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Dedicated to the Memory of Professor Zdzisław Kamont

ON A SINGULAR NONLINEAR NEUMANN PROBLEM

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Abstract. We investigate the solvability of the Neumann problem involving two critical exponents: Sobolev and Hardy-Sobolev. We establish the existence of a solution in three cases: (i) $2 < p+1 < 2^*(s)$, (ii) $p+1 = 2^*(s)$ and (iii) $2^*(s) < p+1 \leq 2^*$, where $2^*(s) = \frac{2(N-s)}{N-2}$, $0 < s < 2$, and $2^* = \frac{2N}{N-2}$ denote the critical Hardy-Sobolev exponent and the critical Sobolev exponent, respectively.

Keywords: Neumann problem, critical Sobolev exponent, Hardy-Sobolev exponent Neumann problem.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with a smooth boundary $\partial\Omega$. Throughout this paper we assume that $0 \in \partial\Omega$. In this paper we investigate the solvability of the following nonlinear Neumann problem

$$\begin{cases} -\Delta u + \lambda u^p = \dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $2^*(s) = \frac{2(N-s)}{N-2}$, $N \geq 3$, $0 < s < 2$, is the critical Hardy-Sobolev exponent and $\lambda > 0$ is a parameter. It is assumed that $0 \in \partial\Omega$ and $2 < p+1 \leq 2^*$, where 2^* is a critical Sobolev exponent given by $2^* = \frac{2N}{N-2}$, $N \geq 3$. Obviously $2^*(0) = 2^*$.

Solutions to problem (1.1) are sought in the Sobolev space $H^1(\Omega)$ equipped with norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx.$$

A nonnegative function $u \in H^1(\Omega)$ is said to be a weak solution of problem (1.1) if

$$\int_\Omega \big(\nabla u \nabla v + \lambda u^p v\big)\, dx = \int_\Omega \frac{u^{2^*(s)-1}}{|x|^s} v\, dx \tag{1.2}$$

for every $v \in H^1(\Omega)$. Problem (1.1) is characterized by lack of compactness because embeddings of the space $H^1(\Omega)$ into spaces $L^{2^*}(\Omega)$ and $L^{2^*(s)}(\Omega, |x|^{-s})$ are continuous but not compact. The literature on problems involving the critical Sobolev exponent and the Hardy-Sobolev potential is very extensive. The pioneering paper by Brezis and Nirenberg [6] has greatly inspired research on nonlinear elliptic problems involving these critical exponents. For further developments we refer to survey articles [4, 19] and the monograph [24]. The results of the paper [6], which deals with the Dirichlet problem have been extended by many authors to the Neumann problem. We mention here some of them [1, 2, 7–12, 15, 16, 22] and [23]. This paper has been inspired by the recent article [17]. The authors of this paper considered a number nonlinear problems, with the Dirichlet boundary conditions, involving the critical Sobolev exponent and the Hardy-Sobolev potential. In particular, they considered the following problems:

$$\begin{cases} -\Delta u + \lambda u^p = \dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ on } \Omega \end{cases} \tag{1.3}$$

and

$$\begin{cases} \Delta u - \lambda u^{\frac{N+2}{N-2}} = \dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{o } \partial\Omega, \quad u > 0 \text{ on } \Omega. \end{cases} \tag{1.4}$$

The following two theorems have been established in [17]:

Theorem 1.1. *Let $\lambda > 0$, $0 \in \partial\Omega$, $1 \le p < \frac{N}{N-2}$, $p + 1 < 2^*(s)$ with $0 < s < 2$. If the mean curvature of $\partial\Omega$ at 0 is negative, then problem (1.3) has a solution.*

Theorem 1.2. *Let $\lambda > 0$, $0 \in \partial\Omega$. Suppose that the mean curvature of $\partial\Omega$ at 0 is negative. Then problem (1.4) has a solution provided that one of the following conditions holds:*

(i) $N = 3$ *and* $0 < s < 1$,
(ii) $N \ge 4$ *and* $0 < s < 2$.

We now observe that equation (1.4) with the Neumann boundary conditions has no positive solution. Indeed, assuming that u is a solution, it follows from the definition of a weak solution of (1.4) that

$$\lambda \int_\Omega u^{\frac{N+2}{N-2}}\, dx + \int_\Omega \frac{u^{2^*(s)-1}}{|x|^s}\, dx = 0$$

which is impossible.

In this paper we focus our attention on problem (1.1) which is an extension of (1.3) to the Neumann boundary conditions. Unlike in paper [17] we consider a full range of exponents p, $2^*(s)$ and distinguish three cases: (i) $2 < p+1 < 2^*(s)$, (ii) $p+1 = 2^*(s)$, (iii) $2^*(s) < p+1 \leq 2^*$. In particular, a solution in the case (iii) has been obtained by a local minimization. However,this method cannot be used for the same equation with the Dirichlet boundary conditions.

The paper is organized as follows. Section 2 contains some information about minimizers for the best Sobolev and Hardy-Sobolev constants that is used in the next sections. The existence results for problem (1.1) in these three cases are given in Sections 3, 4 and 5. In the final Section 6 we discuss the solvability for problem (1.1) with terms u^p and $\frac{u^{2*(s)-1}}{|x|^s}$ interchanged.

Throughout this paper we denote a strong convergence by " $\to$ " and a weak convergence by " $\rightharpoonup$ ".

Let $\phi : X \to \mathbb{R}$ be a C^1 functional on a Banach space X. We recall that a sequence $\{x_n\} \subset X$ is a Palais-Smale sequence for ϕ at a level $c \in \mathbb{R}$ (a $(PS)_c$ sequence for short) if $\phi(x_n) \to c$ and $\phi'(x_n) \to 0$ in X^* as $n \to \infty$. Finally, we say that the functional ϕ satisfies the Palais-Smale condition at level c ($(PS)_c$ condition for short) if each $(PS)_c$ sequence is relatively compact in X.

2. PRELIMINARIES

Solutions to problem (1.1) will be sought as critical points of the variational functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\,dx - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx.$$

It is clear that J_λ is of class C^1 on $H^1(\Omega)$.

