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ASYMPTOTICS OF THE DISCRETE SPECTRUM FOR COMPLEX JACOBI MATRICES

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Abstract. The spectral properties and the asymptotic behaviour of the discrete spectrum for a special class of infinite tridiagonal matrices are given. We derive the asymptotic formulae for eigenvalues of unbounded complex Jacobi matrices acting in $l^2(\mathbb{N})$.

Keywords: tridiagonal matrix, complex Jacobi matrix, discrete spectrum, eigenvalue, asymptotic formula, unbounded operator, Riesz projection.

Mathematics Subject Classification: 47B36, 47B37, 47B06, 47A75, 15A18.

1. INTRODUCTION

Spectral properties of non-symmetric tridiagonal matrices and complex Jacobi matrices are investigated by several authors: Beckermann and Kaliaguine ([1, 2]), Djakov and Mityagin ([7, 8]), Egorova and Golinskii ([10, 11]) and others (see, e.g., [3,12–14,21,23]). The connections of tridiagonal matrices with formal orthogonal polynomials on the complex plane, Mathieu equation and functions, and Bessel functions can be found in [1–3, 7, 13, 21] and [25]. However, systematic research concerning spectral properties of non-selfadjoint tridiagonal operators is difficult because the structure of complex sequences can be more complicated then the structure of real sequences. Moreover, the spectral theorem and its consequences fail in this case. Nevertheless, some properties of real Jacobi matrices can be carried over to the complex tridiagonal matrices. We observe that effective research methods for non-selfadjoint operators use the Riesz projections instead of the spectral theorem (see [8, 16]).

The asymptotic behaviour of eigenvalues for selfadjoint Jacobi matrices was investigated with the use of several methods, which could be found for instance in [4, 5, 9, 15–17] and [24]. In this article we show that the asymptotic formulae for the point spectrum of unbounded discrete operators given by special classes of tridiagonal complex matrices are also true. We generalize the results obtained for selfadjoint Jacobi matrices in [18] and [19].

Consider a complex tridiagonal infinite matrix

$$J((d_n),(a_n),(b_n)) = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & \cdots \\ b_1 & d_2 & a_2 & 0 & \ddots \\ 0 & b_2 & d_3 & a_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.1}$$

where d_n, a_n, $b_n \in \mathbb{C} \setminus \{0\}$. The matrix $J((d_n),(a_n),(b_n))$ defines a linear operator which acts on a maximal domain

$$Dom(J) = \{(f_n)_{n=1}^{\infty} \in l^2 : \ (b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1})_{n=1}^{\infty} \in l^2\}$$

and

$$(Jf)_n = b_{n-1}f_{n-1} + d_n f_n + a_n f_{n+1}, \quad n \geq 1,$$

for $f = (f_n)_{n=1}^{\infty} \in Dom(J)$ and $b_0 = 0$. In the second section we establish primary properties of tridiagonal operators.

Then the paper is organized as follows. In the third section we consider the symmetrization procedure of tridiagonal matrices (1.1) to complex Jacobi matrices

$$J_s = J((d_n),(c_n),(c_n)), \tag{1.2}$$

where

$$c_n^2 = a_n b_n, \ n \geq 1. \tag{1.3}$$

In [2] Beckermann and Kaliaguine proved that the resolvent set of the operator J_s contains the resolvent set of the tridiagonal operator J, for which (1.3) holds. We prove that, for some classes of tridiagonal matrices, the symmetrized complex Jacobi matrices preserve the discrete spectrum.

The fourth section is devoted to a generalized version of the result, which was proved by Janas and Naboko in [16]. If discrete operators are near-similar in the sense of Rozenbljum ([20]), then we expect that their point spectra are asymptotically close. This result concerns asymptotic behaviour of the point spectrum for a compact perturbation of a diagonal discrete operator and it is essential for diagonalization methods.

In the last section we discuss that the method of diagonalization ([18]) can be easily applied for complex Jacobi matrices to obtain the asymptotic formulae for the point spectrum of some tridiagonal matrices.

2. PRELIMINARIES

Assume that the complex sequences $(d_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ satisfy the following conditions:

(A1) $|d_n| \to \infty$ as $n \to \infty$;

(A2)
$$\bigcup_{n=1}^{\infty} B(d_n, r_n) \neq \mathbb{C},$$
where $r_n = |a_n| + |b_n| + |a_{n-1}| + |b_{n-1}|$ and $B(d_n, r_n) = \{\lambda \in \mathbb{C} \colon |d_n - \lambda| \leq r_n\}$ for $n \geq 1$, $a_0 = b_0 = 0$;

(A3)
$$\frac{1}{|d_n|} \sum_{k=-1}^{1} (|a_{n+k}| + |b_{n+k}|) \to 0 \text{ as } n \to \infty.$$

Let $D = \mathrm{Diag}((d_n))$ be an operator in l^2 given by a diagonal matrix

$$\begin{pmatrix} d_1 & 0 & \ddots & & \\ 0 & d_2 & 0 & \ddots & \\ 0 & 0 & d_3 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

Proposition 2.1. *If $J = J((d_n), (a_n), (b_n))$ and* (A1)–(A3) *hold, then:*

(i) *$Dom(J) = Dom(D) = \{(f_n)_{n=1}^{\infty} \in l^2 : \ (d_n f_n)_{n=1}^{\infty} \in l^2\}$;*

(ii) *J is a densely defined and closed operator in l^2;*

(iii) *the spectrum of J is discrete and*
$$\sigma(J) = \{\lambda_n(J) \in \mathbb{C} : \ n \geq 1\},$$
where $\lambda_n(J)$ is an eigenvalue of J $(n \geq 1)$ and the sequence $(\lambda_n(J))_{n=1}^{\infty}$ can be ordered such that
$$|\lambda_n(J)| \to \infty, \ n \to \infty;$$

(iv)
$$\sigma(J) \subset \bigcup_{n=1}^{\infty} B(d_n, \rho_n),$$
where $\rho_n = |a_n| + |b_{n-1}|$ and $B(d_n, \rho_n) = \{z \in \mathbb{C} : |z - d_n| \leq \rho_n\}$;

(v) *J^* is given by the matrix $J((\overline{d_n}), (\overline{b_n}), (\overline{a_n}))$.*

Proof. First we prove that all eigenvalues of J are contained in $\bigcup_{n=1}^{\infty} B(d_n, \rho_n)$. Indeed, let $\lambda \in \mathbb{C}$ be an eigenvalue for J and $Jf = \lambda f$, where $f = (f_n) \in l^2 \setminus \{0\}$. There exists $k \in \mathbb{N}$ such that $|f_k| = \max\{|f_n| : \ n \geq 1\}$ and

$$b_{k-1} f_{k-1} + (d_k - \lambda) f_k + a_k f_{k+1} = 0.$$

Then

$$|d_k - \lambda| \le |a_k||f_{k+1}|/|f_k| + |b_{k-1}||f_{k-1}|/|f_k| \le |a_k| + |b_{k-1}|$$

and $\lambda \in B(d_k, \rho_k)$.

