\infty = \sup\{|p_\mu| : \mu \in \mathbb{N}\} < \infty$. For simplicity we will write $p = \{p_\mu\}$ instead of $p = \{p_\mu\}_{\mu\in\mathbb{N}}$. If $p, q \in l^\infty$, $p = \{p_\mu\}$, $q = \{q_\mu\}$, then we set $p * q = \{p_\mu q_\mu\}$. Put $\mathcal{R}_n^\infty$ to denote the set of all $q = (q_1, \dots, q_n)$, such that $q_j \in l^\infty$, $1 \le j \le n$. We use the symbol $M_{n\times n}$ to denote the set of all real symmetric $n \times n$ matrices. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Analogously we understand the inequalities between infinite sequences. Inequalities between matrices are interpreted by means of quadratic forms.

Let $a > 0,\ b = (b_1, \dots, b_n) \in \mathbb{R}^n,\ b_j > 0$ for $j = 1, \dots, n$, be given. Define the sets

$$E = [0,a] \times [-b,b], \quad E_0 = \{0\} \times [-b,b], \quad \partial_0 E = E \setminus \Big([0,a] \times (-b,b)\Big).$$

Write

$$\partial_{j.+}E = \Big\{(t,x) \in \partial_0 E : x_j = b_j\Big\}, \quad \partial_{j.-}E = \Big\{(t,x) \in \partial_0 E : x_j = -b_j\Big\}, \quad 1 \le j \le n.$$

Set $\Omega = E \times C(E, l^\infty) \times R^n \times M_{n\times n}$. Suppose that

$$f : \Omega \to l^\infty,\ f = \{f_\mu\}, \quad \varphi : E_0 \to l^\infty,\ \varphi = \{\varphi_\mu\},$$

$$\beta, \psi : \partial_0 E \to \mathcal{R}_n^\infty,$$

$$\beta = (\beta_1, \dots, \beta_n),\ \psi = (\psi_1, \dots, \psi_n),\ \beta_j = \{\beta_{j.\mu}\},\ \psi_j = \{\psi_{j.\mu}\},\ 1 \le j \le n,$$

are given functions. For the function $z : E \to l^\infty$, $z = \{z_\mu\}$, of the variables (t,x), $x = (x_1, \dots, x_n)$, and for $1 \le j \le n$ we write

$$\partial_t z = \{\partial_t z_\mu\}, \quad \partial_{x_j} z = \{\partial_{x_j} z_\mu\}, \quad F[z] = \{F^{(\mu)}[z]\},$$

$$F^{(\mu)}[z](t,x) = f_\mu(t, x, z, \partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)),$$

where $\partial_x z_\mu = (\partial_{x_1} z_\mu, \dots, \partial_{x_n} z_\mu)$, $\partial_{xx} z_\mu = [\partial_{x_i x_j} z_\mu]_{i,j=1,\dots,n}$, $\mu \in \mathbb{N}$. We consider the infinite countable system of differential functional equations

$$\partial_t z(t,x) = F[z](t,x) \tag{1.1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0, \tag{1.2}$$

and with the following boundary conditions

$$\beta_j(t,x) * z(t,x) + \partial_{x_j} z(t,x) = \psi_j(t,x) \quad \text{on} \quad \partial_{j.+}E, \tag{1.3}$$

$$\beta_j(t,x) * z(t,x) - \partial_{x_j} z(t,x) = \psi_j(t,x) \quad \text{on} \quad \partial_{j.-}E, \tag{1.4}$$

where $1 \le j \le n$.

We will assume that the functional dependence in (1.1) is of the Volterra type.

Assumption H[V]. The function $f : \Omega \to l^\infty$ satisfies the Volterra condition, i.e. for each $(t, x) \in E$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}$ and $w, \bar{w} \in C(E, l^\infty)$ such that $w(\tau, y) = \bar{w}(\tau, y)$, $(\tau, y) \in E$, $\tau \le t$, we have $f(t, x, w, q, r) = f(t, x, \bar{w}, q, r)$.

We say that a function $v : E \to l^\infty$, $v = \{v_\mu\}$, is a *regular solution* of the system (1.1) if the derivatives $\partial_t v = \{\partial_t v_\mu\}$, $\partial_{x_i x_j} v = \{\partial_{x_i x_j} v_\mu\}$, $1 \le i, j \le n$, exist on E, $\partial_t v, \partial_{x_i x_j} v \in C(E, l^\infty)$, $1 \le i, j \le n$, and v satisfies (1.1) on E.

A regular solution v of (1.1) is said to be *parabolic* if for any two symmetric matrices $r = [r_{ij}]_{i,j=1,\dots,n}$, $\bar{r} = [\bar{r}_{ij}]_{i,j=1,\dots,n}$ such that $r - \bar{r} \le 0$ the inequality $f_\mu(t, x, v, \partial_x v_\mu(t, x), r) \le f_\mu(t, x, v, \partial_x v_\mu(t, x), \bar{r})$ is true for $(t, x) \in E$, $\mu \in \mathbb{N}$. The parabolic solution v of (1.1) such that the conditions (1.2)–(1.4) hold, is called a $\mathcal{P}$*-solution* of (1.1)–(1.4).

Approximate methods for parabolic differential or functional differential equations were considered by many authors and under various assumptions. The main problem in these investigations is to find suitable difference or functional difference equations which are consistent with respect to the original problem and stable. It is not our aim to show a full review of papers concerning difference methods for parabolic functional differential problems. Bibliographical information can be found in [6–8, 10].

We propose difference explicit Euler type schemes which consist of replacing partial derivatives in (1.1) by suitable difference operators. Quasilinear parabolic equations with the Robin conditions are considered in [3]. In the case of quasilinear equations, the choice of the difference operators approximating mixed derivatives is locally determined by the sign of the coefficients in the differential equations and upwind difference schemes are used. In the present paper, the choice of suitable difference operators depends on global assumptions on given functions (see the definitions (2.5), (2.6) and the conditions 2) of Assumption $H_0[\Delta]$). By using explicit schemes, the approximation of the Robin boundary conditions (1.3), (1.4) requires an extention of the mesh outside the set E.

In the first part of the present paper we consider an infinite system of functional difference equations generated by (1.1)–(1.4). If the original differential problem is reduced to the finite one, then the difference method is practically solvable.

The next part of the paper deals with truncated finite differential functional problems corresponding to (1.1)–(1.4) and difference functional methods related to them. We show results of numerical experiments.

Results presented in the paper are new also in the case of infinite systems without a functional dependence.