Problems investigated in this paper are closely related to optimal constants of the Hardy-Sobolev type. The best Sobolev constant is defined by

$$S = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\,dx \colon u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{2^*}\,dx = 1 \Big\},$$

where $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \colon \nabla u \in L^2(\mathbb{R}^N)\}$. S is attained by a family of functions (see [21])

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U\Big(\frac{x-y}{\epsilon}\Big), \quad \epsilon > 0,\, y \in \mathbb{R}^N,$$

called instantons, where

$$U(x) = \left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}}.$$

We also have

$$\int_{\Omega} |\nabla U|^2\, dx = \int_{\mathbb{R}^N} U^{2^*}\, dx = S^{\frac{N}{2}}$$

and moreover U satisfies the equation

$$-\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

The best Sobolev constant can be defined on every domain Ω. It is well-known that S is independent of Ω and is only attained when $\Omega = \mathbb{R}^N$.

The best Hardy-Sobolev constant for the domain $\Omega \subset \mathbb{R}^N$ is defined by

$$M_s(\Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^2\, dx \colon \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s}\, dx = 1,\, u \in H_0^1(\Omega) \right\}.$$

If $\Omega = \mathbb{R}^N$, we write M_s instead of $M_s(\Omega)$. If $s = 0$, then $M_0 = S$. In the case $0 < s < 2$, $M_s(\Omega)$ depends on Ω (see [16]). If $s = 2$, we obtain the Hardy constant and M_2 is independent of Ω and is given by $M_2 = \left(\frac{N-2}{2}\right)^2$. The constant M_2 is not attained.

If $0 < s < 2$, then M_s is attained by a family of functions

$$W_\epsilon(x) = \frac{C_N \epsilon^{\frac{N-2}{2(2-s)}}}{\left(\epsilon + |x|^{2-s}\right)^{\frac{N-2}{2-s}}},$$

where $C_N > 0$ is normalizing constant depending on N and s. Moreover, W_ϵ satisfies the equation

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} \quad \text{in } \mathbb{R}^N - \{0\}.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla W_\epsilon|^2\, dx = \int_{\mathbb{R}^N} \frac{W_\epsilon^{2^*(s)}}{|x|^s}\, dx = M_s^{\frac{N-s}{2-s}}.$$

3. CASE $p + 1 < 2^*(s)$

First we show that the functional J_λ has a mountain-pass structure. The following result is well-known (see [16]).

Lemma 3.1. *Let $0 \in \partial\Omega$. Then there exists a constant $S_H > 0$ such that*

$$\left(\int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s}\, dx \right)^{\frac{2}{2^*(s)}} \le S_H \int_{\Omega} \left(|\nabla u|^2 + u^2 \right) dx$$

for every $u \in H^1(\Omega)$.

Proposition 3.2. *Let* $2 < p+1 < 2^*(s)$ *and* $\lambda > 0$. *Then there exist constants* $\kappa > 0$ *and* $\rho > 0$ *such that*

$$J_\lambda(u) \geq \kappa \quad \text{for } \|u\| = \rho. \tag{3.1}$$

Proof. It follows from the Hölder inequality that

$$\int_\Omega u^2\,dx \leq \left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}} |\Omega|^{1-\frac{2}{p+1}}.$$

Hence

$$\int_\Omega |u|^{p+1}\,dx \geq \left(\int_\Omega u^2\,dx\right)^{\frac{p+1}{2}} |\Omega|^{1-\frac{p+1}{2}}.$$

Thus

$$J_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{p+1}|\Omega|^{1-\frac{p+1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{p+1}{2}} - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx.$$

If $\|u\| = \rho < 1$, then $\int_\Omega |\nabla u|^2\,dx < 1$ and

$$\int_\Omega |\nabla u|^2\,dx \geq \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p+1}{2}}$$

as $p+1 > 2$. From this we obtain the following estimate of J_λ for $\|u\| = \rho$:

$$J_\lambda(u) \geq \frac{1}{2}\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p+1}{2}} + \frac{\lambda}{p+1}|\Omega|^{1-\frac{p+1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{p+1}{2}} - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx.$$

Let $c_1 = \min\left(\frac{1}{2}, \frac{\lambda}{p+1}|\Omega|^{1-\frac{p+1}{2}}\right)$. Then using Lemma 3.1 we get

$$J_\lambda(u) \geq c_1\left[\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p+1}{2}} + \left(\int_\Omega u^2\,dx\right)^{\frac{p+1}{2}}\right] - \frac{1}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx \geq$$

$$\geq c_1 2^{\frac{1-p}{2}}\left(\int_\Omega \left(|\nabla u|^2 + u^2\right)dx\right)^{\frac{p+1}{2}} - \frac{S_H^{\frac{2^*(s)}{2}}}{2^*(s)}\|u\|^{2^*(s)}.$$

Taking $\rho > 0$ sufficiently small the estimate (3.1) follows. □

We now observe that if $u = t\phi$ with $\phi \in H^1(\Omega)$ and $\phi \neq 0$ then $J_\lambda(t\phi) < 0$ for $t > 0$ sufficiently large. Thus the functional J_λ has a mountain-pass structure (see [3]).

Proposition 3.3. *Let $\lambda > 0$ and $2 < p+1 < 2^*(s)$. Then J_λ satisfies the $(PS)_c$ condition for*

$$c < \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) M_s^{\frac{N-s}{2-s}}. \tag{3.2}$$

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence with c satisfying (3.2). First we show that $\{u_n\}$ is bounded in $H^1(\Omega)$. We have

$$J_\lambda(u_n) - \frac{1}{p+1}\langle J'_\lambda(u_n), u_n\rangle = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int\limits_\Omega |\nabla u_n|^2\, dx + \\ + \lambda\Big(\frac{1}{p+1} - \frac{1}{2^*(s)}\Big)\int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}\, dx = c + o(\|u_n\|).$$

Since $\frac{1}{p+1} - \frac{1}{2^*(s)} > 0$ we see that

$$\int\limits_\Omega |\nabla u_n|^2\, dx + \int\limits_\Omega \frac{|u_n|^{2^*(s)}}{|x|^s}\, dx \leq C + o(\|u_n\|)$$

for some constant $C > 0$. This obviously shows that $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*(s)}(\Omega, |x|^{-s})$ and $u_n \to u$ in $L^{p+1}(\Omega)$. By the concentration-compactness principle (see [18]) there exist constants $\mu_0 > 0$ and $\nu_0 > 0$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \mu_0\delta_0$$

and

$$\frac{|u_n|^{2^*(s)}}{|x|^s} \rightharpoonup \nu = \frac{|u|^{2^*(s)}}{|x|^s} + \nu_0\delta_0$$

in the sense of measures, where δ_0 denotes the Dirac measure assigned to 0. The constants ν_0 and μ_0 satisfy the inequality