Conditions (A1)–(A3) entail

$$\mathbb{C} \neq \bigcup_{n=1}^{\infty} B(d_n, \rho_n).$$

Therefore, there exists

$$z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B(d_n, \rho_n) \tag{2.2}$$

and $(D - z)^{-1}$ is a compact operator.

From definition of $Dom(J)$ we observe that $Dom(D) \subset Dom(J)$. For $f \in Dom(D)$

$$(J - z)f = \left(I + (J - D)(D - z)^{-1}\right)(D - z)f. \tag{2.3}$$

Moreover, $(J - D)(D - z)^{-1}$ is also compact under (A1) and (A3).

If $I + (J - D)(D - z)^{-1}$ is not invertible then there exists an eigenvector $g \neq 0$ for this operator. Put $g' = (D - z)^{-1} g$. From (2.3) we derive

$$(J - z)g' = (I + (J - D)(D - z)^{-1})g = 0$$

but this relation is impossible because of (2.2).

Thus $I + (J - D)(D - z)^{-1}$ is an invertible operator on l^2 and from (2.3) we derive

$$(J - z)^{-1} = (D - z)^{-1}(I + (J - D)(D - z)^{-1})^{-1}. \tag{2.4}$$

Equation (2.4) implies that $(J - z)^{-1}$ is compact and J is closed. Moreover, from (2.3) we deduce that $Dom(J) = Dom(D)$. So (i) and (ii) are proved.

Because $(J - z)^{-1}$ is compact, then we know the structure of its spectrum

$$\sigma((J - z)^{-1}) = \{0\} \cup \{z_n :, n \ge 1\},$$

where z_n, $n = 1, 2, \ldots$, is an eigenvalue of $(J - z)^{-1}$ and $z_n \to 0$ as $n \to \infty$. Also

$$\sigma(J) = \left\{ \frac{1}{z_n} + z : \; n \ge 1 \right\}$$

and the sequence of eigenvalues tends to infinity. So (iii) is satisfied. Notice that (iv) is also true because the spectrum of J consists of the eigenvalues only.

To prove (v) denote $J^+ = J((\overline{d_n}), (\overline{b_n}), (\overline{a_n}))$ and notice that $(Jf, g) = (f, J^+ g)$ for $f \in Dom(D)$ and $g \in l^2$. Then $g \in Dom(J^*)$ if and only if $J^+ g \in {}^2$. Therefore, $Dom(J^*) = Dom(J^+)$ and $J^* = J^+$. $\square$

Similar results on tridiagonal matrices are included in [7, 8] and [23]. The Gershgorin type theorem for infinite matrices acting as operators in l^∞ or l^1 can be find in [22].

3. COMPLEX JACOBI MATRICES AND SYMMETRIZATION

Let $J = J((d_n), (a_n), (b_n))$ and consider a complex Jacobi matrix

$$J_s = J((d_n), (c_n), (c_n)),$$

where $c_n \in \mathbb{C} \setminus \{0\}$ and $c_n^2 = a_n b_n,\ n \geq 1$. Choose a complex sequence (α_n) such that $\alpha_1 = 1$ and

$$\alpha_n^2 = \frac{a_{n-1}}{b_{n-1}}\alpha_{n-1}^2 = \frac{a_{n-1}a_{n-2}\cdot \ldots \cdot a_1}{b_{n-1}b_{n-2}\cdot \ldots \cdot b_1},\ \ n \geq 2, \tag{3.1}$$

and put

$$A = \mathrm{Diag}((\alpha_n)).$$

Then the formal matrix equation

$$AJ = J_s A. \tag{3.2}$$

is satisfied.

In [2] Beckermann and Kaliaguine proved that if J is bounded then the resolvent set of J is contained in the resolvent set of the symmetrized operator J_s.

Proposition 3.1. *If* (A1)–(A3) *are satisfied and* J, J_s *are operators associated with* (1.1) *and* (1.2)*, respectively, acting on the maximal domains, then*

$$\sigma(J) = \sigma(J_s).$$

Proof. Due to Proposition 2.1, under conditions (A1)–(A3), the spectra of J and J_s are discrete. Let $\lambda \in \sigma(J)$ and $f = (f_n) \in Dom(J)$ be an eigenvector of J associated with λ. From (A1) and (A3) we deduce that $|d_n - \lambda| > |a_n|$ and $\frac{|b_{n-1}|}{|d_n-\lambda|-|a_n|} < 1$ for $n \geq n_0$, where n_0 is large enough.

Let $n \geq n_0$. The sequence $(f_k) \in l^2$ converges to 0, so there is $k \geq n$ such that $|f_{k+1}| \leq |f_k|$. Then from the spectral equation $Jf = \lambda f$ we derive

$$|b_{k-1}f_{k-1}| = |(d_k - \lambda)f_k + a_k f_{k+1}| \geq |(d_n - \lambda)f_k| - |a_k f_{k+1}| \geq (|d_n - \lambda| - |a_n|)|f_k|,$$

so

$$|f_k| \leq \frac{|b_{k-1}|}{|d_k - \lambda| - |a_k|}|f_{k-1}| \leq |f_{k-1}|.$$

Then by the mathematical induciton reasoning we deduce that

$$|f_j| \leq \frac{|b_{j-1}|}{|d_j - \lambda| - |a_j|}|f_{j-1}| \leq |f_{j-1}|$$

for $j \in \{n_0, \ldots, k\}$. Thus

$$|f_n| \leq \frac{|b_{n-1}|}{|d_n - \lambda| - |a_n|}|f_{n-1}|$$

and

$$|f_n| \le \frac{|b_{n-1}|}{|d_n-\lambda|-|a_n|} \cdot \frac{|b_{n-2}|}{|d_{n-1}-\lambda|-|a_{n-1}|} \cdot \ldots \cdot \frac{|b_{n_0-1}|}{|d_{n_0}-\lambda|-|a_{n_0}|}|f_{n_0-1}|, \tag{3.3}$$

for $n > n_0$.