2. INFINITE DIFFERENCE SCHEMES

To formulate a difference problem corresponding to (1.1)–(1.4) we introduce the following notation and assumptions. Denote by $\mathcal{F}(A, B)$ the class of all functions defined on A and taking values in B, where A and B are arbitrary sets. If $x \in \mathbb{R}^n$, $x = (x_1, \dots x_n)$ then we put $\|x\| = |x_1| + \dots + |x_n|$. We define a mesh on the set E in the following way. Suppose that (h_0, h'), where $h' = (h_1, \dots, h_n)$, $h_i > 0$, $0 \le i \le n$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbb{Z}^{1+n}$, where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1h_1, \ldots, m_nh_n).$$

Denote by Δ the set of all $h = (h_0, h')$ such that there are $N_0 \in \mathbb{N}$ and $N = (N_1, \ldots, N_n) \in \mathbb{N}^n$ with the properties: $N_0h_0 = a$ and $(N_1h_1, \ldots, N_nh_n) = b$. Let

$$R_h^{1+n} = \{ (t^{(r)}, x^{(m)}) : \ (r, m) \in \mathbb{Z}^{1+n} \},$$

$$E_h = E \cap R_h^{1+n}, \quad E_{0.h} = E_0 \cap R_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap R_h^{1+n}$$

and

$$E'_h = \Big\{ (t^{(r)}, x^{(m)}) \in E_h : \quad 0 \le r \le N_0 - 1 \Big\}.$$

For every $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$ we define the set $S^{(m)}$ of $s = (s_1, \ldots, s_n)$ such that $\|s\| = 1$ or $\|s\| = 2$ and

$$\text{if} \quad x_j^{(m_j)} = b_j, \quad \text{then} \quad s_j \in \{0, 1\}, \qquad \text{if} \quad x_j^{(m_j)} = -b_j, \quad \text{then} \quad s_j \in \{0, -1\},$$

$$\text{and} \quad \text{if} \quad -b_j < x_j^{(m_j)} < b_j, \quad \text{then} \quad s_j = 0,$$

where $1 \le j \le n$. Let

$$\partial_0^+ E_h = \{(t^{(r)}, x^{(m+s)}) : (t^{(r)}, x^{(m)}) \in \partial_0 E_h, \ s \in S^{(m)}\} \qquad \text{and} \qquad E_h^+ = \partial_0^+ E_h \cup E_h.$$

If $A_h \subset R_h^{1+n}$ and $z : A_h \to l^\infty$, $\omega : A_h \to \mathbb{R}$, then we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and $\omega^{(r,m)} = \omega(t^{(r)}, x^{(m)})$ on A_h. Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 standing on i-th place. We define the difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ and δ_i^+, δ_i^-, $1 \le i \le n$, in the following way. For $z : E_h^+ \to l^\infty$, $z = \{z_\mu\}$, and $(t^{(r)}, x^{(m)}) \in E'_h$ set

$$\delta_0 z_\mu^{(r,m)} = \frac{1}{h_0}(z_\mu^{(r+1,m)} - z_\mu^{(r,m)}), \tag{2.1}$$

$$\delta_i z_\mu^{(r,m)} = \frac{1}{2h_i}(z_\mu^{(r,m+e_i)} - z_\mu^{(r,m-e_i)}), \tag{2.2}$$

$$\delta_i^+ z_\mu^{(r,m)} = \frac{1}{h_i}(z_\mu^{(r,m+e_i)} - z_\mu^{(r,m)}), \qquad \delta_i^- z_\mu^{(r,m)} = \frac{1}{h_i}(z_\mu^{(r,m)} - z_\mu^{(r,m-e_i)}), \tag{2.3}$$

where $1 \le i \le n$, $\mu \in \mathbb{N}$. The difference operator $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$, is defined as follows. Write

$$\delta_{ii} z_\mu^{(r,m)} = \delta_i^+ \delta_i^- z_\mu^{(r,m)}, \quad 1 \le i \le n, \quad \mu \in \mathbb{N}. \tag{2.4}$$

Put $J = \Big\{(i,j) : 1 \le i,j \le n, \, i \ne j\Big\}$ and suppose that for each $\mu \in \mathbb{N}$ we have defined two disjoint sets $J_{\mu.+}, J_{\mu.-} \subset J$ such that $J_{\mu.+} \cup J_{\mu.-} = J$. Then for $(i,j) \in J$

$$\text{if} \quad (i,j) \in J_{\mu.+}, \quad \text{then} \quad \delta_{ij} z_\mu^{(r,m)} = \frac{1}{2}\Big(\delta_i^+ \delta_j^+ z_\mu^{(r,m)} + \delta_i^- \delta_j^- z_\mu^{(r,m)}\Big), \tag{2.5}$$

$$\text{if} \quad (i,j) \in J_{\mu.-}, \quad \text{then} \quad \delta_{ij} z_\mu^{(r,m)} = \frac{1}{2}\Big(\delta_i^+ \delta_j^- z_k^{(r,m)} + \delta_i^- \delta_j^+ z_\mu^{(r,m)}\Big). \tag{2.6}$$

Solutions of difference equations will be defined on the set E_h^+. Since system (1.1) contains the functional variable z which is an element of the space $C(E, l^\infty)$, we need an interpolating operator $\mathcal{T}_h : \mathcal{F}(E_h^+, l^\infty) \to C(E, l^\infty)$. Additional assumptions on $\mathcal{T}_h$ will be required in the next part of this paper. For $z : E_h^+ \to l^\infty$, $z = \{z_\mu\}$, we put on E_h'

$$\delta_0 z = \{\delta_0 z_\mu\}, \quad F_h[z] = \{F_{h.\mu}[z]\},$$

$$F_{h.\mu}[z]^{(r,m)} = f_\mu(t^{(r)}, x^{(m)}, \mathcal{T}_h z,\, \delta z_\mu^{(r,m)}, \delta^{(2)} z_\mu^{(r,m)}), \quad \mu \in \mathbb{N}.$$

If $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$, $s \in S^{(m)}$, then we write

$$g_h[z]^{(r,m,s)} = 2\sum_{j=1}^{n} s_j^2 h_j \psi_j(t^{(r)}, x^{(m)}) - (z^{(r,m+s)} + z^{(r,m-s)}) * \sum_{j=1}^{n} s_j^2 h_j \beta_j(t^{(r)}, x^{(m)}).$$

We will approximate solutions of (1.1)–(1.4) by means of solutions of the difference functional problem

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \quad \text{on} \quad E_h', \tag{2.7}$$

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h}, \tag{2.8}$$

$$z^{(r,m+s)} - z^{(r,m-s)} = g_h[z]^{(r,m,s)} \quad \text{on} \quad \partial_0 E_h,\ s \in S^{(m)}, \tag{2.9}$$

where $\varphi_h : E_{0.h} \to l^\infty$, $\varphi_h = \{\varphi_{h.\mu}\}$, is a given function.

Now we introduce first assumptions which allow us to obtain the existence of the unique solution of the difference problem (2.7)–(2.9). Under these assumptions we also obtain useful estimates for the solution of (2.7)–(2.9) and for the $\mathcal{P}$-solution of the differential problem (1.1)–(1.4).

