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Dedicated to the Memory of Professor Zdzisław Kamont

CLASSICAL SOLUTIONS OF MIXED PROBLEMS FOR QUASILINEAR FIRST ORDER PFDEs ON A CYLINDRICAL DOMAIN

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Abstract. We abandon the setting of the domain as a Cartesian product of real intervals, customary for first order PFDEs (partial functional differential equations) with initial boundary conditions. We give a new set of conditions on the possibly unbounded domain Ω with Lipschitz differentiable boundary. Well-posedness is then reliant on a variant of the normal vector condition. There is a neighbourhood of $\partial\Omega$ with the property that if a characteristic trajectory has a point therein, then its every earlier point lies there as well. With local assumptions on coefficients and on the free term, we prove existence and Lipschitz dependence on data of classical solutions on $(0,c)\times\Omega$ to the initial boundary value problem, for small c. Regularity of solutions matches this domain, and the proof uses the Banach fixed-point theorem. Our general model of functional dependence covers problems with deviating arguments and integro-differential equations.

Keywords: partial functional differential equations, classical solutions, local existence, characteristics, cylindrical domain, a priori estimates.

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1. INTRODUCTION

We consider the functional differential problem:

$$\partial_t z(t,x) + \sum_{j=1}^{n} \rho_j(t,x,z_{\alpha(t,x)})\,\partial_{x_j} z(t,x) = G(t,x,z_{\alpha(t,x)}) \quad \text{on} \quad E, \tag{1.1}$$

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E, \tag{1.2}$$

where $E=(0,a)\times\Omega$, $a>0$, $E_0=[-b_0,0]\times\overline{\Omega}$, $b_0\geq 0$, and $\partial_0 E$ stands for $(0,a]\times\partial\Omega$. The domain $\Omega\subset\mathbb{R}^n$ has C^1 boundary. Given functions G, ρ_j, $j=1,\ldots,n$, are defined on $E\times X$, with X – the set of bounded uniformly continuous real functions on $\mathscr{D} = [-b_0 - a, 0] \times \{p - q : p \in \overline{\Omega}, q \in \overline{\Omega}\}$. The deviating function $\alpha : E \to E$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, with $\alpha_0(t,x) \leq t$, allows for (a wide class of) delays in (1.1), as just one of a particular case.

We define the Hale operator $E \ni (t,x) \mapsto z_{(t,x)} \in X$ by $z_{(t,x)}(\tau, y) = \tilde{z}(t+\tau, x+y)$, where $\tilde{z}$ is a continuous extension of $z\colon E_0 \cup \partial_0 E \cup E \to \mathbb{R}$ onto $\mathbb{R}^{1+n}$. To make this definition usable, we require

Condition 1.1. For any $(t,x) \in E$ and any w, $\bar{w} \in X$,

$$w \equiv \bar{w} \quad \text{on} \quad (\mathscr{D} + \alpha(t,x)) \cap (E_0 \cup \partial_0 E \cup E)$$

implies

$$\begin{cases} G(t,x,w) = G(t,x,\bar{w}), \\ \rho_j(t,x,w) = \rho_j(t,x,\bar{w}), \quad j = 1, \ldots, n. \end{cases}$$

Thanks to the above condition, the original values of z (and not $\tilde{z}$) are sufficient for the unique definition of $G(t,x,z_{\alpha(t,x)})$, $\rho_j(t,x,z_{\alpha(t,x)})$. This construction is due to [25]. Note that if the Hale operator had been defined in a standard way ($z_{(t,x)}(s,y) = z(t+s, x+y)$, $(t,x) \in E$, $(s,y) \in \mathscr{D}$, $\mathscr{D}$ fixed), then it would have required z to be given on the algebraic sum $E + \mathscr{D} = \{p + q : p \in E, q \in \mathscr{D}\}$. Such a type ($E + \mathscr{D}$) of the domain of z is adopted in many papers on PFDEs. In many interesting cases of functional dependence, the model requires $\partial_0 E$ to be larger than in our setting.

For convenience, let E_c stand for $E \cap ((-\infty, c) \times \mathbb{R}^n)$, and let $\partial_0 E_c = \partial_0 E \cap ((-\infty, c) \times \mathbb{R}^n)$, where $0 < c \leq a$. Additionally, we will denote by E_c^* the sum $E_0 \cup \partial_0 E_c \cup E_c$.

We will discuss the question of local existence and continuous dependence on the initial boundary data of classical $C^{1.L}$ solutions to (1.1), (1.2). A function $\tilde{z}\colon E_c^* \to \mathbb{R}$ is a classical $C^{1.L}$ solution of (1.1), (1.2), if it is bounded and has Lipschitz continuous derivative everywhere in the domain, and if it satisfies (1.1) on E_c and (1.2) on $E_0 \cup \partial_0 E_c$. Assumed regularity is natural in the case of x-dependent α_0, as we have pointed out in [9].

Our existence result can be treated as a continuation of the author's study [10]. In the latter, we have chosen the domain to be a Cartesian product of real intervals; this was common to all other works on mixed problems for hyperbolic PFDEs, with the only exception being our recent paper [11]. The important point to note here is that taking a cylindrical domain instead, one has to ensure the well-posedness of the problem. Our aim is therefore to find fairly modest conditions on the regularity of Ω, taking into account its possible unboundedness. The task is additionally complicated by the regularity requirements on the (defined later) left-end of characteristic, which lies at $\partial\Omega$. Unfortunately, the cone condition variant formulated in [11], is not well adapted to $C^{1.L}$ solutions.

Let us mention various larger solution classes for first order partial functional differential problems. First results on C^1 solutions were obtained in [3, 21] by means of the method of successive approximations. This method is due to T. Ważewski, who introduced it for systems without functional dependence in [28]. In addition to classical solutions, the following classes of generalised solutions to hyperbolic PFDEs are present in the literature. Mixed problems for almost linear systems in two independent variables were treated in [24], see also [19]. A continuous function is a solution of a mixed problem if it satisfies an integral functional system, which arises from the functional differential system by integrating along characteristics. The paper [24] initiated investigations of first order PFDEs. Distributional solutions of almost linear problems were considered in [26]. The method used in this paper is constructive; it is based on a difference scheme.

The class of Carathéodory solutions consists of all functions which are continuous and have their partial derivatives almost everywhere in a domain. The set of all points where the differential functional equation is not fulfilled is of Lebesgue measure zero. The existence and uniqueness results for quasilinear systems with initial or initial-boundary conditions, in the class of Carathéodory solutions, can be found in [13, 27]. Initial problems for non-linear equations were considered in [14].

An essential extension of some ideas concerning classical solutions of first order PFDEs is given in [3, 4], where the Cinquini Cibrario solutions are considered. This class of solutions is placed between classical solutions and solutions in the Carathéodory sense. Its name stems from S. Cinquini and M. Cinquini Cibrario, who introduced and widely studied the method of characteristics for quasilinear problems in a nonfunctional setting, see [6–8].

The monograph [20] contains an exposition of results on existence and uniqueness of generalised and classical solutions to hyperbolic functional differential equations.

First order partial differential equations with deviating variables and differential integral equations find applications in different fields of knowledge. We give a few examples.

In the theory of the distribution of wealth, the density of households at time t depending on their estates is governed by an equation with deviating variables; for details see [16].

As remarked in [2], there are various problems in non-linear optics which lead to hyperbolic integro-differential problems. One of such physical phenomena is the harmonic generation of laser radiation through piezoelectric crystals for non dispersive materials and of the Maxwell-Hopkinson type. This non-linear problem is modelled in [2] by an equation perturbed by dissipative integral terms of Volterra type.