Equation (3.2) entails also the formal matrix equality

$$J_s Af = AJf = \lambda Af \tag{3.4}$$

for the eigenvector f of J; therefore, it is enough to prove that $Af \in l^2$. Notice that by (3.1) and (3.3) we obtain

$$|\alpha_n f_n|^2 \le \frac{|a_{n-1}||b_{n-1}|}{(|d_n-\lambda|-|a_n|)^2} \frac{|a_{n-2}||b_{n-2}|}{(|d_{n-1}-\lambda|-|a_{n-1}|)^2} \cdot \ldots \cdot \frac{|a_{n_0-1}||b_{n_0-1}|}{(|d_{n_0}-\lambda|-|a_{n_0}|)^2} \cdot C_0 =: P_n,$$

$n > n_0$. By (A1) and (A3),

$$P_{n+1}/P_n = \frac{|a_n||b_n|}{(|d_{n+1}-\lambda|-|a_{n+1}|)^2} \to 0,\ n \to \infty,$$

so $\sum_{n=n_0}^{\infty} |\alpha_n f_n|^2 < +\infty$ and $Af \in l^2$. Also from (3.4) we derive that $Af \in Dom(J_s)$, so $\lambda \in \sigma(J_s)$.

Now, assume $\lambda \in \mathbb{C}$ and $\lambda \in \sigma(J_s)$ and $J_s f = \lambda f$, where $f \in Dom(J_s) \setminus \{0\} \subset l^2$. Then the estimates

$$|f_n| \le \frac{|c_{n-1}|}{|d_n-\lambda|-|c_n|}|f_{n-1}| \le$$

$$\le \frac{|c_{n-1}|}{|d_n-\lambda|-|c_n|} \cdot \frac{|c_{n-2}|}{|d_{n-1}-\lambda|-|c_{n-1}|} \cdot \ldots \cdot \frac{|c_{n_0-1}|}{|d_{n_0}-\lambda|-|c_{n_0}|}|f_{n_0-1}|,$$

for $n \ge n_0$, where n_0 is large enough, can be obtained by the same method as inequality (3.3). Moreover, from (3.2) we derive the matrix equation

$$JA^{-1}f = A^{-1}J_s f = \lambda A^{-1} f,$$

where $A^{-1} = \mathrm{Diag}((\frac{1}{\alpha_n}))$. Next, the estimates

$$|\frac{1}{\alpha_n} f_n|^2 \le \frac{|b_{n-1}|^2}{(|d_n-\lambda|-|c_n|)^2} \cdot \frac{|b_{n-2}|^2}{(|d_{n-1}-\lambda|-|c_{n-1}|)^2} \cdot \ldots \cdot \frac{|b_{n_0-1}|^2}{(|d_{n_0}-\lambda|-|c_{n_0}|)^2} C_1,$$

satisfied for $n \ge n_0$, yield $A^{-1}f \in l^2$. Finally $\lambda \in \sigma(J)$ and $A^{-1}f \in Dom(J)$ is an eigenvector of J corresponding to the eigenvalue λ. □

4. ASYMPTOTIC BEHAVIOUR OF THE POINT SPECTRUM OF A COMPACTLY PERTURBED DIAGONAL OPERATOR

Let us introduce the following condition.

(CΓ) Let Γ_n, $n \geq 1$, be a sequence of closed curves on $\mathbb{C}$ and there exists an increasing non-negative sequence (p_n) such that

$$\{z \in \mathbb{C} : |z| \leq p_n\} \subset \mathrm{int}\Gamma_n, \quad n \geq 1,$$

and

$$\lim_{n\to\infty} \frac{|\Gamma_n|}{p_n^2} = 0, \quad \lim_{n\to\infty} p_n = +\infty,$$

where $\mathrm{int}\Gamma_n$ denotes the set surrounded by Γ_n and $|\Gamma_n|$ means the length of Γ_n.

The following theorem generalize the lemma, which concerns asymptotic behaviour of the point spectrum for a compact perturbation of a diagonal discrete operator, given by Janas and Naboko ([16]).

Theorem 4.1 ([16])**.** *Let H be a separable complex Hilbert space and $\{e_n : n \geq 1\}$ be an orthonormal basis for H. Let D be a diagonal operator in H given by a diagonal matrix $Diag((d_n))$ with respect $\{e_n : n \geq 1\}$. Assume that the complex sequence $(d_n)_{n=1}^{\infty}$ satisfies:*

1. $|d_n - d_k| \geq \epsilon_0 > 0$ *for* $d_n \neq d_k$;
2. *there is a sequence of closed Jordan curves $\{\Gamma_n\}_{n\geq 1}$, satisfying* (CΓ)*, such that*

$$\mathrm{dist}(\Gamma_n, d_k) \geq \epsilon_0/4 \quad for \quad k, n \geq 1$$

and

$$\int\limits_{\Gamma_n} \|(D-\lambda)^{-1}\|^2 |d\lambda| \leq C, \quad n \geq 1,$$

where $\mathrm{dist}(\Gamma_n, d_k) = \inf\{|\lambda - d_k| : \lambda \in \Gamma_n\}$;

where ϵ_0, $C > 0$ are independent on n.

Let K be a compact operator in H and $T = D + K$. Then the spectrum of T is discrete and consists of the complex eigenvalues $\lambda_n(T)$, $n \geq 1$, which can be arranged such that

$$\lambda_n(T) = d_n + O(\|K^* P_n\|) \quad \text{as} \quad n \to \infty,$$

where P_n is an orthogonal projection on the finite-dimensional space generated by $\{e_k : d_k = d_n\}$.

The proof of Theorem 4.1 is a consequence of the next two lemmas.