$$2^{-\frac{2-s}{N-s}}\nu_0^{\frac{2}{2^*(s)}} M_s \leq \mu_0. \tag{3.3}$$

To complete the proof it is sufficient to show that $\nu_0 = 0$. Arguing by contradiction assume that $\nu_0 > 0$. Testing $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ by a family of functions ϕ_δ, $\delta > 0$, concentrating at 0 we derive the inequality $\mu_0 \leq \nu_0$. From this and (3.3) we get that $\nu_0 \geq \frac{1}{2}M_s^{\frac{N-s}{2-s}}$. It then follows again from (3.3) that

$$\mu_0 \geq \frac{1}{2}M_s^{\frac{N-s}{2-s}}. \tag{3.4}$$

Thus

$$J_\lambda(u_n) - \frac{1}{2^*(s)}\langle J'_\lambda(u_n), u_n\rangle = \lambda\Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big)\int\limits_\Omega |\nabla u_n|^2\, dx + \\ + \lambda\Big(\frac{1}{p+1} - \frac{1}{2^*(s)}\Big)\int\limits_\Omega |u_n|^{p+1}\, dx.$$

Letting $n \to \infty$ we deduce from this that

$$c \geq \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) M_s^{\frac{N-s}{2-s}},$$

which is impossible. Since $\nu_0 = 0$, $u_n \to u$ in $L^{2^*(s)}(\Omega, |x|^{-s})$. This and the fact that $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ imply that $u_n \to u$ in $H^1(\Omega)$. □

A solution to problem (1.1) always exists for λ belonging to a small interval $(0, \Lambda)$. Indeed, for $t \geq 0$ we have

$$J_\lambda(t) = \frac{\lambda}{p+1}|\Omega| t^{p+1} - \frac{t^{2^*(s)}}{2^*(s)} \int_\Omega \frac{dx}{|x|^s}$$

and

$$\max_{t \geq 0} J_\lambda(t) = J_\lambda(t_{\max}) = \Big(\frac{1}{p+1} - \frac{1}{2^*(s)}\Big) \frac{\big(\lambda|\Omega|\big)^{\frac{2^*(s)}{2^*(s)-p-1}}}{\Big(\int_\Omega \frac{dx}{|x|^s}\Big)^{\frac{p+1}{2^*(s)-p-1}}},$$

where

$$t_{\max} = \left(\frac{\lambda|\Omega|}{\int_\Omega \frac{dx}{|x|^s}}\right)^{\frac{1}{2^*(s)-p-1}}.$$

If $\lambda > 0$ satisfies the following inequality

$$\Big(\frac{1}{p+1} - \frac{1}{2^*(s)}\Big) \frac{\big(\lambda|\Omega|\big)^{\frac{2^*(s)}{2^*(s)-p-1}}}{\Big(\int_\Omega \frac{dx}{|x|^s}\Big)^{\frac{p+1}{2^*(s)-p-1}}} < \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) M_s^{\frac{N-s}{2-s}},$$

then problem (1.1) has a solution. It is clear that this inequality holds for λ belonging to some interval $(0, \Lambda)$.

To verify the validity of the condition (3.2) for each $\lambda > 0$, we need the following asymptotic properties of W_ϵ. Let

$$I(u) = \frac{\int_\Omega |\nabla u|^2\, dx}{\Big(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\, dx\Big)^{\frac{N-2}{2-s}}},$$

then we have

$$I(W_\epsilon) = \begin{cases} \frac{M_s}{2^{\frac{2-s}{N-s}}} - H(0) a_N \epsilon^{\frac{1}{2-s}} + o\big(\epsilon^{\frac{1}{2-s}}\big) & \text{for } N \geq 4, \\ \frac{M_s}{2^{\frac{2-s}{N-s}}} - H(0) b_N \epsilon^{\frac{1}{2-s}} |\log \epsilon| + o\big(\epsilon^{\frac{1}{2-s}}\big) & \text{for } N = 3, \end{cases} \tag{3.5}$$

where $H(0)$ denotes the mean curvature of $\partial\Omega$ at 0, and a_N, b_N are positive constants depending on N and s (see [16]).

Theorem 3.4. *Let $\lambda > 0$ and $H(0) > 0$.*

(i) *If $N \geq 4$, $1 < p < \frac{N}{N-2}$ and $0 < s < 1$, then problem (1.1) has a solution.*
(ii) *If $N = 3$ and $2 < p < 3$ and $0 < s < 1$, then problem (1.1) has a solution.*

Proof. We may assume that $\lambda = 1$. It suffices to verify the condition (3.2). Then the existence of a solution follows from the mountain-pass theorem [3]. Since $p+1 < 2^*(s)$, there exists a constant $t_\epsilon > 0$ such that

$$\max_{t\geq 0} J_\lambda(tW_\epsilon) = \frac{t_\epsilon^2}{2}\int_\Omega |\nabla W_\epsilon|^2\,dx - \frac{t_\epsilon^{2^*(s)}}{2^*(s)}\int_\Omega \frac{W_\epsilon^{2^*(s)}}{|x|^s}\,dx + \frac{t_\epsilon^{p+1}}{p+1}\int_\Omega W_\epsilon^{p+1}\,dx.$$

It is easy to show that t_ϵ is bounded independently of $\epsilon > 0$, that is, there exists a constant $T > 0$ such that $t_\epsilon \leq T$ for every $\epsilon > 0$ (small). From this we deduce that

$$\max_{t\geq 0} J_\lambda(tW_\epsilon) \leq \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\left[\frac{\int_\Omega |\nabla W_\epsilon|^2\,dx}{\left(\int_\Omega \frac{W_\epsilon^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{N-2}{N-s}}}\right]^{\frac{N-s}{2-s}} + \frac{T^{p+1}}{p+1}\int_\Omega W_\epsilon^{p+1}\,dx. \quad (3.6)$$

We now observe that

$$\int_\Omega W_\epsilon^{p+1}\,dx = O\left(\epsilon^{\frac{2N-(N-2)(p+1)}{2(2-s)}}\right), \quad (3.7)$$

if $\frac{2}{N-2} < p$. Since $p < \frac{N}{N-2}$ we see that $\int_\Omega W_\epsilon^{p+1}\,dx = o\left(\epsilon^{\frac{1}{2-s}}\right)$. We point out here that conditions $p < \frac{N}{N-2}$ and $0 < s < 1$ yield $p+1 < 2^*(s)$. Finally, combining (3.5) with inequalities (3.6) and (3.7) we get condition (3.2) and assertions (i) and (ii) follow. According to Theorem 10 in [5] these mountain-pass solutions can be taken to be nonnegative and by the strong maximum principle these solutions are positive on Ω (see [14]). □