The following problems in population dynamics have mathematical models involving hyperbolic functional differential equations. Age dependent epidemics of vertically transmitted diseases are driven, as investigated in [17], by a non-linear functional differential system. Non-linear equations describe also the growth of a population of cells which constantly differentiate (change their properties) in time; for example, a model of the production of erythrocytes based on a continuous maturation-proliferation scheme is developed in [22]. A more simple, almost linear problem is considered in [5] as a description of another proliferating cell population dynamics. The paper [18] discusses optimal harvesting policies for age-structured population harvested with effort independent of age.

The class of problems we consider, appears also in the non-linear theory describing the motion of viscoelastic media, see [23]. For further bibliography on applications of PFDEs see the monographs [20, 29].

2. WELL-POSEDNESS AND CERTAIN NEIGHBORHOOD OF $\partial\Omega$

2.1. DOMAIN AND DATA REGULARITY

For $U \subset \mathbb{R}^{1+n}$ and a normed space $(Y, \|\cdot\|_Y)$, we define $C^m(\overline{U}, Y)$ to be the set of Y-valued functions, which have on U bounded and uniformly continuous partial derivatives up to the order $m \geq 0$; with the usual meaning $C^0 = C$. We write it simply $C^m(\overline{U})$, if no confusion can arise.

By a convenient abuse of notation, we will use the symbol $|\cdot|$ for the Euclidean norm in $\mathbb{R}^n$, mostly in our geometrical considerations; this norm lies under the notion of distance $\operatorname{dist}(\cdot,\cdot)$ between a point and a set, each time we use it. For k, l being arbitrary positive integers, we denote by $M_{k\times l}$ the class of all $k \times l$ matrices with real elements, and we choose the norms in $\mathbb{R}^k$ and $M_{k\times l}$ to be ∞-norms: $\|y\| = \|y\|_\infty = \max_{1\leq i\leq k} |y|$ and $\|A\| = \|A\|_\infty = \max_{1\leq i\leq k} \sum_{j=1}^{l} |a_{ij}|$, respectively, where $A = [a_{ij}]_{i=1,\dots,k, j=1,\dots,l}$. The product of two matrices is denoted by "$*$".

We assume that the domain Ω satisfies a uniform $C^{1.L}$-regularity condition, which is a variant of the one from [1], concerning C^1-regularity. Let Φ be a one-to-one transformation of a domain $U \subset \mathbb{R}^n$ onto a domain $V \subset \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We say that Φ is 1-*smooth* if all components of Φ and Ψ belong to $C^1(\overline{U})$ and to $C^1(\overline{V})$, respectively. We denote by Ω_δ the set of points in Ω within distance δ of the boundary of Ω:

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}\,.$$

Assumption H[Ω]**.** There exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and a corresponding sequence $\{\Phi_j\}$ of 1-smooth transformations, with Φ_j taking U_j onto the unit ball $B = \{y \in \mathbb{R}^n : |y| < 1\}$ and having inverse $\Psi_j = \Phi_j^{-1}$, such that:

(i) For some $\delta > 0$, $\Omega_\delta \subset \bigcup_{j=1}^{\infty} \Psi_j(\{y \in \mathbb{R}^n : |y| < 1/2\})$.

(ii) For each j, $\Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$.

(iii) There is a finite constant $\widetilde{M}$ such that for every j

$$\|\partial_x \Phi_j(x)\| \leq \widetilde{M} \quad \text{for} \quad x \in U_j,$$

$$\|\partial_y \Psi_j(y)\| \leq \widetilde{M} \quad \text{for} \quad y \in B,$$

that is, the norms of Jacobi matrices for Φ_j and Ψ_j are uniformly bounded by $\widetilde{M}$.

(iv) There is a finite constant $\widetilde{L}$ such that for any j, $x, \bar{x} \in U_j$, $1 \leq i \leq n-1$, there is $\|\partial_x \phi_{ji}(x) - \partial_x \phi_{ji}(\bar{x})\| \leq \widetilde{L}\|x - \bar{x}\|$.

We are now in a position to formulate the regularity condition for data φ. To this end, define, for each j, a function

$$F_j\colon (0,a)\times\left\{(y_1,\dots,y_{n-1})\in\mathbb{R}^{n-1} : (y_1,\dots,y_{n-1},0)\in B\right\}\to\mathbb{R}$$

by

$$F_j(t,y_1,\dots,y_{n-1})=\varphi(t,\Psi_j(y_1,\dots,y_{n-1},0)). \tag{2.1}$$

Assumption H[φ]**.** The function $\varphi : E_0\cup\partial_0 E\to\mathbb{R}$ is bounded by a constant p_0, and Lipschitz-continuously differentiable in the following sense:

1. If $b_0>0$ then φ is differentiable on $(-b_0,0)\times\Omega$.
2. Function $\varphi(0,\cdot)$ is differentiable on Ω.
3. Each F_j is differentiable on $(0,a)\times\{y\in B : y_n=0\}$.
4. In the above cases, the corresponding partial derivatives are bounded by p_1/n, and Lipschitz continuous, with the constant $p_2/(n+1)$; both constants are uniform in j.
5. Partial derivatives with respect to $y_1,\dots,y_{n-1}$ of $\varphi(0,\Psi_j(y))$ are continuous on $\{y\in B : y_n\ge 0\}$, for each j.

2.2. INTEGRAL FUNCTIONAL SYSTEM AND A SOLUTION ESTIMATE

Let U be an open subset of $\mathbb{R}^{1+n}$. For $z\colon U\to\mathbb{R}$ and $(t,x)\in U$, denote

$$\partial_x z(t,x)=(\partial_{x_1}z(t,x),\dots,\partial_{x_n}z(t,x))$$

and

$$Dz(t,x)=(\partial_t z(t,x),\partial_{x_1}z(t,x),\dots,\partial_{x_n}z(t,x)).$$

Similarly, for $z\colon U\to\mathbb{R}^n$, $z=(z_1,\dots,z_n)$, and $(t,x)\in U$, denote

$$\partial_x z(t,x)=[\partial_{x_j}z_i(t,x)]_{i,j=1,\dots,n}\quad\text{and}\quad Dz(t,x)=[(Dz_1(t,x))^T,\dots,(Dz_n(t,x))^T]^T.$$

Write, for $z\colon U\to Y$, $\|z\|_{C(U,Y)}$ for the usual supremum norm, and

$$\begin{aligned}
|z|_{C^{0.L}(U,Y)} &= \sup\left\{\frac{\|z(t,x)-z(t,\bar{x})\|_Y}{\max\{|t-\bar{t}|,\|x-\bar{x}\|\}} : (t,x),(\bar{t},\bar{x})\in U,(t,x)\neq(\bar{t},\bar{x})\right\},\\
\|z\|_{C^1(U,Y)} &= \|z\|_{C(U,Y)}+\|Dz\|_{C(U,Y^{1+n})},\\
\|z\|_{C^{1.L}(U,Y)} &= \|z\|_{C^1(U,Y)}+|Dz|_{C^{0.L}(U,Y^{1+n})},\\
\|\cdot\|_0 &= \|\cdot\|_{C(\mathscr{D},\mathbb{R})}\quad\text{or}\quad \|\cdot\|_0=\|\cdot\|_{C(\mathscr{D},\mathbb{R}^{1+n})},\\
\|\cdot\|_1 &= \|\cdot\|_{C^1(\mathscr{D},\mathbb{R})},\\
|\cdot|_L &= |\cdot|_{C^{0.L}(\mathscr{D},\mathbb{R}^{1+n})}.
\end{aligned}$$

The symbol $C^{1.L}(U,Y)$ stands for the set of all $z\in C^1(\overline{U})$ with $\|z\|_{C^{1.L}(U,Y)}<\infty$.

Let us introduce the notation for the function space, where we seek a solution to (1.1), (1.2). Given φ satisfying Assumption H[φ], we set

$$C^{1.L}_{\varphi.c}=\left\{z\in C^{1.L}(E^*_c,\mathbb{R}) : z\equiv\varphi\ \text{ on }\ E_0\cup\partial_0 E_c\right\}.$$

As our main result, we aim to prove that, under suitable assumptions on $\rho = (\rho_1, \ldots, \rho_n)^T$, G, α, φ and for sufficiently small $c \in (0, a]$, there exists a solution $\bar{z}$ of problem (1.1), (1.2) such that $\bar{z} \in C^{1.L}_{\varphi.c}$.