Lemma 4.2. *Assume the operators D, K and T are operators described in Theorem 4.1. Let*

$$P_n f = \sum_{k:d_k=d_n} (f, e_k) e_k \quad for \quad f \in H, \tag{4.1}$$

$$r_n = 12\|K^*P_n\| \tag{4.2}$$

and

$$\gamma_n = \{z \in \mathbb{C} : |z - d_n| = r_n\}.$$

Let

$$P_{n,D} = \frac{1}{2\pi i}\int_{\gamma_n} (\lambda - D)^{-1} d\lambda, \quad P_{n,T} := \frac{1}{2\pi i}\int_{\gamma_n} (\lambda - T)^{-1} d\lambda \tag{4.3}$$

be Riesz projections. Then, under assumption (1) *of Theorem 4.1, there exists* n_0 *such that* $\gamma_n \cap \sigma(T) = \emptyset$ *and*

$$\|P_{n,D} - P_{n,T}\| < 1$$

for $n \geq n_0$.

Proof. The idea of this proof directly comes from the paper [16]. K^* is compact, so

$$K^* = \sum_{k=1}^{\infty} s_k(\cdot, \psi_k)\varphi_k, \tag{4.4}$$

where $\{\psi_k : k = 1, 2, \ldots\}$ and $\{\varphi_k : k = 1, 2, \ldots\}$ are suitable orthogonal bases in H and the sequence of singular numbers s_k, $k \geq 1$, is decreasing and tends to 0. Moreover, $\|K^*P_n\| \to 0$ as $n \to \infty$, and

$$r_n = 12\|K^*P_n\| < \epsilon_0/2 \tag{4.5}$$

for $n \geq n_0$, where n_0 is large enough.

Let P_n be given by (4.1) and define

$$P_n^{\perp} := I - P_n.$$

For $\lambda \in \gamma_n$ consider the following estimate

$$\|K^*(D^* - \bar{\lambda})^{-1}\| \leq \|K^*(D^* - \bar{\lambda})^{-1}P_n\| + \|K^*(D^* - \bar{\lambda})^{-1}P_n^{\perp}\|. \tag{4.6}$$

Then

$$\|K^*(D^* - \bar{\lambda})^{-1}P_n f\| = \|K^*P_n f\|/|d_n - \lambda| \leq (\|K^*P_n\|/|d_n - \lambda|)\|f\|,$$

so

$$\|K^*(D^* - \bar{\lambda})^{-1}P_n\| \leq \|K^*P_n\|/|d_n - \lambda| \leq 1/12 \tag{4.7}$$

for $\lambda \in \gamma_n$. Obviously,

$$\|(D - \lambda)^{-1}\| = \|(D^* - \bar{\lambda})^{-1}\| = 1/r_n$$

and

$$\|(D^* - \bar{\lambda})^{-1}P_n^{\perp}\| = 1/(\min_{d_k \neq d_n} |d_k - \lambda|) \leq 1/(\min_{d_k \neq d_n} |d_k - d_n| - r_n) \leq 2/\epsilon_0 \tag{4.8}$$

for $\lambda \in \gamma_n$.

For $f \in H$

$$\begin{aligned}\|K^*(D^*-\bar\lambda)^{-1}P_n^\perp f\|^2 &= \|\sum_{k=1}^{\infty} s_k((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)\varphi_k\|^2 = \\ &= \sum_{k=1}^{N} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 + \\ &\quad + \sum_{k=N+1}^{\infty} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2\end{aligned}$$

and

$$\begin{aligned}B_2(\lambda,N,n) := \sum_{k=N+1}^{\infty} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \\ \le s_{N+1}^2 \sum_{k=N+1}^{\infty} |((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \\ \le s_{N+1}^2\|(D^*-\bar\lambda)^{-1}P_n^\perp f\|^2 \le \\ \le \frac{4s_{N+1}^2}{\epsilon_0^2}\|f\|^2 \le \frac{1}{32}\|f\|^2\end{aligned}$$

for large enough N and $\lambda \in \gamma_n$, $n \ge n_0$.

Now we are going to prove that

$$B_1(\lambda,N,n) := \sum_{k=1}^{N} s_k^2|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)|^2 \le \frac{1}{32}\|f\|^2$$

for $n \ge N_0$ and large N_0. Let

$$Q_l f = (f,e_l)e_l, \quad l \ge 1.$$

The following estimates are true:

$$\begin{aligned}&|((D^*-\bar\lambda)^{-1}P_n^\perp f,\psi_k)| \le \\ &\le |((D^*-\bar\lambda)^{-1}P_n^\perp f,\sum_{l=1}^{L} P_l\psi_k)| + |((D^*-\bar\lambda)^{-1}P_n^\perp f,(I-\sum_{l=1}^{L} P_l)\psi_k)|\end{aligned}$$

and, by (4.8),

$$|((D^*-\bar\lambda)^{-1}P_n^\perp f,(I-\sum_{l=1}^{L} P_l)\psi_k)| \le 2/\epsilon_0\|f\|\|(I-\sum_{l=1}^{L} P_l)\psi_k\| \le \frac{1}{16s_1\sqrt{N}}\|f\|$$

for $k = 1,2,\ldots,N$ and large enough L.

If L is as large as above, then

$$
\begin{aligned}
|((D^*-\bar\lambda)^{-1}P_n^{\perp}f,\sum_{l=1}^{L}P_l\psi_k)| &\le \sum_{l=1}^{L}|(P_l(D^*-\bar\lambda)^{-1}P_n^{\perp}f,\psi_k)|\le\\
&\le \sum_{l=1}^{L}|(P_l(D^*-\bar\lambda)^{-1}\|\|f\|\le\|f\|\sum_{l=1}^{L}\frac{1}{|\overline{d_l}-\bar\lambda|}\le\\
&\le \|f\|\sum_{l=1}^{L}\frac{1}{||d_n|-|d_l||-|\overline{d_n}-\bar\lambda|}\le\\
&\le \|f\|\sum_{l=1}^{L}\frac{1}{|d_n|-|d_l|-r_n}\le\|f\|\frac{1}{16s_1\sqrt{N}}
\end{aligned}
$$

for $\lambda\in\gamma_n$, $n\ge N_0$ and large enough N_0 because $|d_k|\to+\infty$ as $k\to\infty$. Thus

$$|((D^*-\bar\lambda)^{-1}P_n^{\perp}f,\psi_k)|\le\frac{1}{8s_1\sqrt{N}}\|f\|$$

and

$$B_1(\lambda,N,n)\le\sum_{k=1}^{N}s_k^2\left(\frac{1}{8s_1\sqrt{N}}\|f\|\right)^2\le\frac{1}{32}\|f\|^2.$$