For $w \in C(E, \mathbb{R})$ and for $z \in \mathcal{F}(E_h^+, \mathbb{R})$ we put

$$|w|_t = \max\Big\{|w(\tau, x)| : (\tau, x) \in E,\ \tau \le t\Big\}, \qquad t \in [0, a],$$

$$|z|_{(r)} = \max\Big\{|z^{(\nu,m)}| : (t^{(\nu)}, x^{(m)}) \in E_h,\ \nu \le r\Big\}, \qquad 0 \le r \le N_0.$$

If $w \in C(E, l^\infty)$, $w = \{w_\mu\}$, and $z \in \mathcal{F}(E_h^+, l^\infty)$, $z = \{z_\mu\}$, then we set $|w|_t = \{|w_\mu|_t\}$, $t \in [0, a]$, and $|z|_{(r)} = \{|z_\mu|_{(r)}\}$, $0 \le r \le N_0$.

Assumption H[$\mathcal{T}_\mathrm{h}$]. The operator $\mathcal{T}_h : \mathcal{F}(E_h^+, l^\infty) \to C(E, l^\infty)$ is linear, $\mathcal{T}_h z = \{T_h z_\mu\}$ for $z \in \mathcal{F}(E_h^+, l^\infty)$, $z = \{z_\mu\}$, and the mapping $T_h : \mathcal{F}(E_h^+, \mathbb{R}) \to C(E, \mathbb{R})$ satisfies the conditions:

1) if $\omega, \bar\omega \in \mathcal{F}(E_h^+, \mathbb{R})$ and $\omega = \bar\omega$ on E_h then $T_h\omega = T_h\bar\omega$,
2) for $\omega : E_h^+ \to R$ and $0 \le r \le N_0$ we have $|T_h\omega|_{t^{(r)}} = |\omega|_{(r)}$,
3) if $w : E \to \mathbb{R}$ is of class C^1 and w_h is the restriction of w to the set E_h then there exists $\tilde\gamma : \Delta \to \mathbb{R}_+$ such that $|\,T_h w_h - w\,|_a \le \tilde\gamma(h)$ and $\lim_{h\to 0} \tilde\gamma(h) = 0$.

Remark 2.1. To define an example of $T_h : \mathcal{F}(E_h^+, \mathbb{R}) \to C(E, \mathbb{R})$ satisfying the above conditions we can use the interpolating operator proposed in [5] for the construction of difference scheme corresponding to first order partial differential functional equations.

If $p \in l^\infty$, $p = \{p_\mu\}$, then we write $|p| = \{|p_\mu|\}$. Let $\mathbf{0} \in l^\infty$ and $\mathbf{1} \in l^\infty$ be the sequences with all the elements equal to 0 and 1 respectively. Put $\mathbb{R}_+ = [0, +\infty)$ and

$$l^\infty_+ = \Big\{ p \in l^\infty : \; p = \{p_\mu\}, \; p_\mu \in \mathbb{R}_+, \; \mu \in \mathbb{N} \Big\},$$

$$l^\infty_0 = \Big\{ p \in l^\infty_+ : \; p = \{p_\mu\}, \; \lim_{\mu \to \infty} p_\mu = 0 \Big\}.$$

Assumption H$[\sigma_0]$. The functions $f : \Omega \to l^\infty$, $\beta, \psi : \partial_0 E \to \mathcal{R}^\infty_n$ and $\varphi : E_0 \to l^\infty$ satisfy the conditions:

1) there is $A_0 \in l^\infty_+$ such that $|\varphi(t,x)| \le A_0$ on E_0,
2) there is $\tilde{b} \in l^\infty$, $\tilde{b} = \{\tilde{b}_\mu\}$, such that $\beta_j(t,x) \ge \tilde{b} > \mathbf{0}$ on $\partial_0 E$, $1 \le j \le n$,
3) there exist $\sigma_0 \in C([0,a] \times l^\infty_+, l^\infty_+)$ of variables (t,p), and $L_0 \in l^\infty_+$ such that
   (i) σ_0 is nondecreasing with respect to both variables and $\sigma_0(t,p) \le L_0$ for $(t,p) \in [0,a] \times l^\infty_+$,
   (ii) there exists on $[0,a]$ a maximal solution $\omega_0 = \{\omega_{0.\mu}\}$ of the Cauchy problem

$$\omega'(t) = \sigma_0(t, \omega(t)), \quad \omega(0) = A_0, \tag{2.10}$$

4) the estimates

$$|f(t,x,w,0,0)| \le \sigma_0(t, |w|_t), \quad (t,x,w) \in E \times C(E, l^\infty),$$
$$|\psi_j(t,x)| \le \tilde{b} * \omega_0(t), \quad (t,x) \in \partial_0 E, \quad 1 \le j \le n, \tag{2.11}$$

   are satisfied.

Remark 2.2. Suppose that Assumption H$[\sigma_0]$ is satisfied. Then $\mathcal{P}$-solution $v : E \to l^\infty$ of problem (1.1)–(1.4) satisfies the estimate

$$|v(t,x)| \le \omega_0(t) \quad \text{on} \quad E$$

where ω_0 is the maximal solution of (2.10). This assertion follows from the comparison theorem for infinite systems of parabolic functional differential equations (see [2]).

Let $E^+ = [0,a] \times (-b^+, b^+)$, where $b^+ \in \mathbb{R}^n_+$ and $b^+ > b$.
Assumption H$_0[\Delta]$. The functions $f : \Omega \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}^\infty_n$, $\varphi_h : E_{0.h} \to l^\infty$ and $h \in \Delta$ are such that:

1) Assumption H[V] is satisfied,
2) for each $\mu \in \mathbb{N}$ there exist the derivatives

$$\partial_q f_\mu = (\partial_{q_1} f_\mu, \dots, \partial_{q_n} f_\mu) \quad \text{and} \quad \partial_r f_\mu = [\partial_{r_{ij}} f_\mu]_{i,j=1,\dots,n}$$

   on Ω, they are continuous with respect to (q,r) and for $P \in \Omega$

$$\partial_{r_{ij}} f_\mu(P) \ge 0 \quad \text{for} \quad (i,j) \in J_{\mu.+}, \quad \partial_{r_{ij}} f_\mu(P) \le 0 \quad \text{for} \quad (i,j) \in J_{\mu.-},$$

3) there is $A_0 \in l^\infty_+$, $A_0 = \{A_{0.\mu}\}$, such that $|\varphi^{(r,m)}_h| \le A_0$ on $E_{0.h}$,

4) $E_h^+ \subset E^+$ and the inequalities

$$\mathbf{1} - 2\sum_{i=1}^{n} \frac{h_0}{h_i^2}\partial_{r_{ii}} f(P) + \sum_{(i,j)\in J} \frac{h_0}{h_i h_j}|\partial_{r_{ij}} f(P)| \geq \mathbf{0}, \tag{2.12}$$

$$\frac{1}{h_i}\partial_{r_{ii}} f(P) - \sum_{j=1, j\neq i}^{n} \frac{1}{h_j}|\partial_{r_{ij}} f(P)| - \frac{1}{2}|\partial_{q_i} f(P)| \geq \mathbf{0}, \quad 1 \leq i \leq n, \tag{2.13}$$

hold with $P \in \Omega$, where $\partial_{q_i} f = \{\partial_{q_i} f_\mu\}$, $1 \leq i \leq n$, and the inequality

$$\mathbf{1} - \sum_{j=1}^{n} h_j \beta_j(t,x) \geq \mathbf{0} \tag{2.14}$$

holds on $\partial_0 E$.