4. CASE $p + 1 = 2^*(s)$, $0 < s < 2$

In this case we also have $p + 1 < 2^* = \frac{2N}{N-2}$. If $p + 1 = 2^*(s)$ with $0 < s < 2$, then $s = N - \frac{(N-2)(p+1)}{2}$. Obviously if $1 < p < \frac{N+2}{N-2}$, then $0 < s < 2$. In this case we look for a solution of (1.1) as a minimizer of the constrained variational problem

$$I = \inf\left\{\int_\Omega |\nabla u|^2\,dx\colon u \in H^1(\Omega),\ \int_\Omega \left(\frac{1}{|x|^s} - \lambda\right)|u|^{p+1}\,dx = 1\right\}. \quad (4.1)$$

A minimizer u after rescaling $I^{\frac{1}{p-1}}u$ is a solution of problem (1.1). It is assumed that a parameter $\lambda > 0$ satisfies

$$\frac{1}{|\Omega|}\int_\Omega \frac{dx}{|x|^s} < \lambda. \quad (4.2)$$

To justify this assumption let us assume that u is a solution of problem (1.1). Testing (1.2) with $v = 1$ we get

$$\lambda \int_\Omega |u|^p \, dx = \int_\Omega \frac{|u|^p}{|x|^s} \, dx \geq d^{-s} \int_\Omega |u|^p \, dx,$$

where $d =$ diam Ω. This inequality implies that λ satisfies

$$\lambda > d^{-s}. \tag{4.3}$$

Obviously inequality (4.2) yields inequality (4.3).

To proceed further we need the following decomposition of the space $H^1(\Omega)$. Since 0 is the first eigenvalue of the operator "$-\Delta$" with the Neumann boundary conditions, we have the following decomposition of $H^1(\Omega)$:

$$H^1(\Omega) = V \oplus \mathbb{R} \quad \text{with} \quad V = \left\{ v \in H^1(\Omega) \colon \int_\Omega v \, dx = 0 \right\}.$$

Using this decomposition we can define an equivalent norm on $H^1\Omega)$ given by

$$\|u\|_V^2 = \|\nabla v\|_2^2 + t^2 \quad \text{for} \quad u = v + t \quad \text{with} \quad v \in V, \ t \in \mathbb{R}.$$

Lemma 4.1. *Let $p + 1 = 2^*(s)$ for some $0 < s < 2$. Suppose that (4.2) holds. Then $I > 0$.*

Proof. Arguing by contradiction, assume that $I = 0$. Let $u_n = v_n + t_n$, $v_n \in V$, $t_n \in \mathbb{R}$ be a minimizing sequence for $I = 0$. Since $\|\nabla v_n\|_2^2 \to 0$, we see that $v_n \to 0$ in $L^2(\Omega)$. We now show that the sequence $\{t_n\}$ is bounded. In the contrary case we may assume that $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated in a similar way). We have

$$1 + \lambda \int_\Omega |v_n + t_n|^{p+1} \, dx = \int_\Omega |x|^{-s} |v_n + t_n|^{p+1} \, dx, \tag{4.4}$$

that is,

$$t_n^{-p-1} + \lambda \int_\Omega \left|\frac{v_n}{t_n} + 1\right|^{p+1} dx = \int_\Omega |x|^{-s} \left|\frac{v_n}{t_n} + 1\right|^{p+1} dx.$$

Since V is continuously embedded into $L^{p+1}(\Omega)$ and $L^{2^*(s)}(\Omega, |x|^{-s})$, letting $n \to \infty$ in the above equation, we obtain

$$\lambda |\Omega| = \int_\Omega |x|^{-s} \, dx,$$

which is impossible. Thus $\{t_n\}$ is bounded and we may assume that $t_n \to t_0$. Using this, we derive a contradiction from (4.4). This contradiction completes the proof. □

Proposition 4.2. *Let $p+1 = 2^*(s)$ for some $0 < s < 2$ and suppose that (4.2) holds. If*

$$I < \frac{M_s}{2^{\frac{2-s}{N-s}}}, \tag{4.5}$$

then problem (1.1) has a solution.

Proof. Let $\{u_n\}$ be a minimizing sequence for I such that $\int_\Omega \big(|x|^{-s} - \lambda\big)|u_n|^{p+1}\,dx = 1$ for each n. We have $u_n = v_n + t_n$, $v_n \in V$, $t_n \in \mathbb{R}$. Assuming that the sequence $\{t_n\}$ is unbounded, we obtain a contradiction, as in the proof of Lemma 4.1. Thus the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$ and we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*(s)}(\Omega, |x|^{-s})$ and $u_n \to u$ n $L^{p+1}(\Omega)$. It then follows from the concentration-compactness principle that there exist constants $\mu_0 \geq 0$ and $\nu_0 \geq 0$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \mu_0\delta_0$$

and

$$\frac{|u_n|^{p+1}}{|x|^s} - \lambda|u_n|^{p+1} \rightharpoonup |u|^{p+1}\Big(\frac{1}{|x|^s} - \lambda\Big) + \nu_0\delta_0$$

in the sense of measures. The constants μ_0 and ν_0 satisfy the following inequality

$$\frac{M_s\nu_0^{\frac{2}{p+1}}}{2^{\frac{2-s}{N-s}}} \leq \mu_0. \tag{4.6}$$

Moreover, there holds

$$1 = \int_\Omega \Big(\frac{1}{|x|^s} - \lambda\Big)|u|^{p+1}\,dx + \nu_0. \tag{4.7}$$

First we show that

$$\int_\Omega \Big(\frac{1}{|x|^s} - \lambda\Big)|u|^{p+1}\,dx > 0.$$

In the contrary case we would have

$$\int_\Omega \Big(\frac{1}{|x|^s} - \lambda\Big)|u|^{p+1}\,dx \leq 0.$$

By (4.7), we would have $\nu_0 \geq 1$. It then follows from (4.6) that $\mu_0 \geq \frac{M_s}{2^{\frac{2-s}{N-s}}}$. Consequently,

$$I \geq \int_\Omega |\nabla u|^2\,dx + \mu_0 \geq \frac{M_s}{2^{\frac{2-s}{N-s}}}$$

which is impossible. From the definition of I we derive, using (4.5) and (4.6) that

$$I \geq I\Big(\int_\Omega \Big(\frac{|u|^{p+1}}{|x|^s} - \lambda |u|^{p+1}\Big)\,dx\Big)^{\frac{2}{p+1}} + \frac{M_s \nu_0^{\frac{2}{p+1}}}{2^{\frac{2-s}{N-s}}} >$$

$$> I\Big(\int_\Omega \Big(\frac{|u|^{p+1}}{|x|^s} - \lambda |u|^{p+1}\Big)\,dx\Big)^{\frac{2}{p+1}} + I\nu_0^{\frac{2}{p+1}}.$$