Suppose that φ satisfies Assumption H$[\varphi]$ and $z \in C^{1.L}_{\varphi.c}$. For a point $(t, x) \in E_c$, we consider the Cauchy problem

$$\eta'(\tau) = \rho(\tau, \eta(\tau), z_{\alpha(\tau,\eta(\tau))}), \quad \eta(t) = x, \tag{2.2}$$

and denote by $g[z](\cdot, t, x) = (g_1[z](\cdot, t, x), \ldots, g_n[z](\cdot, t, x))^T$ its classical solution. This function is the characteristic of the equation (1.1), corresponding to z. Let $\delta[z](t, x)$ be the left-end of the maximal interval on which the characteristic $g[z](\cdot, t, x)$ is defined (more briefly: the left-end of characteristic). Write

$$Q[z](\tau, t, x) = (\tau, g[z](\tau, t, x), z_{\alpha(\tau, g[z](\tau,t,x))}),$$
$$S[z](t, x) = (\ \delta[z](t, x),\ g[z](\delta[z](t, x), t, x)\ ),$$

and

$$\mathbb{F}z(t, x) = \begin{cases} \varphi(S[z](t, x)) + \int\limits_{\delta[z](t,x)}^{t} G(Q[z](s, t, x))ds, & (t, x) \in E_c, \\ \varphi(t, x), & (t, x) \in E_0 \cup \partial_0 E_c. \end{cases}$$

Note that the characteristic $g[z]$ satisfies the integral equation

$$g[z](\tau, t, x) = x + \int_t^{\tau} \rho(Q[z](s, t, x))ds \quad \text{for} \quad \tau \in [\delta[z](t, x),\, t], \quad (t, x) \in E_c. \tag{2.3}$$

We consider the functional integral system, consisting of (2.3) and

$$z(t, x) = \mathbb{F}z(t, x), \qquad (t, x) \in E_c^*. \tag{2.4}$$

The right-hand side of (2.4) is obtained as in [10, Section 3].

Assumption H$_0[G]$**.** The function $G\colon E \times X \to \mathbb{R}$ is continuous, and it has at most sub-linear growth in w: there are K_G, A_G, such that

$$|G(t, x, w)| \le A_G + K_G \|w\|_0 \quad \text{on} \quad E \times X.$$

Application of Gronwall's inequality to (2.4) leads to the following a priori estimate.

Lemma 2.1. *Suppose that Assumptions* H$_0[G]$, H$[\varphi]$ *hold. If* $\bar{z}\colon E_c^* \to \mathbb{R}$ *is bounded, and if* $\bar{z}$, $g[\bar{z}]$ *(the set of characteristics corresponding to* $\bar{z}$*) satisfy* (2.3), (2.4)*, then* $|\bar{z}(t, x)| \le \mu(t)$ *on* E_c*, where*

$$\mu(t) = (p_0 + A_G t)\exp(K_G t). \tag{2.5}$$

Note that μ is independent of c, $0 < c \le a$. In the sequel, we will localise our assumptions on ρ and G using the following notation:

$$C^{1.L}_{\varphi.c}[\mu] = \{z \in C^{1.L}_{\varphi.c} : |z(t,x)| \le \mu(t) \quad \text{for} \quad (t,x) \in E_c\}$$

and

$$X_0 = \{w \in C^{1.L}(\mathscr{D}, \mathbb{R}) : \|w\|_0 \le \mu(a)\}.$$

2.3. PROPERTIES OF CHARACTERISTICS NEAR BOUNDARY

We are now able to state the main results of this section.

Assumption $\mathrm{H}_0[\rho]$**.** The function $\rho\colon E \times X \to \mathbb{R}^n$ is, on $E \times X_0$, continuous and bounded uniformly by K. Moreover, there is $\kappa > 0$ such that for every j

$$\sum_{k=1}^{n} \partial_{x_k}\phi_{jn}(x)\rho_k(t,x,w) \ge \kappa \tag{2.6}$$

for $t \in [0,a]$, $x \in U_j \cap \Omega$, and for $w \in X_0$, where $(\phi_{j1}, \ldots, \phi_{jn})$ are the components of Φ_j.

Assumption $\mathrm{H}_0[\alpha]$**.** There is r_1 such that, for $(t,x) \in E$,

$$|\alpha_0(t,x) - \alpha_0(\bar{t},\bar{x})| + \|\alpha(t,x) - \alpha(\bar{t},\bar{x})\| \le r_1 \max\{|t-\bar{t}|, \|x-\bar{x}\|\}.$$

Note that (2.6) is a variant of the normal vector condition. It is important for the well-posedness of the initial-boundary value problem (1.1), (1.2), established by our next lemma.

Lemma 2.2. *Suppose that Assumptions* $\mathrm{H}[\varphi]$, $\mathrm{H}[\Omega]$, $\mathrm{H}_0[G]$, $\mathrm{H}_0[\rho]$, $\mathrm{H}_0[\alpha]$ *are satisfied. Given* $z \in C^{1.L}_{\varphi.c}[\mu]$*, there exists a solution* $g[z](\cdot,t,x)$ *of* (2.2) *on* $[\delta[z](t,x), c]$*, and if* $\xi = \delta[z](t,x) > 0$ *then* $g[z](\xi,t,x) \in \partial\Omega$*. Moreover, every characteristic touches* $\partial\Omega$ *no more than once, that is,*

$$g[z](\tau,t,x) \in \Omega \quad \textit{for} \quad \tau \in (\delta[z](t,x), c]. \tag{2.7}$$

Proof. The existence, up to the boundary, of solutions to (2.2) follows from the theorem on classical solutions of initial problems. We prove (2.7). Conversely, suppose that for some $(t,x) \in E_c$ and for some $\tilde{a}$, $\delta[z](t,x) < \tilde{a} \le c$, we have

$$g[z](\tau,t,x) \in \Omega \quad \text{for} \quad \tau \in (\delta[z](t,x), \tilde{a}) \quad \text{and} \quad g[z](\tilde{a},t,x) \in \partial\Omega.$$

Recall that, by Assumption $\mathrm{H}[\Omega]$, there is j such that $g[z](\tilde{a},t,x) \in U_j$. Since U_j is open and $g[z](\cdot,t,x)$ is continuous, there is also $\varepsilon > 0$ such that

$$g[z](\tau,t,x) \in U_j \cap \Omega \quad \text{for} \quad \tau \in [\tilde{a}-\varepsilon, \tilde{a}).$$

By the chain rule,

$$\phi_{jn}(g[z](\tilde{a},t,x)) - \phi_{jn}(g[z](\tilde{a}-\varepsilon,t,x)) = \int_{\tilde{a}-\varepsilon}^{\tilde{a}} \frac{d}{d\tau}\big[\phi_{jn}(g[z](\tau,t,x))\big]d\tau =$$

$$= \int_{\tilde{a}-\varepsilon}^{\tilde{a}} \partial_x \phi_{jn}(g[z](\tau,t,x)) * \rho(\tau, g[z](\tau,t,x), z_{\alpha(\tau, g[z](\tau,t,x))})d\tau.$$

According to the condition (2.6) of Assumption $H_0[\rho]$, the integrand is positive. Moreover, the number $\phi_{jn}(g[z](\tilde{a},t,x))$ is zero because the n-th component of the image of $\partial\Omega$ through Φ_j (where $g[z](\tilde{a},t,x)$ belongs) vanishes – see Assumption H[Ω], condition (ii). Hence the number $\phi_{jn}(g[z](\tilde{a}-\varepsilon,t,x))$ is negative, which contradicts the condition (ii), just mentioned. □

We now present a fact concerning the behaviour of characteristics near the boundary of Ω. This result is important for showing Lipschitz continuity of $\delta[z](t,x)$ in z and in (t,x).