Then for $n\ge N_0$ and $\lambda\in\gamma_n$

$$\|K^*(D^*-\bar\lambda)^{-1}P_n^{\perp}f\|^2\le B_1(\lambda,N,n)+B_2(\lambda,N,n)\le\frac{1}{16}\|f\|^2$$

so

$$\|K^*(D^*-\bar\lambda)^{-1}P_n^{\perp}\|\le 1/4 \tag{4.9}$$

and, by (4.6),(4.7) and (4.9),

$$\|K^*(D^*-\bar\lambda)^{-1}\|\le 1/3$$

for $n\le N_0$ and $\lambda\in\gamma_n$.

Therefore,

$$\sup_{\lambda\in\gamma_n}\|(D-\lambda)^{-1}K\|=\sup_{\lambda\in\gamma_n}\|K^*(D^*-\bar\lambda)^{-1}\|\le 1/3$$

for $n\ge N_0$. Moreover, $T-\lambda=(D-\lambda)^{-1}(I+(D-\lambda)^{-1}K)$ is invertible for $\lambda\in\gamma_n$.

Finally we observe that the Riesz projections (4.3) satisfy the following estimates

$$
\begin{aligned}
\|P_{n,T}-P_{n,D}\| &= \frac{1}{2\pi}\Big\|\int_{\gamma_n}((\lambda-T)^{-1}-(\lambda-D)^{-1})d\lambda\Big\|=\\
&=\frac{1}{2\pi}\Big\|\int_{\gamma_n}((\lambda-D)^{-1}\left(I-K(\lambda-D)^{-1}\right)^{-1}-(\lambda-D)^{-1})d\lambda\Big\|=
\end{aligned}
$$

$$= \frac{1}{2\pi}\Big\| \int\limits_{\gamma_n} \sum_{k=1}^{\infty}[(\lambda - D)^{-1}K]^k(\lambda - D)^{-1}d\lambda\Big\| \leq$$

$$\leq \frac{1}{2\pi}\int\limits_{\gamma_n} \sum_{k=1}^{\infty} \|(\lambda - D)^{-1}K\|^k\|(\lambda - D)^{-1}\||d\lambda| \leq$$

$$\leq \frac{1}{2\pi}\int\limits_{\gamma_n} \sum_{k=1}^{\infty}(1/3)^k\|(\lambda - D)^{-1}\||d\lambda| \leq \frac{1}{2\pi}|\gamma_n|\frac{1}{2r_n} = 1/2 < 1.$$

$\square$

Lemma 4.3. *Under assumptions of Theorem 4.1*

$$\Gamma_n \cap \sigma(T) = \Gamma_n \cap \sigma(D) = \emptyset$$

and the Riesz projections

$$\widetilde{P_{n,T}} = \frac{1}{2\pi i}\int_{\Gamma_n}(\lambda - T)^{-1}d\lambda, \quad \widetilde{P_{n,D}} = \frac{1}{2\pi i}\int\limits_{\Gamma_n}(\lambda - D)^{-1}d\lambda \tag{4.10}$$

satisfy

$$\|\widetilde{P_{n,T}} - \widetilde{P_{n,D}}\| < 1$$

for enough large n.

Proof. For $\lambda \in \Gamma_n$ and $f \in H$ we can write that

$$\|K^*(D^* - \bar{\lambda})^{-1}f\|^2 = \sum_{k=1}^{N} s_k^2|((D^* - \bar{\lambda})^{-1}f, \psi_k)|^2 + \sum_{k=N+1}^{\infty} s_k^2|((D^* - \bar{\lambda})^{-1}f, \psi_k)|^2.$$

Notice that $|d_l - \lambda| \geq \epsilon_0/4$, $l \geq 1$, so $\|(D^* - \bar{\lambda})^{-1}\| \leq 4/\epsilon_0$ because $\lambda \in \Gamma_n$. Then

$$\sum_{k=N+1}^{\infty} s_k^2|((D^* - \bar{\lambda})^{-1}f, \psi_k)|^2 \leq s_{N+1}^2\|(D^* - \bar{\lambda})^{-1}f\|^2 \leq$$

$$\leq s_{N+1}^2(4/\epsilon_0)^2\|f\|^2 \leq \frac{1}{18}\|f\|^2,$$

if N is large enough because $s_N \to 0$ as $N \to \infty$.

Let $g \in H$. Then

$$(D^* - \bar{\lambda})^{-1}g = (D^* - \bar{\lambda})^{-1}\left(\sum_{l=1}^{\infty}(g, e_l)e_l\right) = \sum_{l=1}^{\infty}\frac{(g, e_l)}{\overline{d_l} - \bar{\lambda}}e_l, \tag{4.11}$$

$$\|(D^* - \bar{\lambda})^{-1}g\|^2 = \sum_{l=1}^{\infty}\frac{|(g, e_l)|^2}{|\overline{d_l} - \bar{\lambda}|^2} \leq (4/\epsilon_0)^2\|g\|^2. \tag{4.12}$$

For a fixed large N we have

$$\sum_{k=1}^{N} s_k^2|((D^*-\bar{\lambda})^{-1}f,\psi_k)|^2 = \sum_{k=1}^{N} s_k^2|(f,(D-\lambda)^{-1}\psi_k)|^2 \le N s_1^2\|f\| \max_{1\le k\le N}\|(D-\lambda)^{-1}\psi_k\|^2$$

and

$$\sup_{\lambda\in\Gamma_n}\|(D-\lambda)^{-1}\psi_k\|^2 \le \sum_{l=1}^{\infty}\left[|(\psi_k,e_l)|^2\left(\sup_{\lambda\in\Gamma_n}\frac{1}{|d_l-\lambda|^2}\right)\right].$$