Thus

$$1 > \Big(\int_\Omega \Big(\frac{|u|^{p+1}}{|x|^s} - \lambda |u|^{p+1}\Big)\,dx\Big)^{\frac{2}{p+1}} + \nu_0^{\frac{2}{p+1}}.$$

This is obviously in contradiction with (4.7). Therefore $\mu_0 = \nu_0 = 0$ and the minimizing sequence $\{u_n\}$ converges in $H^1(\Omega)$ to u. A minimizer u, up to a multiplicative constant, is a solution of problem (1.1). Indeed, let $\phi \in H^1(\Omega)$ and set

$$f(t) = \frac{\int_\Omega |\nabla(u+t\phi)|^2\,dx}{\Big(\int_\Omega \big(|x|^{-s} - \lambda\big)|u+t\phi|^{2^*(s)}\,dx\Big)^{\frac{2}{2^*(s)}}}$$

for t small. Since $f'(0) = 0$, we get

$$\int_\Omega \nabla u \nabla \phi\,dx = I \int_\Omega \frac{|u|^{2^*(s)-2}u}{|x|^s}\,dx.$$

We now set $u = \frac{1}{I^{\frac{1}{p-1}}} v$ and it is easy to check that v is a solution of problem (1.1). Since $|u|$ is also a minimizer for I, we may assume that u is nonnegative and by the strong maximum principle $u(x) > 0$ on Ω. □

Theorem 4.3. *Let $p+1 = 2^*(s)$ for some $1 < s < 2$ and $H(0) > 0$. Suppose that* (4.2) *holds. Then* (4.5) *holds and problem* (1.1) *has a solution.*

Proof. The assumption that $1 < s < 2$ implies that $p < \frac{N}{N-2}$. To verify (4.5) we need the following asymptotic properties of W_ϵ (see [16]). Let $K_1(\epsilon) = \int_\Omega |\nabla W_\epsilon|^2\,dx$ and $K_2(\epsilon) = \int_\Omega \frac{W_\epsilon^{2^*(s)}}{|x|^s}\,dx$. We then have (see [16])

$$K_1(\epsilon) = \frac{1}{2}K_1 - I(\epsilon) + o\big(\epsilon^{\frac{1}{2-s}}\big),$$

$$K_2(\epsilon) = \frac{1}{2}K_2 - \Pi(\epsilon) + o\big(\epsilon^{\frac{1}{2-s}}\big),$$

where

$$K_1 = c_N^2 (N-2)^2 \int_{\mathbb{R}^N} \frac{|y|^{2-2s}\,dy}{\big(1+|y|^{2-s}\big)^{\frac{2(N-s)}{2-s}}},$$

$$K_2 = c_N^{2^*(s)} \int\limits_{\mathbb{R}^N} \frac{dy}{|y|^s\big(1+|y|^{2-s}\big)^{\frac{2(N-s)}{2-s}}},$$

$$\lim_{\epsilon\to 0} \epsilon^{-\frac{1}{2-s}} I(\epsilon) = H(0)A_N \quad \text{and} \quad \lim_{\epsilon\to 0} \epsilon^{-\frac{1}{2-s}} \Pi(\epsilon) = H(0)B_N,$$

where $A_N > 0$ and $B_N > 0$ are constants depending on N and s. We also have

$$\lim_{\epsilon\to 0} \frac{I(\epsilon)}{\Pi(\epsilon)} > \frac{(N-2)K_1}{(N-s)K_2}.$$

Since $1 < s < 2$, it is easy to check that

$$\int\limits_{\Omega} W_\epsilon^{p+1}\, dx = O\big(\epsilon^{\frac{2N-(N-2)(p+1)}{2(2-s)}}\big) = O\big(\epsilon^{\frac{s}{2-s}}\big) = o\big(\epsilon^{\frac{1}{2-s}}\big).$$

Using these asymptotic formulae we can write

$$\frac{\int\limits_\Omega |\nabla W_\epsilon|^2\, dx}{\Big(\int\limits_\Omega \big(\frac{W_\epsilon^{2^*(s)}}{|x|^s} - \lambda W_\epsilon^{2^*(s)}\big)\, dx\Big)^{\frac{2}{2^*(s)}}} = \frac{\frac{1}{2}K_1 - I(\epsilon) + o\big(\epsilon^{\frac{1}{2-s}}\big)}{\Big(\frac{1}{2}K_2 - \Pi(\epsilon) + o\big(\epsilon^{\frac{1}{2-s}}\big)\Big)^{\frac{2}{2^*(s)}}} =$$

$$= \frac{M_s}{2^{\frac{2-s}{N-s}}} - H(0)a_N\epsilon^{\frac{1}{2-s}} + o\big(\epsilon^{\frac{1}{2-s}}\big)$$

for some constant a_N depending on N and s. This obviously yields (4.5). □

5. CASE $2^*(s) < p+1 \le 2^*$, $0 < s < 2$

In this case we modify equation (1.1) by moving a parameter λ to the term $\frac{|u|^{2^*(s)-1}}{|x|^s}$, that is, we consider the following problem

$$\begin{cases} -\Delta u + u^p = \lambda \dfrac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ on } \Omega. \end{cases} \tag{5.1}$$

In fact, problem (1.1) can be reduced to (5.1) by introducing a new unknown function $u = \lambda^{-\frac{1}{p-1}} v$. Then v satisfies the equation

$$-\Delta v + v^p = \lambda^{-\frac{2^*(s)-2}{p-1}} \frac{v^{2^*(s)-1}}{|x|^s}.$$

The variational functional for problem (5.1) is given by

$$I_\lambda(u) = \frac{1}{2}\int\limits_\Omega |\nabla u|^2\, dx + \frac{1}{p+1}\int\limits_\Omega |u|^{p+1}\, dx - \frac{\lambda}{2^*(s)}\int\limits_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\, dx.$$

Theorem 5.1. *Let $2^*(s) < p+1 \le 2^*$. Then there exists $\lambda_0 > 0$ such that problem* (5.1) *has a solution for each $0 < \lambda < \lambda_0$ (consequently problem* (1.1) *has a solution for $\lambda > \lambda_0^{-\frac{p-1}{2^*(s)-2}}$).*