Lemma 2.3. *If Assumptions* $\mathrm{H}[\,\mathcal{T}_h\,]$, $\mathrm{H}[\,\sigma_0\,]$ *and* $\mathrm{H}_0[\,\Delta\,]$ *are satisfied then there exists exactly one solution* $u_h : E_h^+ \to l^\infty$ *of problem* (2.7)–(2.9) *and*

$$|u_h^{(r,m)}| \leq \omega_0(t^{(r)}) \quad on \quad E_h, \tag{2.15}$$

where ω_0 *is the maximal solution of* (2.10).

To prove Lemma 2.3 we can use the techniques from [3] for quasilinear problems. We need relations

$$\delta_0 u_{h.\mu}^{(r,m)} = \sum_{i,j=1}^{n} \partial_{r_{ij}} f_\mu(P_\mu^{(r,m)}[u_h])\, \delta_{ij} u_{h.\mu}^{(r,m)} + \\ + \sum_{i=1}^{n} \partial_{q_i} f^{(\mu)}(P_\mu^{(r,m)}[u_h])\, \delta_i u_{h.\mu}^{(r,m)} + f^{(\mu)}(t^{(r)}, x^{(m)}, \mathcal{T}_h u_h, 0, 0), \quad \mu \in \mathbb{N},$$

where $(t^{(r)}, x^{(m)}) \in E_h'$ and $P_\mu^{(r,m)}[u_h] = (t^{(r)}, x^{(m)}, \mathcal{T}_h u_h, \xi\delta u_{h.\mu}^{(r,m)}, \xi\delta^{(2)} u_{h.\mu}^{(r,m)})$, $\xi \in (0,1)$, instead of (21) in [3].

3. CONVERGENCE OF DIFFERENCE METHODS

Now we formulate general conditions for the convergence of the method (2.7)–(2.9). For $p \in l_+^\infty$ we define $C_p(E, l^\infty) = \{w \in C(E, l^\infty) : |w|_a \leq p\}$.

Assumption $\mathrm{H}[\sigma]$. The function $f : \Omega \to l^\infty$ is continuous, Assumption $\mathrm{H}[\sigma_0]$ is satisfied and

1) the sequence $A \in l_+^\infty$ is such that $A > \mathbf{0}$ and $A \geq \omega_0(a)$,
2) there exists $\sigma \in C([0,a] \times l_+^\infty, l_+^\infty)$, $\sigma = \{\sigma_\mu\}$, and $L \in l_+^\infty$ such that
   (i) σ is nondecreasing with respect to both variables, $\sigma(t, \mathbf{0}) = \mathbf{0}$, $t \in [0,a]$, and $\sigma(t,p) \leq L$ on $[0,a] \times l_+^\infty$,

(ii) the function $\tilde{\omega}(t) = \mathbf{0}$, $t \in [0, a]$, is the unique solution of the problem

$$\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = \mathbf{0},$$

3) the estimate

$$|\, f(t, x, w, q, r) - f(t, x, \bar{w}, q, r)\,| \leq \sigma(t, |w - \bar{w}|_t)$$

is satisfied for $(t, x) \in E$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}$ and $w, \bar{w} \in C_A(E, l^\infty)$.

Assumption $\mathrm{H}_1[\Delta]$. The functions $f : \Omega \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}_n^\infty$, $\varphi_h : E_{0.h} \to l^\infty$ and $h \in \Delta$ satisfy Assumption $\mathrm{H}_0[\Delta]$ and

1) there is a sequence $B \in l_+^\infty$ such that $\beta_j(t, x) \leq B$ on $\partial_0 E$, $1 \leq j \leq n$,
2) there is a constant $\tilde{C} > 0$ such that $\|h'\|^2 \leq \tilde{C} h_0$.

Theorem 3.1. *Suppose that Assumptions* $\mathrm{H}[\mathcal{T}_h]$, $\mathrm{H}_1[\,\Delta\,]$ *and* $\mathrm{H}[\sigma]$ *are satisfied and*

1) *the function* $v : E^+ \to l^\infty$, $v = \{v_\mu\}$, *is such that* $v(\cdot, x) : [0, a] \to l^\infty$, $x \in (-b^+, b^+)$, *is of class* C^1, $v(t, \cdot) : (-b^+, b^+) \to l^\infty$, $t \in [0, a]$, *is of class* C^3 *and there are* $c_0, c_1 \in l_+^\infty$ *such that*

$$|\partial_{x_i x_j} v(t, x)| \leq c_0, \quad |\partial_{x_i x_j x_k} v(t, x)| \leq c_1 \quad on \quad E^+, \quad 1 \leq i, j, k \leq n,$$

and v *is* $\mathcal{P}$*-solution of* (1.1)–(1.4) *on* E,
2) *the function* $u_h : E_h^+ \to l^\infty$, $u_h = \{u_{h.\mu}\}$, *is the solution of problem* (2.7)–(2.9),
3) *there exists a function* $\gamma_\varphi : \Delta \to l_+^\infty$ *such that* $\lim_{h\to 0} \gamma_\varphi(h) = \mathbf{0}$ *and*

$$|\, \varphi_h^{(r,m)} - \varphi(t^{(r)}, x^{(m)})\,| \leq \gamma_\varphi(h) \quad on \quad E_{0.h}.$$

Then there is $\gamma : \Delta \to l_+^\infty$ *such that* $\lim_{h\to 0} \gamma(h) = \mathbf{0}$ *and*

$$|\, u_h^{(r,m)} - v(t^{(r)}, x^{(m)})\,| \leq \gamma(h) \quad on \quad E_h^+. \tag{3.1}$$

We omit the proof of Theorem 3.1. The case of quasilinear problems is proved in [3].

4. FINITE SYSTEMS OF DIFFERENCE EQUATIONS

The main task in investigations presented in this part of the paper is to find a finite difference scheme corresponding to the original infinite problem (1.1)–(1.4). We will apply the truncation method.

Fix $k \in \mathbb{N}$. Let $\tilde{\varphi} \in C(E, l^\infty)$, $\tilde{\varphi} = \{\tilde{\varphi}_\mu\}$, be such that $\tilde{\varphi} = \varphi$ on E_0. For $w : E \to l^\infty$, $w = \{w_\mu\}$, we put

$$[w]_{k.\tilde{\varphi}} = \{\bar{w}_\mu\}, \quad \text{where} \quad \bar{w}_\mu = w_\mu \text{ for } 1 \leq \mu \leq k \quad \text{and} \quad \bar{w}_\mu = \tilde{\varphi}_\mu \text{ for } \mu > k.$$

If $D \subset E$ and $w : D \to l^\infty$, $w = \{w_\mu\}$, then the symbol $w^{[k]}$ denotes the function $w^{[k]} : D \to \mathbb{R}^k$ given by $w^{[k]} = (w_1, \ldots, w_k)$. We will treat an element $p \in \mathbb{R}^k$, $p = (p_1, \ldots, p_k)$, also as the sequence $p = \{p_\mu\}$ with $p_\mu = 0$ for $\mu > k$. Write

$$F^{[k]}[z] = (F_1^{[k]}[z], \ldots, F_k^{[k]}[z]),$$

$$F_\mu^{[k]}[z](t,x) = f_\mu(t, x, [z]_{k.\tilde{\varphi}}, \partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)),$$

where $z : E \to \mathbb{R}^k$, $z = (z_1, \ldots, z_k)$, $1 \le \mu \le k$.