It is clear that

$$\sup_{\lambda\in\Gamma_n}\left(\frac{1}{|d_l-\lambda|^2}\right)\to 0,\ n\to\infty, \tag{4.13}$$

for all $l\ge 1$, so

$$\Sigma(n,k) := \sum_{l=1}^{\infty}\left[|(\psi_k,e_l)|^2\left(\sup_{\lambda\in\Gamma_n}\frac{1}{|d_l-\lambda|^2}\right)\right] \le$$

$$\le \sum_{l=1}^{\infty}|(\psi_k,e_l)|^2(4/\epsilon_0)^2 \le (4/\epsilon_0)^2\|\psi_k\|^2 < +\infty$$

because of (4.11) and (4.12) applied to ψ_k. Moreover, by the dominated Lebesgue theorem, (4.13) and the above estimates,

$$\Sigma(n,k)\to 0 \quad \text{as} \quad n\to\infty.$$

Therefore,

$$\sup_{\lambda\in\Gamma_n}\sum_{k=1}^{N} s_k^2|((D^*-\bar{\lambda})^{-1}f,\psi_k)|^2 \le N s_1^2\|f\|^2\epsilon_n,$$

where

$$\epsilon_n = \max_{1\le k\le N}\left(\sup_{\lambda\in\Gamma_n}\|(D-\lambda)^{-1}\psi_k\|^2\right)\to 0,\ n\to\infty.$$

Let n_0 be such that $\epsilon_n \le \frac{1}{18 s_1^2 N}$ for $n\ge n_0$. Then

$$\|K^*(D^*-\bar{\lambda})^{-1}f\|^2 \le \frac{1}{9}\|f\|^2,$$

for $f\in H$ and $\lambda\in\Gamma_n$.

Finally,

$$\sup_{\lambda\in\Gamma_n}\|(D-\lambda)^{-1}K\| \le 1/3$$

for $n\ge n_0$.

Then $I + (D - \lambda)^{-1}K$ $(\lambda \in \Gamma_n)$ is invertible in H and the equation $T - \lambda = (D - \lambda)(I + (D - \lambda)^{-1}K)$ yields $\sigma(T) \cup \Gamma_n = \emptyset$ for large n. Moreover,

$$\begin{aligned}
\|\widetilde{P_{n,T}} - \widetilde{P_{n,D}}\| &\le \frac{1}{2\pi} \int\limits_{\Gamma_n} \|(\lambda - T)^{-1} - (\lambda - D)^{-1}\| |d\lambda| = \\
&= \frac{1}{2\pi} \int\limits_{\Gamma_n} \|(\lambda - T)^{-1} K (\lambda - D)^{-1}\| |d\lambda| = \\
&= \frac{1}{2\pi} \int\limits_{\Gamma_n} \|(I + \sum_{k=1}^{\infty} [(\lambda - D)^{-1}K]^k)(\lambda - D)^{-1} K (\lambda - D)^{-1}\| |d\lambda| \le \\
&\le \frac{1}{2\pi} \int\limits_{\Gamma_n} \left(1 + \sum_{k=1}^{\infty} \|(\lambda - D)^{-1}K\|^k\right) \|(\lambda - D)^{-1} K (\lambda - D)^{-1}\| |d\lambda| \le \\
&\le \frac{1}{2\pi} \int\limits_{\Gamma_n} \left(\sum_{k=0}^{\infty} (1/3)^k\right) \|(\lambda - D)^{-1} K (\lambda - D)^{-1}\| |d\lambda| \le \\
&\le \frac{3}{4\pi} \int\limits_{\Gamma_n} \|(\lambda - D)^{-1} K (\lambda - D)^{-1}\| |d\lambda|.
\end{aligned}$$

From (4.4) we also derive

$$K = \sum_{k=1}^{\infty} s_k(\cdot, \varphi_k)\psi_k = K_1 + \sum_{k=1}^{M} s_k(\cdot, \varphi_k)\psi_k,$$

where $\|K_1\| \le 1/C$. Then

$$\begin{aligned}
&\|\widetilde{P_T} - \widetilde{P_D}\| \le \\
&\le \frac{3}{4\pi} \left(\int\limits_{\Gamma_n} \|K_1\| \|(D - \lambda)^{-1}\|^2 |d\lambda| + \right. \\
&\quad \left. + \sum_{k=1}^{M} \int\limits_{\Gamma_n} s_k \|(\cdot, (D^* - \overline{\lambda})^{-1}\varphi_k)(D - \lambda)^{-1}\psi_k\| |d\lambda| \right) \le \\
&\le \frac{3}{4\pi} \|K_1\| C + \frac{3}{4\pi} \sum_{k=1}^{M} \int\limits_{\Gamma_n} s_k \|(D^* - \overline{\lambda})^{-1}\varphi_k\| \|(D - \lambda)^{-1}\psi_k\| |d\lambda| \le \\
&\le \frac{3}{4\pi} + \\
&\quad + \frac{3 s_1 M}{4\pi} \max_{1 \le k \le M} \left[\left(\int\limits_{\Gamma_n} \|(D^* - \overline{\lambda})^{-1}\varphi_k\|^2 |d\lambda| \right)^{1/2} \left(\int\limits_{\Gamma_n} \|(D - \lambda)^{-1}\psi_k\|^2 |d\lambda| \right)^{1/2} \right].
\end{aligned}$$

For $k \geq 1$ and $g \in H$, due to (CΓ) and assumption (2),

$$\int\limits_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2}|d\lambda| \leq |(g,e_k)|^2 \int_{\Gamma_n} \|(D-\lambda)^{-1}\|^2 |d\lambda| \leq C|(g,e_k)|^2 \tag{4.14}$$

and

$$\int\limits_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2}|d\lambda| \leq \|g\|^2 \int\limits_{\Gamma_n} \frac{1}{|d_k-\lambda|^2}|d\lambda| \leq \|g\|^2 |\Gamma_n|/(p_n-|d_k|)^2 \to 0 \tag{4.15}$$

as $n \to \infty$.