Proof. First we consider the case $2^*(s) < p+1 = 2^*$. As in the proof of Proposition 3.2 we obtain the following estimate

$$I_\lambda(u) \ge c_1 2^{\frac{1-p}{2}} \rho^{p+1} - \lambda \frac{S_H^{\frac{2^*(s)}{2}}}{2^*(s)} \rho^{2^*(s)}$$

for $\|u\| = \rho < 1$, where $c_1 = \min\left(\frac{1}{2}, \frac{|\Omega|^{1-\frac{p+1}{2}}}{p+1}\right)$. Let

$$c_2 = \frac{c_1 2^{\frac{1-p}{2}} 2^*(s)}{2 S_H^{\frac{2^*(s)}{2}}} \quad \text{and} \quad 0 < \rho < \min\left(1, \left[\frac{M_s^{\frac{N-s}{2-s}}}{2 c_2^{\frac{N-2}{2-s}}}\right]^{\frac{2-s}{4}}\right).$$

We choose λ_0 satisfying

$$\lambda_0 \frac{S_H^{\frac{2^*(s)}{2}} \rho^{2^*(s)}}{2^*(s)} = \frac{1}{2} c_1 2^{\frac{1-p}{2}} \rho^{2^*},$$

that is,

$$\lambda_0 = \frac{c_1 2^{\frac{1-p}{2}} 2^*(s)}{2 S_H^{\frac{2^*(s)}{2}}} \rho^{\frac{2s}{N-2}} = c_2 \rho^{\frac{2s}{N-2}}.$$

Then

$$I_\lambda(u) \ge \frac{1}{2} c_1 2^{\frac{1-p}{2}} \rho^{2^*(s)}$$

for $\|u\| = \rho$ and $0 < \lambda < \lambda_0$. We also have $d = \inf_{\|u\| \le \rho} I_\lambda(u) < 0$ for each $0 < \lambda < \lambda_0$. By the Ekeland variational principle (see [13]) there exists a sequence $\{u_n\} \subset \{u\colon \|u\| \le \rho\}$ such that $I_\lambda(u_n) \to d$ and $I'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$. Applying the P.L. Lions' concentration-compactness principle (see [18]) there exist points $\{x_j\} \subset \bar{\Omega}$ and constants ν_j, μ_j, $j \in J \cup \{0\}$ such that

$$|\nabla u_n|^2 \, dx \rightharpoonup d\mu \ge |\nabla u|^2 \, dx + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0, \tag{5.2}$$

$$|u_n|^{2^*} \, dx \rightharpoonup d\nu = |u|^{2^*} \, dx + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0, \tag{5.3}$$

$$\frac{|u_n|^{2^*(s)}}{|x|^s} \, dx \rightharpoonup d\gamma = \frac{|u|^{2^*(s)}}{|x|^s} + \gamma_0 \delta_0, \tag{5.4}$$

$$S \nu_j^{\frac{2}{2^*}} \le \mu_j \quad \text{if } x_j \in \Omega,\ j \in J, \tag{5.5}$$

$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \le \mu_j \ \text{ if } \ x_j \in \partial\Omega,\ j \in J, \tag{5.6}$$

and

$$\frac{M_s}{2^{\frac{2-s}{N-s}}}\gamma_0^{\frac{2}{2^*(s)}} \le \mu_0. \tag{5.7}$$

Testing $I'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ with $u_n\varphi_\delta$, where φ_δ, $\delta > 0$, is a family of C^1-functions concentrating at x_j as $\delta \to 0$ we deduce that

$$\mu_j + \nu_j = 0 \ \text{ for } \ j \in J.$$

This shows that the concentration can only occur at $0 \in \partial\Omega$. In a similar way we can show that $\mu_0 + \nu_0 \le \lambda\gamma_0$. It suffices to show that $\gamma_0 = 0$. Arguing by contradiction assume that $\gamma_0 > 0$. Since $\mu_0 \le \lambda\gamma_0$, we derive from (5.7) that

$$\frac{1}{2}\left(\frac{M_s}{\lambda}\right)^{\frac{N-s}{2-s}} \le \gamma_0. \tag{5.8}$$

This combined with (5.7) gives

$$\frac{M_s^{\frac{N-s}{2-s}}}{2\lambda^{\frac{N-2}{2-s}}} \le \mu_0. \tag{5.9}$$

Since $\|u_n\| \le \rho$, we get from (5.9) and (5.2) that

$$\rho^2 \ge \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^2 + u_n^2\right) dx \ge \frac{M_s^{\frac{N-s}{2-s}}}{2\lambda^{\frac{N-2}{2-s}}} \ge \frac{M_s^{\frac{N-s}{2-s}}}{2\lambda_0^{\frac{N-2}{2-s}}}. \tag{5.10}$$

According to the choice of λ_0 we derive from (5.10) that

$$\rho^2 \ge \frac{M_s^{\frac{N-s}{2-s}}}{2c_2^{\frac{N-2}{2-s}}\rho^{\frac{2s}{2-s}}}.$$

Hence

$$\rho \ge \left(\frac{M_s^{\frac{N-s}{2-s}}}{2c_2^{\frac{N-2}{2-s}}}\right)^{\frac{2-s}{4}}$$

and we have arrived at a contradiction with the choice of ρ. This completes the proof for the case $2^*(s) < p+1 = 2^*$. If $2^*(s) < p+1 < 2^*$, then the concentration of a minimizing sequence can only occur at $0 \in \partial\Omega$. In this case we choose λ_0 in the following way

$$\lambda_0 = \frac{c_1 2^{\frac{1-p}{2}} 2^*(s)}{2S_H^{\frac{2^*(s)}{2}}}\rho^{p+1-2^*(s)}.$$

Arguing as in the first part of the proof we can show the existence of a solution of problem (5.1). $\square$

6. FINAL REMARKS

In this section we consider problem (1.1) with terms u^p and $\frac{u^{2^*(s)-1}}{|x|^s}$ interchanged, that is, we are concerned with the following problem

$$\begin{cases} -\Delta u + \lambda \dfrac{u^{2^*(s)-1}}{|x|^s} = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \quad u > 0 \text{ on } \Omega, \end{cases} \tag{6.1}$$

where $\lambda > 0$ is a parameter and it is assumed that $0 \in \partial\Omega$. As in the case of problem (1.1) we distinguish three cases: (i) $2 < p+1 < 2^*(s)$, (ii) $p+1 = 2^*(s)$ and (iii) $2^*(s) < p+1 \leq 2^*$. Solutions to problem (6.1) are sought as critical points of the variational functional

$$\Phi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2^*(s)}\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx.$$

Case (i).