Lemma 2.3. *Under the same hypotheses, there is $\tilde{\delta}$, $0 < \tilde{\delta} \le \delta$, such that the region $\Omega_{\tilde{\delta}}$ with the property that if a characteristic trajectory has a point therein, then every earlier point lies there as well. Precisely, if $x \in \Omega_{\tilde{\delta}}$, then for any j such that $|\Phi_j(x)| < 1/2$ (such j exists by the Assumption* H[Ω]*), and for any $t \in [0,c]$, $z \in C^{1.L}_{\varphi.c}[\mu]$, $\varphi \in C^{1.L}_{\partial}[p]$,*

$$g[z](\tau,t,x) \in U_j \quad \textit{for} \quad \tau \in [\delta[z](t,x), t].$$

Proof. The proof is divided into two steps. First, we show that for a subset of Ω_δ, any point lying in it belongs to the same U_j as its certain companion point lying in $\partial\Omega$. In the second step, the claimed proximity $\tilde{\delta}$ of $\partial\Omega$ is found.

Step I. Recall that by B we denote the unit ball in $(\mathbb{R}^n, |\cdot|)$. Fix $x \in \Omega_\delta$; by Assumption H[Ω], there is $j = j(x)$ such that $|\Phi_j(x)| < 1/2$. We will show that

$$\Big\{ y \in \mathbb{R}^n : |y - x| < 1/(4n\widetilde{M}) \Big\} \subset U_{j(x)}. \tag{2.8}$$

Suppose the above is not true; take $y \in \mathbb{R}^n \setminus U_j$, such that $|y - x| < 1/(4n\widetilde{M})$. Let $y^* \in \partial U_j$ be the realisation of the distance $\mathrm{dist}(y, U_j)$. Note that Φ_j and Ψ_j admit Lipschitz continuous extensions onto $\overline{U_j}$ and $\overline{B}$, respectively; obviously the extensions map between ∂U_j and ∂B. Denote the extension of Φ_j by the same symbol, for simplicity. By the definition of y^*, $|y - y^*| \le |y - x|$, so the triangle inequality yields $2|y - x| \ge |x - y^*|$, and hence

$$\frac{1}{2n\widetilde{M}} > 2|y - x| \ge |x - y^*| \ge \frac{1}{n\widetilde{M}} |\Phi_j(x) - \Phi_j(y^*)| \ge \frac{1}{n\widetilde{M}} \cdot \frac{1}{2},$$

a contradiction.

Having shown (2.8), our aim is now to find $\Omega_\gamma \subset \Omega_\delta$, such $b(x) \in U_{j(x)}$ for any $x \in \Omega_\gamma$, where $b(x) \in \partial\Omega$ is the realisation of the distance $d = \operatorname{dist}(x, \partial\Omega)$ (it exists by precompactness of any bounded subset of $\partial\Omega$).

Define $\gamma = \min\{\delta, 1/(4n\widetilde{M})\}$ and let $d < \gamma$ be the distance just mentioned. The point $b(x)$ may be approached from within Ω. Take $z \in \Omega$ close enough to $b(x)$, $|b(x) - z| \le (\gamma - d)/2$, so that

$$|x - z| \le |x - b(x)| + |b(x) - z| = d + |b(x) - z| \le (d + \gamma)/2 < \gamma,$$

and, by (2.8), x, $z \in U_{j(x)} \cap \Omega$. Hence, with $j = j(x)$, $|\Phi_j(x) - \Phi_j(z)| \le n\widetilde{M}|x - z| \le n\widetilde{M}(d + \gamma)/2$. Denoting the last bound by q, we may write

$$|\Phi_j(z)| \le |\Phi_j(x)| + |\Phi_j(x) - \Phi_j(z)| \le \frac{1}{2} + q < 1.$$

Repeating the above argument with z_n in place of z, such that $\{z_n\}$ converges to $b(x)$, and then passing to the limit, we obtain $|\Phi_j(b(x))| < 1$, and hence $b(x) \in U_{j(x)}$.

Step II. Let $\tilde{\delta} = \min\{\delta, 1/(4n\widetilde{M}), \kappa/(2n\widetilde{M}^2 K)\}$, and $x \in \Omega_{\tilde{\delta}}$, $t \in [0, c]$, $z \in C^{1.L}_{\varphi.c}[\mu]$. By what we have proved in step I, we may write, for $j = j(x)$,

$$\phi_{jn}(x) = |\phi_{jn}(x) - \phi_{jn}(b(x))| \le \widetilde{M}\|x - b(x)\| < \frac{\kappa}{2n\widetilde{M}K}. \tag{2.9}$$

Define $\Gamma(\tau) = \Phi_j(g[z](\tau, t, x))$, $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$. The proof is completed by showing that the curve $\{\Gamma(\tau), \tau < t\}$ cannot escape a finite cone, wholly included in B.

Thanks to the differentiability of Γ, if only $t_1 < t_2$ have the property

$$g[z](\tau, t, x) \in U_j \quad \text{for} \quad \tau \in [t_1, t_2], \tag{2.10}$$

then

$$|\Gamma(t_2) - \Gamma(t_1)| \le \int_{t_1}^{t_2} \left| \frac{d}{d\tau} \Phi_j(g[z](\tau, t, x)) \right| d\tau \le n\widetilde{M}K(t_2 - t_1),$$

$$\Gamma_n(t_2) - \Gamma_n(t_1) = \int_{t_1}^{t_2} \frac{d}{d\tau} \phi_{jn}(g[z](\tau, t, x)) d\tau \ge \kappa(t_2 - t_1).$$

Dividing these inequalities yields

$$\Gamma(t_2) - \Gamma(t_1) \in C = \left\{ y \in \mathbb{R}^n : |y| \le \frac{n\widetilde{M}K}{\kappa} y_n \right\},$$

whenever (2.10) holds. Recall that $g[z](t, t, x) = x$ (it is just the initial condition for a characteristic), and hence $\Gamma(t) = \Phi_j(x)$. Let W denote the closed set

$$(\Gamma(t) - C) \cap \{y \in \mathbb{R}^n : y_n \ge 0\},$$

where $\Gamma(t) - C$ means an algebraic difference. We claim that the region W is included in B. Indeed, since inequality (2.9) reads $\frac{n\widetilde{M}K}{\kappa}\Gamma_n(t) < \frac{1}{2}$, and $j = j(x)$ is so chosen that $|\Gamma(t)| < 1/2$, an easy calculation involving the definition of C and possibly supported by Figure 1, proves this inclusion.