Because of (4.14), (4.15) and the dominated Lebesgue theorem, we deduce

$$\int\limits_{\Gamma_n} \|(D-\lambda)^{-1}g\|^2 |d\lambda| = \int\limits_{\Gamma_n} \|(D^*-\overline{\lambda})^{-1}g\|^2 |d\lambda| = \sum_{k=1}^{\infty} \left(\int\limits_{\Gamma_n} \frac{|(g,e_k)|^2}{|d_k-\lambda|^2} \right) |d\lambda| \to 0,$$

as $n \to \infty$ for all $g \in H$. Therefore, if

$$\widetilde{\epsilon_n} = \max_{1\leq k\leq M} \left[\left(\int\limits_{\Gamma_n} \|(D^*-\overline{\lambda})^{-1}\varphi_k\|^2 |d\lambda| \right)^{1/2} \left(\int\limits_{\Gamma_n} \|(D-\lambda)^{-1}\psi_k\|^2 |d\lambda| \right)^{1/2} \right],$$

then $\widetilde{\epsilon_n} \to 0$ as $n \to \infty$.

Then, finally,

$$\|\widetilde{P_T} - \widetilde{P_D}\| \leq \frac{3}{4\pi}(1 + Ms_1\widetilde{\epsilon_0}) < 1$$

for $n \geq \widetilde{n_0}$, where $\widetilde{n_0}$ is large enough. □

Proof of Theorem 4.1. Denote rank$P$ = dim$P(H)$, where P is a projection in H. Lemma 4.2 implies

$$\text{rank}P_T = \text{rank}P_D = M_n, \tag{4.16}$$

where

$$M_n = \sharp\{k : \, d_k = d_n\},$$

so the sum of the algebraic multiplicities of eigenvalues of T in $\{z \in \mathbb{C} : \, |z-d_n| \leq r_n\}$ equals M_n for $n \geq N_0$.

Moreover, from Lemma 4.3 for (4.10) we deduce

$$\text{rank}\widetilde{P_{n,T}} = \text{rank}\widetilde{P_{n,D}} < +\infty \tag{4.17}$$

for $n \geq \widetilde{n_0}$.

Taking into account (4.16) and (4.17) we see that the sequences, in which the algebraic multiplicities of the eigenvalues of T and D are taken into consideration, coincide and

$$|\lambda_n(T) - d_n| \leq r_n$$

for large n, where r_n is given by (4.2). □

Remark 4.4. Consider a complex sequence $(d_n)_{n=1}^{\infty}$ and the following conditions:

1. $|d_{n+1}| > |d_n|$ for $n \geq n_0$, $\lim_{n\to\infty} |d_n| = +\infty$, and $\left(\frac{|d_{n+1}|+|d_n|}{(|d_{n+1}|-|d_n|)^2}\right)_{n=1}^{\infty}$ is bounded;
2. $d_n = \epsilon_n n^{\alpha}(1 + o(\frac{1}{n}))$, $|\epsilon_n| = \epsilon > 0$ for $n \geq 1$, where $\alpha \geq 2$;
3. $d_n = \epsilon_n \left([\frac{n+T-1}{T}]\right)^{\alpha} (1 + o(\frac{1}{n}))$, $|\epsilon_n| = \epsilon > 0$ for $n \geq 1$, where $\alpha \geq 1$, and (ϵ_n) is a periodic sequence with the period equal to T, $\epsilon_n \neq \epsilon_k$ for $k \neq n$, and $k, n = 1, \ldots, T$, and $[x] = \max\{k \in \mathbb{N} :\ k \leq x\}$ for $x \in \mathbb{R}$.

If one of the above conditions is true then the assumptions of Theorem 4.1 are satisfied for the sequence $(d_n)_{n=1}^{\infty}$.

5. DIAGONALIZATION FOR COMPLEX JACOBI MATRICES

In this section we assume (A1)–(A3) and (1.3). Due to Proposition 3.1, we assume without loss of generality that $J = J((d_n), (c_n), (c_n))$ is a complex Jacobi matrix.

Let $1 \leq k \leq l$. Then denote

$$J_l^k = \begin{pmatrix} d_k & c_k & & \\ c_k & d_{k+1} & \ddots & \\ & \ddots & \ddots & c_{l-1} \\ & & c_{l-1} & d_l \end{pmatrix} \tag{5.1}$$

and

$$D_l^k(\lambda) = \det(J_l^k - \lambda). \tag{5.2}$$

Assume also $D_{k-1}^k(\lambda) = 1$.

For $n > q$ denote

$$D_n = \begin{pmatrix} d_{n-q} & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & d_{n+q} \end{pmatrix}. \tag{5.3}$$

Let

$$c_n'(q) = \max\{|c_k| :\ k \leq n+q\}, \quad R_n = 6c_n'(q) \tag{5.4}$$

and let

$$C(d_n, R_n) = \{z \in \mathbb{C} :\ |z - d_n| = R_n\}$$

be positively oriented on the complex plane.

Define the Riesz projection for J_{n+q}^{n-q} and D_n:

$$P_{1n} = \frac{1}{2\pi i} \int\limits_{C(d_n,R_n)} (\lambda - J_{n+q}^{n-q})^{-1} d\lambda, \quad P_{2n} = \frac{1}{2\pi i} \int\limits_{C(d_n,R_n)} (\lambda - D_n)^{-1} d\lambda. \tag{5.5}$$

Lemma 5.1. *If $|d_n - d_{n+j}| \geq 2R_n$ for $j \in \{\pm 1, \pm 2, \ldots, \pm q\}$, then $\|P_{1n} - P_{2n}\| < 1$ and there exists exactly one eigenvalue λ_n of J_{n+q}^{n-q} such that $|d_n - \lambda_n| \leq R_n$.*

Proof. At first notice that $\|(\lambda - D_n)^{-1}\| \leq 1/R_n$ and $\|C_n(\lambda - D_n)^{-1}\| \leq 1/3$, where

$$C_n = J_{n+q}^{n-q} - D_n, \tag{5.6}$$

provided that $|\lambda - d_n| = R_n$. Indeed,

$$C_n(\lambda - D_n)^{-1} = \begin{pmatrix} 0 & \frac{c_{n-q}}{\lambda - d_{n-q+1}} & & \\ \frac{c_{n-q}}{\lambda - d_{n-q}} & 0 & \ddots & \\ & \ddots & \ddots & \frac{c_{n+q-1}}{\lambda - d_{n+q}} \\ & & \frac{c_{n+q-1}}{\lambda - d_{n+q-1}} & 0 \end{pmatrix},$$

$$\frac{|c_{n+j}|}{|\lambda - d_{n+j+1}|} \leq \frac{c'_n(q)}{R_n}$$

and

$$\frac{|c_{n+j}|}{|\lambda - d_{n+j}|} \leq \frac{c'_n(q)}{R_n}$$

for $\lambda = d_n + R_n e^{it} \in C(d_n, R_n)$ and $-q \leq j \leq q-1$. Thus $\|C_n(\lambda - D_n)^{-1}\| \leq 2c'_n(q)/R_n \leq 1/3$.