Theorem 6.1. *Let $1 < p+1 < 2^*(s)$ for some $0 < s < 2$. Then for each $\lambda > 0$ problem (6.1) has a solution. Let u_λ be a solution corresponding to $\lambda > 0$. Then $\|u_\lambda\| \to 0$ as $\lambda \to \infty$.*

Proof. We commence by showing that functional Φ_λ is coercive for each $\lambda > 0$. Let $d =$ diam Ω. We then have

$$\Phi_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2^*(s)d^s}\int_\Omega |u|^{2^*(s)}\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx.$$

Using the Young inequality for each $\delta > 0$ we have

$$\int_\Omega |u|^{p+1}\,dx \leq \frac{\delta^{\frac{2^*(s)}{p+1}}(p+1)}{2^*(s)}\int_\Omega |u|^{2^*(s)}\,dx + \frac{2^*(s)-p-1}{2^*(s)}\delta^{-\frac{2^*(s)}{2^*(s)-p-1}}|\Omega|.$$

We choose δ so that

$$\frac{(p+1)\delta^{\frac{2^*(s)}{p+1}}}{2^*(s)} = \frac{\lambda}{22^*(s)d^s}.$$

Thus

$$\Phi_\lambda(t) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\lambda}{22^*(s)d^s}\int_\Omega |u|^{2^*(s)}\,dx - \frac{2^*(s)-p-1}{2^*(s)(p+1)}\delta^{-\frac{2^*(s)}{2^*(s)-p-1}}|\Omega|.$$

This inequality shows that Φ_λ is coercive. It is clear that Φ_λ is weakly lower semicontinuous in $H^1(\Omega)$. Moreover, for $t > 0$ small enough

$$\Phi_\lambda(t) = \frac{\lambda t^{2^*(s)}}{2^*(s)}\int_\Omega \frac{dx}{|x|^s} - \frac{t^{p+1}}{p+1}|\Omega| < 0.$$

Hence $\infty < \inf_{u\in H^1(\Omega)}, \Phi_\lambda(u) < 0$ and the existence of a minimizer follows from Theorem 1.2 in [20]. The second part of this theorem follows from the following inequality

$$\frac{\lambda}{d^s}\int_\Omega |u_\lambda|^{2^*(s)}\,dx \le \int_\Omega |\nabla u_\lambda|^2\,dx + \lambda\int_\Omega \frac{|u_\lambda|^{2^*(s)}}{|x|^s}\,dx = \\ = \int_\Omega |u_\lambda|^{p+1}\,dx \le \frac{p+1}{2^*(s)}\int_\Omega |u_\lambda|^{2^*(s)}\,dx + \frac{2^*(s)-p-1}{2^*(s)}|\Omega|. \qquad \square$$

Case (ii).

In this case we were unable to find a solution for problem (6.1) through a constrained minimization. Following the argument used for problem (1.1) in this case, we observe that if u is a solution of problem (6.1) then

$$\lambda\int_\Omega \frac{|u|^{2^*}}{|x|^s}\,dx = \int_\Omega |u|^{p+1}\,dx.$$

This yields $\lambda d^{-s} < 1$. As in the case of problem (1.1) we introduce a stronger condition

$$\lambda\int_\Omega \frac{dx}{|x|^s} < |\Omega| \tag{6.2}$$

which obviously implies that $\lambda d^{-s} < 1$. Under assumption (6.2) the constrained minimization does not produce a solution for problem (6.1). Indeed, let

$$m = \inf\left\{\int_\Omega |\nabla u|^2\,dx\colon u \in H^1(\Omega),\ \int_\Omega \Big(1 - \frac{\lambda}{|x|^s}\Big)|u|^{p+1}\,dx = 1\right\}.$$

By (6.2) a constant function $\big(\int_\Omega \big(1 - \frac{\lambda}{|x|^2}\big)\,dx\big)^{-\frac{1}{p+1}}$ belongs to the set of constraints and consequently $m = 0$.

Case (iii).

First, we show that the functional Φ_λ has a mountain-pass structure. For $2 < p+1 \le 2^*$ we set

$$S_p = \inf_{u\in H^1(\Omega)-\{0\}} \frac{\int_\Omega \big(|\nabla u|^2 + u^2\big)\,dx}{\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}.$$

Proposition 6.2. *Let $2^*(s) < p+1 \le 2^*$. Then for every $\lambda > 0$ there exist constants $0 < \rho < 1$ and $\kappa > 0$ such that*

$$\Phi_\lambda(u) \ge \kappa \ \text{ for } \|u\| = \rho.$$

Proof. Since $\|u\| = \rho < 1$, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\int\limits_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2^*(s)d^s}\int\limits_\Omega |u|^{2^*(s)}\,dx - \frac{1}{p+1}\int\limits_\Omega |u|^{p+1}\,dx \geq$$

$$\geq \frac{1}{2}\int\limits_\Omega |\nabla u|^2\,dx + \frac{\lambda}{2^*(s)d^s}|\Omega|^{1-\frac{2^*(s)}{2}}\Big(\int\limits_\Omega u^2\,dx\Big)^{\frac{2^*(s)}{2}} - \frac{1}{p+1}\int\limits_\Omega |u|^{p+1}\,dx \geq$$

$$\geq \frac{1}{2}\Big(\int\limits_\Omega |\nabla u|^2\,dx\Big)^{\frac{2^*(s)}{2}} + \frac{\lambda}{2^*(s)d^s}|\Omega|^{1-\frac{2^*(s)}{2}}\Big(\int\limits_\Omega u^2\,dx\Big)^{\frac{2^*(s)}{2}} -$$

$$- \frac{1}{p+1}\int\limits_\Omega |u|^{p+1}\,dx.$$

Let $c_1 = \min\big(\frac{1}{2}, \frac{\lambda}{2^*(s)d^s}|\Omega|^{1-\frac{2^*(s)}{2}}\big)$. Then

$$\Phi_\lambda(u) \geq c_1 2^{\frac{2-2^*(s)}{2}}\Big(\int\limits_\Omega \big(|\nabla u|^2 + u^2\big)\,dx\Big)^{\frac{2^*(s)}{2}} -$$

$$- \frac{1}{p+1}S_p^{-\frac{p+1}{2}}\Big(\int\limits_\Omega \big(|\nabla u|^2 + u^2\big)\,dx\Big)^{\frac{p+1}{2}} =$$

$$= c_1 2^{\frac{2-2^*(s)}{2}}\rho^{2^*(s)} - \frac{1}{p+1}S_p^{-\frac{p+1}{2}}\rho^{p+1}.$$