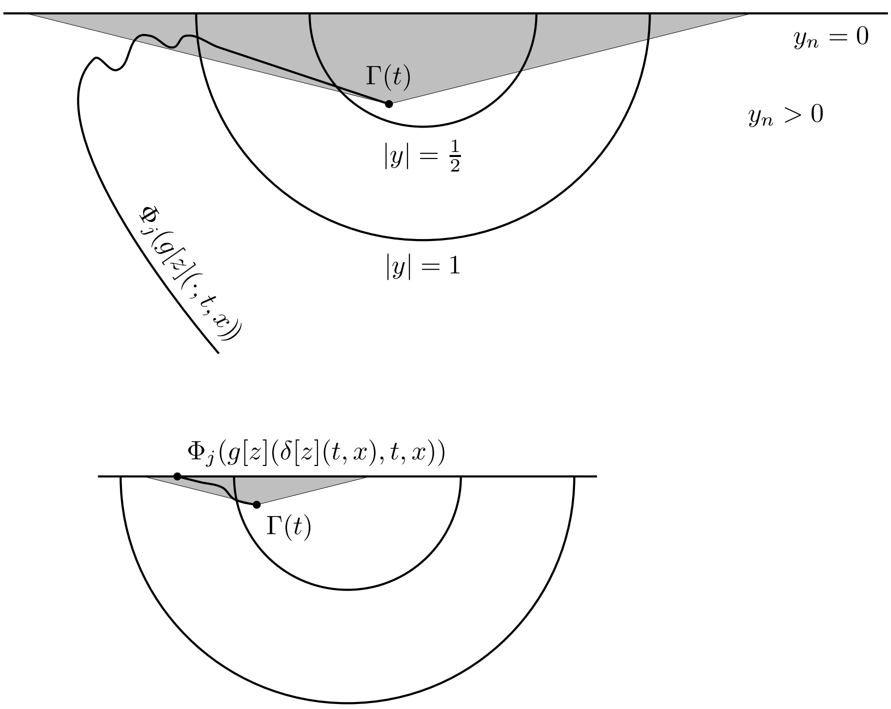

Fig. 1. Cone $\Gamma(t)+C$ so placed, that the characteristic may escape from U_j (upper sketch); cone $\Gamma(t) + C$, $\Gamma(t) = (\gamma_1, \ldots, \gamma_n)$, with γ_n small enough to keep the characteristic inside U_j, all way long to the left, up to $\delta[z](t,x)$ (lower sketch). In both cases, the gray set is W.

Now, suppose that $g[z](\tau^*, t, x) \notin U_j$ for some $\tau^* \in [\delta[z](t,x), t)$, meaning that Γ is not defined at τ^*. But $\Gamma(t) \in W$, and W is well-separated from ∂B. Thus there exists $t_1 \in (\tau^*, t)$, satisfying

$$\Gamma(t_1) \in B \setminus W \qquad \text{and} \qquad \Gamma(\tau) \in B \quad \text{for} \quad \tau \in [t_1, t].$$

Clearly, $\Gamma(t_1) \notin W$ and $\Gamma_n(t_1) \geq 0$. Concluding, for $t_2 = t$, we have (2.10) together with the relation $\Gamma(t_2) - \Gamma(t_1) \notin C$, which is impossible. $\square$

3. REGULARITY OF CHARACTERISTICS

Assumption H$[\rho]$**.** Assumption $\mathrm{H}_0[\rho]$ is satisfied and, for $(t,x) \in E$, $w \in X_0$, the derivative $\partial_x\rho(t,x,w)$ and the Fréchet derivative $\partial_w\rho(t,x,w)$ exists. Furthermore, there are M, L, such that for $(t,x) \in E$, $w \in X_0$, and for any Lipschitz continuous $h\colon \mathscr{D} \to \mathbb{R}$,

1. the derivatives are bounded, in the sense that
$$\|\partial_x\rho(t,x,w)\| \le M(1+\|Dw\|_0), \qquad \|\partial_w\rho(t,x,w)h\| \le M\|h\|_0,$$
2. the Fréchet derivative is Lipschitz continuous in w: for any $\bar{w} \in X_0$,
$$\|\partial_w\rho(t,x,w)h - \partial_w\rho(t,x,\bar{w})h\| \le L\|h\|_0\,\|w-\bar{w}\|_0,$$
3. with respect to the topology induced by the norm $\|\cdot\|_{C^1(\mathscr{D},\mathbb{R})}$, $\partial_x\rho(t,x,w)$ is continuous in w,
4. $\partial_x\rho(t,x,w)$, $\partial_w\rho(t,x,w)h$, are continuous in (t,x).

Assumption H$[\alpha]$**.** The function α satisfies Assumption $\mathrm{H}_0[\alpha]$ and is differentiable on E, $D\alpha$ is continuous, and there is r_2 such that
$$\|D\alpha(t,x) - D\alpha(t,\bar{x})\| \le r_2\|x-\bar{x}\|$$
for (t,x), $(t,\bar{x}) \in E$.

Define, for $u \in C(\overline{E_c^*},\mathbb{R}^{1+n})$, $u=(u_0,\bar{u})$, $\bar{u}=(u_1,\dots,u_n)$, and for $(t,x,w) \in E_c \times X$,
$$\mathbb{K}[u](t,x,w) = \partial_x\rho(t,x,w) + \partial_w\rho(t,x,w)\Big(u_{\alpha(t,x)} * \partial_x\alpha(t,x)\Big),$$
where the Hale operator is understood component-wise.

Since the above two Assumptions guarantee the continuity of $\mathbb{K}[Dz](t,x,z_{\alpha(t,x)})$ in (t,x), a classical theorem on differentiation of solutions with respect to initial data ensures that $Dg[z]$ exists and fulfils the integral equation
$$Dg[z](\tau,t,x) = \big[-\rho(t,x,z_{\alpha(t,x)})\,\big|\,I\big] + \int_t^\tau \mathbb{K}[Dz](Q[z](s,t,x)) * Dg[z](s,t,x)ds, \tag{3.1}$$
where $\big[-\rho(t,x,z_{\alpha(t,x)})\,\big|\,I\big]$ is the concatenation of the matrix $-\rho(t,x,z_{\alpha(t,x)})$ with the n-by-n identity matrix.

As is easily seen from the definition of $\mathbb{F}z$, its regularity in (t,x) is not higher than this of $g[z]$ and $\delta[z]$. Additionally, we need them to be Lipschitz continuous in z, for the sake of proving that $\mathbb{F}$ be a contraction.

Lemma 3.1. *If Assumptions* H$[\varphi]$*,* H$[\Omega]$*,* $\mathrm{H}_0[G]$*,* H$[\rho]$*,* H$[\alpha]$ *are satisfied, then for any* $z \in C^{1.L}_{\varphi.c}[\mu]$*, the characteristics* $g[z](\cdot,t,x)$ *are unique, and on*
$$\{(\tau,t,x) : (t,x) \in E_c, \tau \in (\delta[z](t,x),c)\}$$

the derivative $Dg[z](\tau,t,x)$ *exists, is continuous in* (t,x)*, and bounded uniformly in* (τ,t,x)*.*

Proof. Put $d = \|Dz\|_{C(E_c^*,\mathbb{R}^{1+n})}$. Whereas uniqueness follows by classical arguments (due to Assumption H[ρ]), a uniform bound for $\|Dg[z]\|$ may be found by applying the Gronwall lemma to

$$\|Dg[z](\tau,t,x)\| \le A_1 + (M + B_1 d)\left|\int_\tau^t \|Dg[z](s,t,x)\|\,ds\right|,$$

$$A_1 = 1 + K, \quad B_1 = 2Mr_1.$$

Of course, $g[z](\tau,t,x)$ is continuous in (t,x); since (3.1) is linear, and its kernel (as well as the free term) is continuous in (t,x), so is $Dg[z](\tau,t,x)$. □

Lemma 3.2. *Under the same hypotheses, for any* $z \in C^{1.L}_{\varphi.c}[\mu]$*,*

$$\delta[z] \in C(\overline{E_c}) \cap C^1(U[z]) \quad \textit{and} \qquad \|D\delta[z]\| \le (n+1)\kappa^{-1}C[z] \quad \textit{on} \quad U[z],$$

where $U[z] = \{(t,x) \in E_c : \delta[z](t,x) > 0\}$ *and* $C[z]$ *is a bound depending on* z*,*

$$C[z] = \sup\{\|Dg[z](\tau,t,x)\| : (t,x) \in E_c, \tau \in (\delta[z](t,x), c)\}.$$