Consider the finite differential functional system

$$\partial_t z(t,x) = F^{[k]}[z](t,x) \tag{4.1}$$

with the initial boundary conditions

$$z(t,x) = \varphi^{[k]}(t,x), \quad (t,x) \in E_0, \tag{4.2}$$

$$\beta^{[k]}(t,x) * z(t,x) + \partial_{x_j} z(t,x) = \psi^{[k]}(t,x), \quad (t,x) \in \partial_{j.+}E, \tag{4.3}$$

$$\beta^{[k]}(t,x) * z(t,x) - \partial_{x_j} z(t,x) = \psi^{[k]}(t,x), \quad (t,x) \in \partial_{j.-}E, \tag{4.4}$$

where $1 \le j \le n$.

To estimate the difference between the solution of the infinite problem (1.1)–(1.4) and the solution of the truncated problem (4.1)–(4.4) we formulate additional assumptions.

Assumption H$[\sigma, \varphi]$. The functions $f : \Omega \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}_n^\infty$ satisfy Assumption H$[\sigma]$ and the function $\varphi \in C(E_0, l^\infty)$ is such that there exists $\tilde{\varphi} \in C(E, l^\infty)$, $\tilde{\varphi} = \{\tilde{\varphi}_\mu\}$, with the properties:

1) $\tilde{\varphi}(t,x) = \varphi(t,x)$ for $(t,x) \in E_0$ and $|\tilde{\varphi}|_a \le \tilde{A}$ with $\tilde{A} = \frac{1}{2}A$,
2) the function $\tilde{\varphi}(\cdot, x) : [0,a] \to l^\infty$, $x \in [-b,b]$, is of class C^1, the function $\tilde{\varphi}(t, \cdot) : [-b,b] \to l^\infty$, $t \in [0,a]$, is of class C^2 and there is $d \in l_+^\infty$, $d = \{d_\mu\}$, such that
$$|\partial_{x_i x_j} \tilde{\varphi}(t,x)| \le d, \quad (t,x) \in E, \quad 1 \le i,j \le n,$$
3) there is $c \in l_0^\infty$, $c = \{c_\mu\}$, such that
$$|\,\partial_t \tilde{\varphi}(t,x) - F[\tilde{\varphi}](t,x)\,| \le c \quad \text{for} \quad (t,x) \in E,$$
and the maximal solution $\tilde{\omega} = \{\tilde{\omega}_\mu\}$ of the problem
$$\omega'(t) = \sigma(t, \omega(t)) + c, \quad \omega(0) = \mathbf{0}, \tag{4.5}$$
exists on $[0,a]$ and $\lim_{\mu \to \infty} \tilde{\omega}_\mu(t) = 0$ uniformly on $[0,a]$,
4) the estimates
$$|\,\beta_j(t,x) * \tilde{\varphi}(t,x) + \partial_{x_j} \tilde{\varphi}(t,x) - \psi_j(t,x)\,| \le \tilde{b} * \tilde{\omega}(t), \quad (t,x) \in \partial_{j.+}E,$$
$$|\,\beta_j(t,x) * \tilde{\varphi}(t,x) - \partial_{x_j} \tilde{\varphi}(t,x) - \psi_j(t,x)\,| \le \tilde{b} * \tilde{\omega}(t), \quad (t,x) \in \partial_{j.-}E,$$
are satisfied for $1 \le j \le n$.

Remark 4.1. If we assume that the function $\tilde{\varphi}$ satisfies the initial boundary conditions (1.2)–(1.4) and for each $\mu \in \mathbb{N}$ there are $\tilde{A}_\mu, \tilde{B}_\mu \in \mathbb{R}_+$ such that for $(t,x) \in E$

$$|f_\mu(t,x,\tilde{\varphi},\partial_x\tilde{\varphi}_\mu(t,x)\partial_{xx}\tilde{\varphi}_\mu(t,x))| \le \tilde{A}_\mu, \quad |\partial_t\tilde{\varphi}_\mu(t,x)| \le \tilde{B}_\mu$$

and $\lim_{\mu\to\infty} \tilde{A}_\mu = \lim_{\mu\to\infty} \tilde{B}_\mu = 0$, then the conditions 3) and 4) of Assumption $\mathrm{H}[\sigma,\varphi]$ are satisfied.

Remark 4.2. Let $a_{\mu j} \in \mathbb{R}_+$, $\mu, j \in \mathbb{N}$, be such that the series $S_\mu = \sum_{j=1}^{\infty} a_{\mu j}$, $\mu \in \mathbb{N}$, are convergent and the sequence $S = \{S_\mu\}$ tends to zero. Fix the sequence $\tilde{p} \in l_+^\infty$, $\tilde{p} = \{\tilde{p}_\mu\}$, such that $\tilde{p}_\mu > 0$ for $\mu \in \mathbb{N}$. Put $I[\tilde{p}] = \Big\{p \in l_+^\infty : p \le \tilde{p}\Big\}$. Then the function $\sigma : [0,a] \times l_+^\infty \to l_+^\infty$, $\sigma = \{\sigma_\mu\}$, given by

$$\sigma_\mu(t,p) = \sum_{j=1}^{\infty} a_{\mu j}\, p_j, \quad p \in I[\tilde{p}], \qquad \text{and} \qquad \sigma_\mu(t,p) = \sum_{j=1}^{\infty} a_{\mu j}\, \tilde{p}_j, \quad p \in l_+^\infty \setminus I[\tilde{p}],$$

where $t \in [0,a]$, $\mu \in \mathbb{N}$, satisfies the required conditions.

The following lemma will be useful in the sequel.

Lemma 4.3. *If Assumption* $\mathrm{H}[\sigma,\varphi]$ *is satisfied and the function* $v : E \to l^\infty$ *is* $\mathcal{P}$*-solution of* (1.1)–(1.4) *then*

$$|v(t,x) - \tilde{\varphi}(t,x)| \le \tilde{\omega}(t), \quad (t,x) \in E,$$

where $\tilde{\omega}$ *is the maximal solution of the problem* (4.5).