Taking $\rho \in (0,1)$ sufficiently small the result follows. □

Proposition 6.3. *The following holds:*

(i) *Let $2^*(s) < p+1 = 2^*$ for some $s \in (0,2)$. Then Φ_λ satisfies the $(PS)_c$ condition for*

$$c < \frac{1}{2}\Big(\frac{1}{2^*(s)} - \frac{1}{p+1}\Big)S^{\frac{N}{2}}.$$

(ii) *If $2^*(s) < p+1 < 2^*$ for some $s \in (0,2)$, then the $(PS)_c$ condition holds for all $c \geq 0$.*

Proof. (i) Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence for Φ_λ, that is $\Phi_\lambda(u_n) \to c$ and $\Phi'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$. First, we show that the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$. We have

$$c + o(1) + o(\|u_n\|) = \Phi_\lambda(u_n) - \frac{1}{2^*(s)}\langle\Phi'_\lambda(u_n), u_n\rangle = \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big)\int\limits_\Omega |\nabla u_n|^2\,dx +$$

$$+ \Big(\frac{1}{2^*(s)} - \frac{1}{p+1}\Big)\int\limits_\Omega |u_n|^{p+1}\,dx.$$

From this we deduce that

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u_n|^{p+1}\,dx \le C\big(1+\|u_n\|\big) \tag{6.3}$$

for some constant $C>0$. Since

$$\int_\Omega u_n^2\,dx \le |\Omega|^{1-\frac{2}{p+1}}\Big(\int_\Omega |u_n|^{p+1}\,dx\Big)^{\frac{2}{p+1}},$$

we deduce that $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{p+1}(\Omega)$ and $L^{2^*(s)}(\Omega,|x|^{-s})$. By the P.L. Lions concentration-compactness principle there exist points $\{x_j\}\subset\bar\Omega$ and constants ν_j, μ_j, $j\in J$, γ_0, ν_0 and μ_0 such that (5.2)–(5.7) hold. Moreover, we have

$$\mu_j \le \nu_j, \quad j\in J, \tag{6.4}$$

and

$$\mu_0+\lambda\gamma_0 \le \nu_0. \tag{6.5}$$

It suffices to show that $\nu_j=\nu_0=0$ for $j\in J$. Assuming that $\nu_j>0$ for some $j\in J$, we derive from (6.4), (5.5) and (5.6) that $S^{\frac{N}{2}}\le\nu_j$ if $x_j\in\Omega$ and $\frac{S^{\frac{N}{2}}}{2}\le\nu_j$ if $x_j\in\partial\Omega$. Similarly, if $\nu_0>0$, then $\frac{S^{\frac{N}{2}}}{2}\le\nu_0$, as μ_0 and ν_0 satisfy the inequality (5.6). We then have

$$\frac12\Big(\frac{1}{2^*(s)}-\frac{1}{p+1}\Big)S^{\frac{N}{2}} > c+o(1) = \Phi_\lambda(u_n)-\frac{1}{2^*(s)}\langle\Phi_\lambda(u_n),u_n\rangle =$$
$$= \Big(\frac12-\frac{1}{2^*(s)}\Big)\int_\Omega|\nabla u_n|^2\,dx + \Big(\frac{1}{2^*(s)}-\frac{1}{p+1}\Big)\int_\Omega|u_n|^{p+1}\,dx.$$

Letting $n\to\infty$ we derive in all these cases that

$$\frac12\Big(\frac{1}{2^*(s)}-\frac{1}{p+1}\Big)S^{\frac{N}{2}} > \Big(\frac12-\frac{1}{2^*(s)}\Big)\int_\Omega|\nabla u|^2\,dx+\frac12\Big(\frac{1}{2^*(s)}-\frac{1}{p+1}\Big)S^{\frac{N}{2}}$$

which is impossible. The proof of assertion (ii) is standard and is omitted. □

Let $\phi\in H^1(\Omega)-\{0\}$. Then for $t>0$ sufficiently large, we have $\Phi_\lambda(t\phi)<0$ and $\|t\phi\|>\rho$. Thus the functional Φ_λ has a mountain-pass structure for every $\lambda>0$. If $2^*(s)<p+1<2^*$, then $(PS)_c$ condition holds for every $c>0$ and we are in a position to formulate the following existence result:

Theorem 6.4. *Let $2^*(s)<p+1<2^*$ for some $s\in(0,2)$. Then problem* (6.1) *has a solution for every $\lambda>0$.*

In the case $2^*(s)<p+1=2^*$ we have the following existence result.

Theorem 6.5. *Let* $2^*(s) < p+1 = 2^*$ *for some* $s \in (0,2)$*. Then there exists a constant* $\Lambda > 0$ *such that for every* $\lambda \in (0,\Lambda)$ *problem* (6.1) *has a solution.*

Proof. We choose a constant $T > 0$ such that $\Phi_\lambda(T) < 0$ and $\|T\| > \rho$. We set

$$\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) \colon \gamma(0) = 0,\, \gamma(1) = T\}.$$

Since the path $\gamma(\sigma) = \sigma T$, $0 \le \sigma \le 1$, belongs to Γ, we have

$$\Phi_\lambda(\sigma T) \le \max_{t \ge 0} \Phi_\lambda(t) = \frac{(p+1-2^*(s))}{(p+1)2^*(s)} \frac{\left(\lambda \int\limits_\Omega \frac{dx}{|x|^s}\right)^{\frac{p+1}{p+1-2^*(s)}}}{|\Omega|^{\frac{2^*(s)}{p+1-2^*(s)}}}.$$

Thus there exists a constant $\Lambda > 0$ such that

$$\inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\gamma(\gamma(t)) \le \frac{(p+1-2^*(s))}{(p+1)2^*(s)} \frac{\left(\lambda \int\limits_\Omega \frac{dx}{|x|^s}\right)^{\frac{p+1}{p+1-2^*(s)}}}{|\Omega|^{\frac{2^*(s)}{p+1-2^*(s)}}} < \frac{1}{2}\Big(\frac{1}{2^*(s)} - \frac{1}{p+1}\Big) S^{\frac{N}{2}}$$

for $0 < \lambda < \Lambda$. Hence Proposition 6.3, together with the mountain-pass principle yield, the existence of a solution of problem (6.1). □

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