Proof. Fix $z \in C^{1.L}_{\varphi.c}[\mu]$. Once it is done, we may introduce the notation $f_0 = \delta[z]$. We first prove that $f_0 \in C^1(U[z])$. Fix now $(\bar t,\bar x) \in U[z]$, and take τ such that $\Omega_{\tilde\delta/2}$ contains the point $g[z](\tau,\bar t,\bar x)$; denote the point briefly by y. Take also j such that $y \in U_j$ and $|\Phi_j(y)| < 1/2$. Let $\xi = \phi_{jn}(y) > 0$. Since $g[z](\tau,\cdot)$ and ϕ_{jn} are continuous, and U_j is open, there is $\varepsilon > 0$ such that, whenever $|t-\bar t| + \|x-\bar x\| \le \varepsilon$,

$$|g[z](\tau,\bar t,\bar x) - g[z](\tau,t,x)| \le \tilde\delta/2$$

and

$$g[z](\tau,t,x) \in U_j \quad \text{and} \quad \phi_{jn}(g[z](\tau,t,x)) \ge \xi/2.$$

Consequently, by Lemma 2.3, $g[z](s,t,x) \in U_j$ for $s \in [\delta[z](t,x),\tau]$ provided (t,x) is close to $(\bar t,\bar x)$. This shows the existence of a neighbourhood V of $(\bar t,\bar x)$ such that the family $f_\eta\colon V \to \mathbb{R}$, $0 \le \eta \le \xi/2$, is defined by

$$\phi_{jn}(g[z](f_\eta(t,x),t,x)) = \eta; \tag{3.2}$$

the uniqueness of this definition follows from the monotonicity (see (2.6)) of $\phi_{jn}(g[z](\cdot,t,x))$ on the left of τ. By the same argument, this family of functions is uniformly continuous in the parameter η: $|f_\eta(t,x) - f_{\tilde\eta}(t,x)| \le \kappa^{-1}|\eta-\tilde\eta|$. Moreover, for $\eta \ne 0$, the implicit function theorem may be applied to (3.2), implying continuity of f_η and of all its partial derivatives: for $i = 0, 1, \ldots, n$,

$$\partial_{x_i} f_\eta(t,x) = -\frac{\sum_{k=1}^n \partial_{x_k}\phi_{jn}(g[z](f_\eta(t,x),t,x)) \cdot \partial_{x_i} g_k[z](f_\eta(t,x),t,x)}{\sum_{k=1}^n \partial_{x_k}\phi_{jn}(g[z](f_\eta(t,x),t,x)) \cdot \rho_k(Q[z](f_\eta(t,x),t,x))}, \tag{3.3}$$

with ∂_t denoted by ∂_{x_0} for simplicity. Previous arguments, together with Lemma 3.1, lead to equicontinuity and uniform boundedness (by $\kappa^{-1}\widetilde{MC}[z]$) on V of those partial derivatives. With the aid of the Arzelà-Ascoli theorem, it is easy to see that f_0 is continuously differentiable on V and its partial derivatives are bounded by $\kappa^{-1}\widetilde{MC}[z]$.

What remains is to prove the continuity of f_0 on E_c, which may be done along the steps of the proof of Lemma 2.2 in [10]. □

4. A COMPATIBILITY CONDITION AND AN A PRIORI ESTIMATE OF Dz

The following compatibility condition for the problem (1.1), (1.2) will be indispensable in the proof of continuity of the derivative of $\mathbb{F}z$.

Assumption $\mathrm{H}_c[\,\varphi, \rho, G\,]$. On the set $\partial_0 E$, the right-hand side of (1.1) depends solely on φ. Precisely, the equivalence, on $\partial_0 E$,

$$\rho(t, x, z_{\alpha(t,x)}) = \rho(t, x, \bar{z}_{\alpha(t,x)}), \qquad G(t, x, z_{\alpha(t,x)}) = G(t, x, \bar{z}_{\alpha(t,x)})$$

holds for any z, $\bar{z} \in C^{1.L}_{\varphi.a}[d]$. Moreover, φ satisfies

$$\partial_t \varphi(t, x) + \varphi_x(t, x) * \rho(t, x, z_{\alpha(t,x)}) = G(t, x, z_{\alpha(t,x)}) \quad \text{on} \quad \partial_0 E, \tag{4.1}$$

where

$$\varphi_x(t, x) = \begin{cases} \sum_{i=1}^{n-1} \partial_{y_i} F_j(t, \Phi_j(x)) \partial_x \phi_{ji}(x) & \text{for} \quad (t, x) \in \partial_0 E, \\ \partial_x \varphi(0, x) & \text{for} \quad x \in \Omega. \end{cases} \tag{4.2}$$

and F_j is defined by (2.1). Additionally, if $b_0 > 0$, then

$$\partial_t \varphi(t, x) + \partial_x \varphi(t, x) * \rho(t, x, z_{\alpha(t,x)}) = G(t, x, z_{\alpha(t,x)}) \quad \text{on} \quad \{0\} \times \Omega. \tag{4.3}$$

Assumption $\mathrm{H}[G]$**.** Assumption $\mathrm{H}_0[G]$ is satisfied and, for $(t, x) \in E$, $w \in X_0$, the derivative $\partial_x G(t, x, w)$ and the Fréchet derivative $\partial_w G(t, x, w)$ exists. Furthermore, for the same M and L as in Assumption $\mathrm{H}[\rho]$, we have for $(t, x) \in E$, $w \in X_0$, and for any Lipschitz continuous $h\colon \mathscr{D} \to \mathbb{R}$:

1. the derivatives are bounded, in the sense that

$$\|\partial_x G(t, x, w)\| \le M(1 + \|Dw\|_0), \qquad \|\partial_w G(t, x, w)h\| \le M\|h\|_0,$$

2. the Fréchet derivative is Lipschitz continuous in w: for any $\bar{w} \in X_0$,

$$\|\partial_w G(t, x, w)h - \partial_w G(t, x, \bar{w})h\| \le L\|h\|_0 \, \|w - \bar{w}\|_0,$$

3. with respect to the topology induced by the norm $\|\cdot\|_{C^1(\mathscr{D},\mathbb{R})}$, $\partial_x G(t, x, w)$ is continuous in w,
4. $\partial_x G(t, x, w)$, $\partial_w G(t, x, w)h$, are continuous in (t, x).

Assumption H[c]**.** With the constants K, A_G, K_G, M, M, r_1, p_1, as in the above assumptions on ρ, G, φ and α, and with

$$A_1 = 1 + K, \quad A_2 = A_G + K_G\mu(a), \quad B_1 = M(3 + r_1),$$
$$A_3 = (M + B_1A_2)A_1, \quad \tilde{\eta} = \max\left\{\frac{M + B_1A_2 + B_1A_1}{2A_3}, \sqrt{\frac{B_1}{A_3}}\right\}, \quad \eta = p_1A_1,$$

the time interval $(0, c)$, for a solution, satisfies $c < \frac{1}{A_3(\tilde{\eta}\eta+1)}$.

Lemma 4.1. *Under the above Assumptions, and the preceding ones, there are* C_1 *and* C_2*, such that for any solution* $(z, g[z])$*,* $z \in C^{1.L}_{\varphi.c}[\mu]$*, to the system* (2.3)*,* (2.4)

$$\|Dg[z](\tau, t, x)\| \le C_1 \quad \textit{and} \quad \|Dz(t, x)\| \le C_2 \tag{4.4}$$

for $(t, x) \in E_c$*,* $\tau \in (\delta[z](t, x), c)$*.*

Proof. Define, for $u \in C(\overline{E^*_c}, \mathbb{R}^{1+n})$, $u = (u_0, \bar{u})$, $\bar{u} = (u_1, \ldots, u_n)$, and for $(t, x, w) \in E_c \times X$,

$$\mathbb{G}[u](t, x, w) = \partial_x G(t, x, w) + \partial_w G(t, x, w)\Big(u_{\alpha(t,x)} * \partial_x\alpha(t, x)\Big).$$

Let us first observe that, for any $z \in C^{1.L}_{\varphi.c}[\mu]$, $\mathbb{G}[Dz](t, x, z_{\alpha(t,x)})$ is bounded in (t, x) on E_c by Assumption H[G]. This allows for differentiation under the integral sign in (2.4), leading to an integral formula for $D\mathbb{F}z$.