Proof. Define $\tilde{v} : E \to l^\infty$, $\tilde{v} = \{\tilde{v}_\mu\}$, by $\tilde{v} = v - \tilde{\varphi}$. Let the function $G = \{G_\mu\}$ be defined on $E \times C_{\tilde{A}}(E, l^\infty) \times \mathbb{R}^n \times M_{n\times n}$ in the following way

$$G_\mu(t,x,w,q,r) = f_\mu(t,x,w+\tilde{\varphi},q+\partial_x\tilde{\varphi}_\mu(t,x)r+\partial_{xx}\tilde{\varphi}_\mu(t,x)) - \partial_t\tilde{\varphi}_\mu(t,x),$$

where $\mu \in \mathbb{N}$ and $r = [r_{ij}]_{i,j=1,\ldots,n}$. Consider the infinite differential functional system

$$\partial_t z_\mu(t,x) = G_\mu(t,x,z,\partial_x z_\mu(t,x),\partial_{xx} z_\mu(t,x)), \quad \mu \in \mathbb{N}, \tag{4.6}$$

where $z = \{z_\mu\}$, $\partial_{xx} z_\mu = [\partial_{x_i x_j} z_\mu]_{i,j=1,\ldots,n}$. It follows that the function $\tilde{v}$ is a parabolic solution of (4.6) such that $\tilde{v}(t,x) = 0$ on E_0 and

$$|\,\beta_j(t,x) * \tilde{v}(t,x) + \partial_{x_j}\tilde{v}(t,x)\,| \le \tilde{b} * \tilde{\omega}(t) \quad \text{on} \quad \partial_{j.+}E,$$

$$|\,\beta_j(t,x) * \tilde{v}(t,x) - \partial_{x_j}\tilde{v}(t,x)\,| \le \tilde{b} * \tilde{\omega}(t) \quad \text{on} \quad \partial_{j.-}E,$$

where $1 \le j \le n$. The following estimate

$$\begin{aligned}
&|G_\mu(t,x,w,0,0)| \le \\
&\le |f_\mu(t,x,w+\tilde{\varphi},\partial_x\tilde{\varphi}_\mu(t,x),\partial_{xx}\tilde{\varphi}_\mu(t,x)) - f_\mu(t,x,\tilde{\varphi},\partial_x\tilde{\varphi}_\mu(t,x),\partial_{xx}\tilde{\varphi}_\mu(t,x))| + \\
&\quad + |F^{(\mu)}[\tilde{\varphi}] - \partial_t\tilde{\varphi}_\mu(t,x)| \le \\
&\le \sigma_\mu(t,|w|_t) + c_\mu
\end{aligned}$$

is satisfied for $(t,x) \in E$, $w \in C_{\tilde{A}}(E, l^\infty)$ and $\mu \in \mathbb{N}$. It follows from the comparison theorem (see [2]) that

$$|\tilde{v}(t,x)| \leq \tilde{\omega}(t) \quad \text{on} \quad E.$$

The proof is complete. □

For a function $w \in C(E, \mathbb{R}^k)$, $w = (w_1, \dots, w_k)$, we write $|w|_t = (|w_1|_t, \dots, |w_k|_t)$, $t \in [0,a]$. Put $C_{\tilde{A}}(E, \mathbb{R}^k) = \{w \in C(E, \mathbb{R}^k) : |w|_a \leq \tilde{A}\}$, where $\tilde{A} \in l^\infty$ is given in Assumption H$[\sigma, \varphi]$.

Lemma 4.4. *Suppose that Assumption* H$[\sigma, \varphi]$ *is satisfied and:*

1) *the function* $v : E \to l^\infty$, $v = \{v_\mu\}$, *is* $\mathcal{P}$*-solution of* (1.1)–(1.4),
2) *for each* $k \in \mathbb{N}$ *the function* $u^{[k]} : E \to \mathbb{R}^k$, $u^{[k]} = (u_1^{[k]}, \dots, u_k^{[k]})$, *is the parabolic solution of* (4.1)–(4.4).

Then there exists $\omega^{[k]} \in C([0,a], \mathbb{R}_+^k)$ *such that*

$$|v^{[k]}(t,x) - u^{[k]}(t,x)| \leq \omega^{[k]}(t), \quad (t,x) \in E,$$

and $\lim_{k\to\infty} \|\omega^{[k]}(t)\|_\infty = 0$ *uniformly on* $[0,a]$.

Proof. Let $k \in \mathbb{N}$ be fixed and let the function $\tilde{v}^{[k]} : E \to \mathbb{R}^k$ be given by $\tilde{v}^{[k]} = u^{[k]} - v^{[k]}$. We define the function $H : E \times C_{\tilde{A}}(E, \mathbb{R}^k) \times \mathbb{R}^n \times M_{n\times n} \to \mathbb{R}^k$, $H = (H_1, \dots, H_k)$, as follows:

$$H_\mu(t,x,w,q,r) =$$

$$= f_\mu(t,x,[w+v]_{k.\tilde{\varphi}}, q + \partial_x v_\mu(t,x), r + \partial_{xx} v_\mu(t,x)) - f_\mu(t,x,v,\partial_x v_\mu(t,x), \partial_{xx} v_\mu(t,x)).$$

Consider the differential functional system

$$\partial_t z_\mu(t,x) = H_\mu(t,x,z,\partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)), \quad 1 \leq \mu \leq k, \tag{4.7}$$

where $z = (z_1, \dots, z_k)$, with the homogeneous initial boundary conditions

$$z(t,x) = 0 \quad \text{on} \quad E_0, \tag{4.8}$$

$$\beta_j^{[k]}(t,x) * z(t,x) + \partial_{x_j} z(t,x) = 0 \quad \text{on} \quad \partial_{j.+}E, \tag{4.9}$$

$$\beta_j^{[k]}(t,x) * z(t,x) - \partial_{x_j} z(t,x) = 0 \quad \text{on} \quad \partial_{j.-}E, \tag{4.10}$$

where $1 \leq j \leq n$. The function $\tilde{v}^{[k]}$ is a parabolic solution of the problem (4.7)–(4.10). We use the comparison theorem for systems of differential functional equations to estimate the values of $\tilde{v}^{[k]}$.

We need the following additional notation. For $p \in l^\infty$, $p = \{p_\mu\}$, we denote by $\mathbf{0}_{k.p}$ the sequence $\{\bar{p}_\mu\}$ such that $\bar{p}_\mu = 0$ for $1 \leq \mu \leq k$ and $\bar{p}_\mu = p_\mu$ for $\mu > k$. Let $\sigma^{[k]} : [0,a] \times \mathbb{R}_+^k \to \mathbb{R}_+^k$, $\sigma^{[k]} = (\sigma_1^{[k]}, \dots, \sigma_k^{[k]})$, be given by

$$\sigma_\mu^{[k]}(t,p) = \sigma_\mu(t,p), \quad 1 \leq \mu \leq k. \tag{4.11}$$

We observe that

$$|f_\mu(t,x,[w+v]_{k.\tilde{\varphi}},\partial_x v_\mu(t,x),\partial_{xx} v_\mu(t,x)) - f_\mu(t,x,v,\partial_x v_\mu(t,x),\partial_{xx} v_\mu(t,x))| \le$$
$$\le \sigma_\mu^{[k]}(t,|w|_t) + \sigma_\mu(t,\mathbf{0}_{k.\tilde{\omega}(t)})$$