To this end, fix $z \in C^{1.L}_{\varphi.c}[\mu]$ and $(t, x) \in E_c$. Once it is done, we may introduce the notation $g = g[z](\cdot, t, x)$ and $\delta = \delta[z](t, x)$. Let us extend the definition (4.2) of φ_x by setting $\varphi_x(0, x) = \partial_x\varphi(0, x)$ on Ω. Due to the compatibility condition $\mathrm{H}_c[\,\varphi, \rho, G\,]$ and thanks to continuity of $\delta[z]$, and also from (the last line of) Assumption H[φ], for $(t, x) \in E_c$

$$D\mathbb{F}z(t, x) = \varphi_x(\delta, g(\delta)) * Dg[z](\delta, t, x) + \big[\, G(t, x, z_{\alpha(t,x)}) \,\big|\, 0 \,\big] + \\ + \int_{\delta[z](t,x)}^{t} \mathbb{G}[Dz](Q[z](s, t, x)) * Dg[z](s, t, x)\, ds, \tag{4.5}$$

where $\big[\, G(t, x, z_{\alpha(t,x)}) \,\big|\, 0 \,\big] = \big(G(t, x, z_{\alpha(t,x)}),\, 0,\, \ldots,\, 0\big) \in \mathbb{R}^{1+n}$.

Let $z = \mathbb{F}z$, $z \in C^{1.L}_{\varphi.c}[\mu]$. Thanks to the regularity of z, the function $u\colon [0, c) \to \mathbb{R}$ defined by

$$u(t) = \sup\{\|Dz(s, x)\| : s \in (0, t], x \in \Omega\}$$

is continuous, even if Ω is unbounded. Following the argument in the proof of Lemma 3.1, we get

$$\|Dg[z](\tau, t, x)\| \le A_1 \exp\left\{\left|\int_{\tau}^{t}(M + B_1u(s))\, ds\right|\right\}.$$

Denote the above right-hand side by $q(\tau, t)$. Using it to estimate $Dg[z](\delta, t, x)$ and $\|Dg[z](s, t, x)\|$ in (4.5), we obtain $u(t) \leq A_2 + \psi(t)$, where

$$\psi(t) = p_1 q(0, t) + \int_0^t (M + B_1 u(s)) q(s, t)\, ds.$$

An easy calculation yields $\psi(0) = \eta$ and

$$\begin{aligned}\psi'(t) &= (M + B_1 u(t))\psi(t) + (M + B_1 u(t))A_1 \leq \\ &\leq (M + B_1 A_2 + B_1 \psi(t))\psi(t) + (M + B_1 A_2 + B_1\psi(t))A_1 \leq A_3(1 + \tilde{\eta}\psi(t))^2.\end{aligned}$$

Solving the quadratic differential inequality, as in [25], gives

$$\|Dz(t, x)\| \leq A_2 + \frac{1}{\tilde{\eta}}\left(\frac{\tilde{\eta}\eta + 1}{1 - A_3(\tilde{\eta}\eta + 1)t} - 1\right) \quad \text{for} \quad t \in \left(0, \frac{1}{A_3(\tilde{\eta}\eta + 1)}\right). \tag{4.6}$$

□

Remark 4.2. Note that (4.6) is the only reason to keep c small. Consequent estimates of Lipschitz constants for $D\mathbb{F}z$, $Dg[z]$ do not rely on the size of time interval. Neither do the contractivity of $\mathbb{F}$, due to the application of the Bielecki norm.

Let us write

$$C^{1.L}_{\varphi.c}[\mu; C_1] = \{z \in C^{1.L}_{\varphi.c} : |z(t, x)| \leq \mu(t), \|Dz(t, x)\| \leq C_1 \quad \text{for} \quad (t, x) \in E_c\}$$

and

$$X_1 = \{w \in C^{1.L}(\mathscr{D}, \mathbb{R}) : \|w\|_0 \leq \mu(a), \|Dw\|_0 \leq C_1\}.$$

The following result on Lipschitz continuity in z is important for the application of the Banach fixed-point theorem to $\mathbb{F}$. We give it without proof; one analogous to this of Lemma 3.5 in [12] works.

Lemma 4.3. *Under the same hypotheses, for $z \in C^{1.L}_{\varphi.c}[\mu; C_1]$ and for $\bar{z} \in C^{1.L}_{\bar{\varphi}.c}[\mu; C_1]$, with $\bar{z}(t, x) = \bar{\varphi}(t, x)$ on $E_0 \cup \partial_0 E_c$, and $\bar{\varphi}$ which satisfies Assumption* H$[\varphi]$ *with the same p_1 as φ does, we have, for any $(t, x) \in E_c$, and with* $\max\{\delta[z](t, x), \delta[\bar{z}](t, x)\} \leq \tau < c$,

$$\|g[z](\tau, t, x) - g[\bar{z}](\tau, t, x)\| \leq \bar{A}\left|\int_t^\tau \|z - \bar{z}\|_{C(E^*_s)}\, ds\right|,$$

$$|\delta[z](t, x) - \delta[\bar{z}](t, x)| \leq \widetilde{M}\bar{A}\kappa^{-1}\int_0^t \|z - \bar{z}\|_{C(E^*_s)}\, ds,$$

where $\bar{A} = 2M\exp(cK_\rho)$, $K_\rho = M + B_1 C_1$.

5. EXISTENCE AND CONTINUOUS DEPENDENCE

Assumption H$[\rho, G]$**.** The Assumptions H$[\rho]$, H$[G]$ are fulfilled and ρ, G are Lipschitz continuous in t with constant M. Moreover, for any (t,x), $(\bar{t},\bar{x}) \in E$, and for any w, $\bar{w} \in X_1$,

$$\begin{aligned}\|\partial_x\rho(t,x,w) - \partial_x\rho(t,\bar{x},w)\| &\leq L(1+|Dw|_L)\,\|x-\bar{x}\|,\\ \|\partial_x G(t,x,w) - \partial_x G(t,\bar{x},w)\| &\leq L(1+|Dw|_L)\,\|x-\bar{x}\|,\\ \|\partial_x\rho(t,x,w) - \partial_x\rho(t,x,\bar{w})\| &\leq L\|w-\bar{w}\|_1,\\ \|\partial_x G(t,x,w) - \partial_x G(t,x,\bar{w})\| &\leq L\|w-\bar{w}\|_1,\end{aligned}$$

and, additionally, with any Lipschitz continuous $h\colon D \to \mathbb{R}$,

$$\begin{aligned}\|\partial_w\rho(t,x,w)h - \partial_w\rho(t,\bar{x},w)h\| &\leq L(\|h\|_0 + |h|_{C^{0.L}(D,\mathbb{R})})\,\|x-\bar{x}\|,\\ |\partial_w G(t,x,w)h - \partial_w G(t,\bar{x},w)h| &\leq L(\|h\|_0 + |h|_{C^{0.L}(D,\mathbb{R})})\,\|x-\bar{x}\|.\end{aligned}$$

Define

$$C^{1.L}_{\varphi.c}[\mu; C_1; L_1] = \{z \in C^{1.L}_{\varphi.c}[\mu; C_1] : |Dz|_{C^{0.L}(E_c,\mathbb{R}^{1+n})} \leq L_1\}.$$

Lemma 5.1. *Under the above Assumptions, and the preceding ones, any solution* $(z, g[z])$ *to the system* (2.3), (2.4)*, has the following property: if* $z \in C^{1.L}_{\varphi.c}[\mu; C_1]$*, then* $z \in C^{1.L}_{\varphi.c}[\mu; C_1; L_1]$ *for some* L_1 *depending on bounds assumed or already proved.*

Proof. Unlike in estimating $\|Dz\|$, the supremum, for which we construct an integral inequality, is not necessarily a continuous function of the time variable. Let

$$f_1(t,\bar{t},x,\bar{x}) = \begin{cases} \frac{\|Dz(t,x)-Dz(\bar{t},\bar{x})\|}{\max\{|t-\bar{t}|,\|x-\bar{x}\|\}} & \text{if } 0 < t \leq \bar{t} < c,\ x \neq \bar{x},\ x,\bar{x} \in \Omega,\\ 0 & \text{for all other } (t,x),(\bar{t},\bar{x}) \in E_c.\end{cases}$$

Note that functions $f_1(\cdot,\bar{t},x,\bar{x})$, where $(t,x) \in E_c$, $(\bar{t},\bar{x}) \in E_c$, are measurable (in fact, piecewise absolutely continuous). Hence there is (see [15]) a measurable function $q_1\colon (0,c) \to \mathbb{R}_+$, uniquely determined up to null sets by the two properties that:

1. for every $(t,x) \in E_c$, $(\bar{t},\bar{x}) \in E_c$, $q_1(t) \geq f_1(t,\bar{t},x,\bar{x})$ for almost all t,
2. if $\bar{q}_1$ is another function with this property, then $\bar{q}_1 \geq q_1$ for almost all t.