with $(t,x)\in E$, $w\in C_{\tilde{A}}(E,\mathbb{R}^k)$, $1\le\mu\le k$, where $\tilde{\omega}$ is the maximal solution of (4.5). Then

$$|H_\mu(t,x,w,0,0)| \le \sigma_\mu^{[k]}(t,|w|_t) + \alpha_\mu^{[k]}$$

with $\alpha_\mu^{[k]} = \sigma_\mu(a,\mathbf{0}_{k.\tilde{\omega}(a)})$, $1\le\mu\le k$. Write $\alpha^{[k]} = (\alpha_1^{[k]},\ldots,\alpha_k^{[k]})$. It follows that

$$|\tilde{v}^{[k]}(t,x)| \le \omega^{[k]}(t) \quad \text{on} \quad E,$$

where $\omega^{[k]}$ is the maximal solution of the problem

$$\omega'(t) = \sigma^{[k]}(t,\omega(t)) + \alpha^{[k]}, \quad \omega(0)=0. \tag{4.12}$$

Since $\lim_{k\to\infty}\|\alpha^{[k]}\|_\infty = 0$, we have that $\lim_{k\to\infty}\|\omega^{[k]}(t)\|_\infty = 0$ uniformly on $[0,a]$. This finishes the proof of Lemma 4.4. □

Now we construct the difference problem related to (4.1)–(4.4). For $z: E_h^+\to\mathbb{R}^k$, $z=(z_1,\ldots,z_k)$, we write

$$F_h^{[k]}[z] = (F_{h.1}^{[k]}[z],\ldots,F_{h.k}^{[k]}[z]),$$
$$F_{h.\mu}^{[k]}[z]^{(r,m)} = f_\mu(t^{(r)},x^{(m)},[\mathcal{T}_h z]_{k.\tilde{\varphi}},\delta z_\mu^{(r,m)},\delta^{(2)} z_\mu^{(r,m)})$$

on E'_h, $1\le\mu\le k$. For $(t^{(r)},x^{(m)})\in\partial_0 E_h$, $s\in S^{(m)}$ we put

$$g_h^{[k]}[z]^{(r,m,s)} = 2\sum_{j=1}^n s_j^2 h_j \psi_j^{[k]}(t^{(r)},x^{(m)}) - (z_\mu^{(r,m+s)} + z_\mu^{(r,m-s)}) * \sum_{j=1}^n s_j^2 h_j \beta_j^{[k]}(t^{(r)},x^{(m)}).$$

Consider the difference functional problem

$$\delta_0 z^{(r,m)} = F_h^{[k]}[z]^{(r,m)} \quad \text{on} \quad E'_h, \tag{4.13}$$

$$z^{(r,m)} = (\varphi_h^{[k]})^{(r,m)} \quad \text{on} \quad E_{0.h}, \tag{4.14}$$

$$z^{(r,m+s)} - z^{(r,m-s)} = g_h^{[k]}[z]^{(r,m,s)} \quad \text{on} \quad \partial_0 E_h, \quad s\in S^{(m)}. \tag{4.15}$$

We formulate the main theorem in this part of the paper.

Theorem 4.5. *Suppose that Assumptions* $\mathrm{H}[\sigma,\varphi]$, $\mathrm{H}[\mathcal{T}_\mathrm{h}]$, $\mathrm{H}_1[\Delta]$ *are satisfied, the function* $v: E\to l^\infty$ *is* $\mathcal{P}$*-solution of* (1.1)–(1.4) *and for each* $k\in\mathbb{N}$*:*

1) *the function* $u^{[k]}: E^+\to\mathbb{R}^k$ *is such that* $u^{[k]}(\cdot,x):[0,a]\to\mathbb{R}^k$, $x\in(-b^+,b^+)$, *is of class* C^1, $u^{[k]}(t,\cdot):(-b^+,b^+)\to\mathbb{R}^k$, $t\in[0,a]$, *is of class* C^3 *and there are* $c_0^{[k]},c_1^{[k]}\in\mathbb{R}_+^k$ *such that*

$$|\partial_{x_ix_j} u^{[k]}(t,x)| \le c_0^{[k]}, \quad |\partial_{x_ix_jx_k} u^{[k]}(t,x)| \le c_1^{[k]}, \quad (t,x)\in E^+, \quad 1\le i,j,k\le n,$$

and $u^{[k]}$ *is the parabolic solution of* (4.1)–(4.4) *on* E*,*

2) *the function* $u_h^{[k]} : E_h^+ \to \mathbb{R}^k$ *is the solution of* (4.13)–(4.15),
3) *there is* $\gamma_\varphi^{[k]} : \Delta \to \mathbb{R}_+^k$ *such that* $\lim_{h\to 0} \gamma_\varphi^{[k]}(h) = 0$ *and*

$$|(\varphi_h^{[k]})^{(r,m)} - \varphi^{[k]}(t^{(r)}, x^{(m)})| \le \gamma_\varphi^{[k]}(h) \quad on \quad E_{0.h}.$$

Then there exist $\gamma^{[k]} : \Delta \to \mathbb{R}_+^k$ *and* $\varepsilon^{[k]} \in \mathbb{R}_+^k$ *such that*

$$|(u_h^{[k]})^{(r,m)} - v^{[k]}(t^{(r)}, x^{(m)})| \le \gamma^{[k]}(h) + \varepsilon^{[k]} \quad on \quad E_h \tag{4.16}$$

and $\lim_{h\to 0} \gamma^{[k]}(h) = 0$, $\lim_{k\to\infty} \|\varepsilon^{[k]}\|_\infty = 0$.

Proof. Let us fix $k \in \mathbb{N}$. Using the methods from the proof of Theorem 3.1 we can prove that

$$|(u_h^{[k]})^{(r,m)} - u^{[k]}(t^{(r)}, x^{(m)})| \le \hat{\omega}_h^{[k]}(t^{(r)}) \quad \text{on} \quad E_h^+$$

where $\hat{\omega}_h^{[k]}$ is the maximal solution of the problem

$$\omega'(t) = \sigma^{[k]}(t, \omega(t)) + \tilde{\gamma}^{[k]}(h), \quad \omega(0) = \gamma_0^{[k]}(h),$$

with $\tilde{\gamma}^{[k]}, \gamma_0^{[k]} : \Delta \to \mathbb{R}_+^k$ satisfying condition $\lim_{h\to 0} \tilde{\gamma}^{[k]}(h) = \lim_{h\to 0} \gamma_0^{[k]}(h) = 0$ and with $\sigma^{[k]}$ given by (4.11). It follows from Lemma 4.4 that

$$|u^{[k]}(t^{(r)}, x^{(m)}) - v^{[k]}(t^{(r)}, x^{(m)})| \le \omega^{[k]}(t^{(r)}) \quad \text{on} \quad E_h$$

where $\omega^{[k]}$ is the maximal solution of (4.12). Thus we obtain the assertion (4.16) with $\gamma^{[k]}(h) = \omega_h^{[k]}(a)$ and $\varepsilon^{[k]} = \omega^{[k]}(a)$. □

5. NUMERICAL EXAMPLES

We consider two examples of functional differential infinite problems. All assumptions of Theorem 4.5 for these problems are satisfied and we show that numerical calculated error estimates are consistent with the theory.

Example 5.1. Let $E = [0, a] \times [-1, 1]^2$ with $a = 0.25$. Suppose that

$$\begin{aligned} f_\mu(t, x, w, q, r) = {} & \arctan(r_{11} + r_{22} - g(t, x) w_\mu(t, x_2, x_1)) + \\ & + (x_1^2 - 1)(x_2^2 - 1) w_\mu(t, x) + g_\mu(w(t, x)), \end{aligned}$$

where

$$g(t, x) = 4t^2 x_1^2 (x_2^2 - 1)^2 + 4t^2 x_2^2 (x_1^2 - 1)^2 + 2t x_1^2 + 2t x_2^2 - 4t,$$

$$g_1(p) = 0, \quad g_\mu(p) = p_{\mu+1} + p_{\mu-1} - 2\mu^6 \frac{\mu^4 + 10\mu^2 + 5}{(\mu^2 - 1)^5} p_\mu, \quad \mu > 1.$$

Consider the functional differential system

$$\partial_t z_\mu(t, x) = f_\mu(t, x, z, \partial_x z_\mu(t, x), \partial_{xx} z_\mu(t, x))$$

with the initial boundary conditions

$$z_\mu(t,x) = \mu^{-5} \quad \text{on} \quad E_0,$$

$$(1+4t(1-x_{3-j}^2))\, z_\mu(t,x) \pm 2\,\partial_{x_j} z_\mu(t,x) = \mu^{-5} \quad \text{on} \quad \partial_{j.\pm}E,$$

where $\mu \in \mathbb{N}$ and $j \in \{1,2\}$. The exact solution is $v_\mu(t,x) = \mu^{-5}\exp[t(x_1^2-1)(x_2^2-1)]$. We take $\tilde{\varphi}_\mu(t,x) = \mu^{-5}$ on E for $\mu \in \mathbb{N}$. Let $u_h^{[k]}$ be the solution of the difference method (4.13)–(4.15) with $\varphi_h = \varphi$. The following Table 1 shows maximal error values $\| e_h^{[k]} \|_\infty$ where $e_h^{[k]} = | u_h^{[k]} - v^{[k]} |_{(N_0)}$, for several steps $h = (h_0, h_1, h_2)$ and system sizes k.

Table 1

k	h_0	$h_1 = h_2$	$\| e_h^{[k]} \|_\infty$	$-\log_2 \| e_h^{[k]} \|_\infty$
4	2^{-7}	2^{-2}	$0.96312739241933 \cdot 10^{-3}$	10.01998574424679
8	2^{-9}	2^{-3}	$0.26850755315921 \cdot 10^{-3}$	11.86274970753216
16	2^{-11}	2^{-4}	$0.06917975692011 \cdot 10^{-3}$	13.81929052997669
32	2^{-13}	2^{-5}	$0.01741260986243 \cdot 10^{-3}$	15.80950801909845

Example 5.2. Let $E = [0,a] \times [-1,1]$ with $a = 0.25$. Suppose that

$$\begin{aligned} &f_\mu(t,x,w,q,r) = \\ &= \arctan\Big(r - \sum_{n=2}^{\mu+1} 4nta_n(t)b_{n-1}(x)[(4n+1)x^2-3]\Big) + \\ &+ \int_{0.5(-x-1)}^{0.5(-x+1)} (w_{\mu+1} - w_\mu)(t,s)\,ds + g_\mu(t,x), \end{aligned}$$

where

$$g_\mu(t,x) = -\frac{ta_{\mu+2}(t)}{2(2\mu+5)}\Big(\frac{3}{4}\Big)^{2\mu+5}\big(\gamma^{2\mu+5}(x) - \beta^{2\mu+5}(x)\big) + \sum_{n=2}^{\mu+1} 2na_n(t)b_n(x),$$

$$\beta(x) = 3x^2 - 1 + 2x \quad \text{for} \quad x \in \Big[-1, \frac{1}{3}\Big] \quad \text{and} \quad \beta(x) = 0 \quad \text{for} \quad x \in \Big(\frac{1}{3}, 1\Big],$$

$$\gamma(x) = 0 \quad \text{for} \quad x \in \Big[-1, -\frac{1}{3}\Big) \quad \text{and} \quad \beta(x) = 3x^2 - 1 - 2x \quad \text{for} \quad x \in \Big[-\frac{1}{3}, 1\Big],$$

$$a_n(t) = (-1)^n \frac{4^n - 4}{(2n)!}\, t^{2n-1}, \qquad b_n(x) = x(x^2-1)^{2n}, \qquad n \geq 2.$$

We consider the integral differential problem

$$\partial_t z_\mu(t,x) = f_\mu(t,x,z,\partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)) \quad \text{on} \quad E,$$

$$z_\mu(t,x) = 0 \quad \text{on} \quad E_0,$$
$$z_\mu(t,x) \pm \partial_{x_j} z_\mu(t,x) = 0 \quad \text{on} \quad \partial_{j.\pm}E,$$

where $\mu \in \mathbb{N}$ and $j \in \{1,2\}$.

The exact solution is $z_\mu(t,x) = \sum_{n=2}^{\mu+1} t\, a_n(t) b_n(x)$, $\mu \in \mathbb{N}$. We take, for $\mu \in \mathbb{N}$,

$$\tilde{\varphi}_\mu(t,x) = 8x \sin^4\left(\frac{1}{2}t(x^2-1)\right) \quad \text{on} \quad E \quad \text{and} \quad \tilde{\varphi}_\mu(t,x) = 0 \quad \text{on} \quad E_0 \cup \partial_0 E.$$

We apply the interpolating operator $T_h : \mathcal{F}(E_h^+, \mathbb{R}) \to C(E, \mathbb{R})$ given in [5]. Then the integrals are calculated by the use of trapezoidal rule. The following Table 2 shows maximal error values $\| e_h^{[k]} \|_\infty$, where $e_h^{[k]} = | u_h^{[k]} - v^{[k]} |_{(N_0)}$ and $u_h^{[k]}$ is the solution of (4.13)–(4.15) with $\varphi_h = \varphi$.

Table 2

k	h_0	h_1	$\| e_h^{[k]} \|_\infty$	$-\log_2 \| e_h^{[k]} \|_\infty$
4	2^{-6}	2^{-2}	$0.88806256460348 \cdot 10^{-4}$	13.45897915544746
8	2^{-8}	2^{-3}	$0.24163663270311 \cdot 10^{-4}$	15.33680128710483
16	2^{-10}	2^{-4}	$0.06391724973343 \cdot 10^{-4}$	17.25536323679841
32	2^{-12}	2^{-5}	$0.01597374728806 \cdot 10^{-4}$	19.25586577443758

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