Function q_1 is called an essential supremum of the class

$$\{f_1(\cdot,\bar{t},x,\bar{x})\colon \bar{t} \in (0,c), x,\bar{x} \in \Omega\}.$$

Similarly, for each $t \in (0,c)$, let $q_2(\cdot,t)\colon (0,c) \to \mathbb{R}_+$ be an essential supremum of the class

$$\{f_2(\cdot,t,\bar{t},x,\bar{x}) : \bar{t} \in (0,c), x,\bar{x} \in \Omega\},$$

where

$$f_2(\tau,t,\bar{t},x,\bar{x}) = \begin{cases} \frac{\|Dg[z](\tau,t,x)-Dg[z](\tau,\bar{t},\bar{x})\|}{\max\{|t-\bar{t}|,\|x-\bar{x}\|\}}, & \text{if } (\tau,t,\bar{t},x,\bar{x}) \in \Theta[z],\\ 0 & \text{for all other } \tau \in (0,c)\\ & \text{and } (t,x),(\bar{t},\bar{x}) \in E_c.\end{cases}$$

and

$$\Theta[z] = \{(\tau, t, \bar{t}, x, \bar{x}) \in (0, c)^3 \times \Omega^2 \colon \max\{\delta[z](t, x), \delta[z](\bar{t}, \bar{x})\} \le \tau \le t \le \bar{t}, \\ (t, x) \neq (\bar{t}, \bar{x})\}.$$

Then, for $(\tau, t, \bar{t}, x, \bar{x}) \in \Theta[z]$,

$$\frac{\|Dg[z](\tau, t, x) - Dg[z](\tau, \bar{t}, \bar{x})\|}{\max\{|t - \bar{t}|, \|x - \bar{x}\|\}} \le C \left(1 + \int_\tau^t q_1(s) + q_2(s, t) ds \right), \tag{5.1}$$

where C is a suitable constant; we shall use this notation subsequently. Note that the value of C may differ from line to line. Denote the right-hand side of the above by $\psi(\tau, t)$. Then $q_2(s, t) \le \psi(s, t)$ almost everywhere in s, on the domain of ψ; the inequality follows right from the definition of q_2. By use of the Gronwall lemma,

$$\psi(\tau, t) \le C \left(1 + (t - \tau) + \int_\tau^t q_1(s)\, ds \right) \exp(C(t - \tau)) \le C \left(1 + \int_\tau^t q_1(s)\, ds \right). \tag{5.2}$$

Take $(\bar{t}, \bar{x}) \neq (t, x) \in E_c$, $t \le \bar{t}$, and let $\zeta = \max\{\delta[z](t, x), \delta[z](\bar{t}, \bar{x})\}$. We may now use (5.1), (5.2) in estimating the difference $\|D\mathbb{F}z(t, x) - D\mathbb{F}z(\bar{t}, \bar{x})\|$ written with the aid of (4.5), obtaining

$$\frac{\|D\mathbb{F}z(t, x) - D\mathbb{F}z(\bar{t}, \bar{x})\|}{\max\{|t - \bar{t}|, \|x - \bar{x}\|\}} \le C \left(1 + \int_0^t q_1(s)\, ds + \int_0^t \int_s^t q_1(\xi)\, d\xi\, ds \right).$$

Again, abbreviating the right-hand side to $\tilde{\phi}(t)$ and using $q_1 \le \tilde{\phi}$ a.e., we get, for all $t \in (0, c)$,

$$\tilde{\phi}(t) \le \phi(t), \quad \text{where} \quad \phi(t) = C \left(1 + \int_0^t \tilde{\phi}(s)\, ds + \int_0^t \int_s^t \tilde{\phi}(\xi)\, d\xi\, ds \right).$$

Hence

$$\phi(0) = C \quad \text{and} \quad \phi'(t) = C\tilde{\phi}(t) + C \int_0^t \tilde{\phi}(t)\, ds \le C(1 + t)\phi(t).$$

It follows easily that $\phi(t) \le C \exp(C(c + c^2/2)) = L_1$. This completes the proof. □

Theorem 5.2. *Let all the preceding Assumptions hold. Then there exists exactly one solution* $\bar{z} \in C^{1.L}_{\varphi.c}[\mu; C_1; L_1]$ *of problem* (1.1), (1.2). *Moreover, there is* $\Lambda_c \in \mathbb{R}_+$ *such that for any* ψ *satisfying Assumptions* H$[\varphi]$, H$_c[\varphi, \rho, G]$,

$$\|\bar{z} - v\|_{C(E_t^*)} \le \Lambda_c \|\varphi - \psi\|_{C(E_0 \cup \partial_0 E_t)}, \quad 0 \le t \le c, \tag{5.3}$$

where $v \in C^{1.L}_{\psi.c}[\mu; C_1; L_1]$ is the solution of (1.1), (1.2) with φ replaced by ψ.

Proof. Consider the space $C^{1.L}_{\varphi.c}[\mu; C_1; L_1]$; we first prove that $\mathbb{F}$ maps it into itself. Indeed, the bounds required for $|\mathbb{F}z|$, $\|D\mathbb{F}z\|$, and for $|D\mathbb{F}z|_{C^{0.L}(E_c,\mathbb{R}^{1+n})}$ are already shown, and the fact that $\mathbb{F}z$ is a continuous extension of φ, is a simple consequence of the definition (2.4); it remains to prove that this extension is of class C^1. From (3.1), (4.5), and from the compatibility condition (4.1) we obtain for $(t,x) \in \partial_0 E$,

$$\lim_{\substack{(\bar{t},\bar{x})\to(t,x)\\(\bar{t},\bar{x})\in E_c}} D\mathbb{F}[z](\bar{t},\bar{x}) = \varphi_x(t,x) * Dg[z](t,t,x) + \big[\, G(t,x,z_{\alpha(t,x)}) \,\big|\, 0 \,\big] =$$
$$= \varphi_x(t,x) * \big[\, -\rho(t,x,z_{\alpha(t,x)}) \,\big|\, I \,\big] + \big[\, G(t,x,z_{\alpha(t,x)}) \,\big|\, 0 \,\big] = (\partial_t\varphi(t,x), \varphi_x(t,x)).$$

If $b_0 > 0$, then similar arguments, involving (4.3), apply to the case $(t,x) \in \{0\} \times \Omega$. To complete the proof, we would like to point out that equality $\partial_x \mathbb{F}z(0,\cdot) = \partial_x\varphi(0,\cdot)$ follows easily from (3.1), (4.5).

Note that our space is a closed subset of $C(E^*_c, \mathbb{R})$. To prove that there exists exactly one $\bar{z}$ therein, satisfying (2.4), we use the Banach fixed point theorem, with the aid of Bielecki norm (Gronwall lemma yields (5.3)); for details, see a similar proof in [10], Theorem 4.1. □

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