Next, we observe that

$$\begin{aligned}(\lambda - J_{n+q}^{n-q})^{-1} &= (\lambda - D_n)^{-1} \left(I - C_n(\lambda - D_n)^{-1}\right)^{-1} = \\ &= (\lambda - D_n)^{-1} \left(I + \sum_{k=1}^{\infty} [C_n(\lambda - D_n)^{-1}]^k\right)\end{aligned}$$

and

$$(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1} = (\lambda - D_n)^{-1} \left(\sum_{k=1}^{\infty} [C_n(\lambda - D_n)^{-1}]^k\right).$$

So

$$\|(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1}\| \leq \frac{1}{R_n} \left(\sum_{k=1}^{\infty} (1/3)^k\right) = 1/(2R_n)$$

and

$$\|P_{1n} - P_{2n}\| \leq \frac{1}{2\pi} \int\limits_{C(d_n, R_n)} \|(\lambda - J_{n+q}^{n-q})^{-1} - (\lambda - D_n)^{-1}\| |d\lambda| \leq 1/2 < 1.$$

Clearly, the Riesz projection P_{2n} has the one-dimensional range generated by an eigenvector associated with the eigenvalue d_n. Then $\operatorname{rank} P_{1n} = \operatorname{rank} P_{2n} = 1$, so J_{n+q}^{n-q} has also a unique eigenvalue λ_n in the ball $\{z : \ |d_n - z| \leq R_n\}$. $\square$

Assume that the complex sequences (d_n) and (c_n) satisfy the following properties. There exist $\alpha, \beta \geq 0$ and $p \in \{1, 2, \ldots\}$ such that $\alpha > \frac{p+1}{p}\beta + 1$ and

(C1) $|d_n - d_{n+j}| \geq \rho_p n^{\alpha-1}$ for $j = \pm 1, \pm 2, \ldots, \pm p$, large n, where $\rho_p > 0$ is independent on n;
(C2) $|c_n| = O(n^\beta)$ as $n \to \infty$.

Choose the complex sequence $(\lambda_n)_{n=1}^\infty$, which satisfy the following condition:

(E) λ_n is an eigenvalue of J_{n+p-1}^{n-p+1} such that $|d_n - \lambda_n| \leq R_n$ for large enough n, where

$$R_n = 6 \max\{|c_k| : \ k \leq n+p\}.$$

(For small n we assume that λ_n is an eigenvalue of J_{n+p-1}^1.)

By Lemma 5.1, the sequence $(\lambda_n)_{n=1}^\infty$ is well defined because (C1) and (C2) imply that $|d_n - d_{n+j}| \geq 2R_n$ for $j = \pm 1, \pm 2, \ldots \pm p$ if n is large.

The method of diagonalization for real Jacobi matrices is described in [15, 16] and [18]. Conditions (C1), (C2) and (E) are sufficient to perform the diagonalization procedure presented in [18].

For fixed n denote

$$\varphi^{(n)} = \begin{pmatrix} f_{n-p+1}^{(n)}(\lambda_n) \\ \vdots \\ f_n^{(n)}(\lambda_n) \\ \vdots \\ f_{n+p-1}^{(n)}(\lambda_n) \end{pmatrix} \tag{5.7}$$

an eigenvector of J_{n+p-1}^{n-p+1} associated with λ_n. Notice that$f_n^{(n)}(\lambda_n) \neq 0$ if n is large enough; therefore, we may assume

$$f_n^{(n)}(\lambda_n) = 1. \tag{5.8}$$

Further, put

$$f_{n-p}^{(n)}(\lambda_n) = \frac{-c_{n-p}}{d_{n-p} - \lambda_n} f_{n-p+1}^{(n)}(\lambda_n), \tag{5.9}$$

$$f_{n+p}^{(n)}(\lambda_n) = \frac{-c_{n+p-1}}{d_{n+p} - \lambda_n} f_{n+p-1}^{(n)}(\lambda_n) \tag{5.10}$$

and

$$f_{n \pm j}^{(n)}(\lambda_n) = 0, \ \text{ for } \ j \geq p+1. \tag{5.11}$$

Finally, for $n \geq 1$, we put

$$f^{(n)} = \left(\overline{f_k^{(n)}(\lambda_n)}\right)_{k=1}^\infty . \tag{5.12}$$

The properties of the system of the sequences $f^{(n)}$, $n \geq 1$, are expressed in the following lemma and proposition.

Lemma 5.2. *Assume* (C1) *and* (C2) *hold. Let* $f_k^{(n)}(\lambda_n)$, $n, k \geq 1$, *be given by* (5.7)-(5.11) *and* $(\lambda_n)_{n=1}^{\infty}$ *satisfy* (E). *Then*

$$f_{n\pm j}^{(n)}(\lambda_n) = O\left(\frac{1}{n^{j(\alpha-\beta-1)}}\right), n \to \infty,$$

for $j \in \{1, 2, \ldots, p\}$.

Proof. In [18] the analogous result is proved for real sequences (d_n) and (c_n) but the proof can be rewritten for a complex case. □

Proposition 5.3. *If* $n \geq 1$ *is large enough, then*

$$\begin{aligned}(J^* - \overline{\lambda_n})f^{(n)} =: R^{(n)} = \overline{c_{n-p-1}}\overline{f_{n-p}^{(n)}(\lambda_n)}e_{n-p-1} + \overline{c_{n-p}}\overline{f_{n-p}^{(n)}(\lambda_n)}e_{n-p+1} + \\ + \overline{c_{n+p-1}}\overline{f_{n+p}^{(n)}(\lambda_n)}e_{n+p-1} + \overline{c_{n+p}}\overline{f_{n+p}^{(n)}(\lambda_n)}e_{n+p+1},\end{aligned} \tag{5.13}$$

where $\{e_k : k \geq 1\}$ *is a canonical basis of* l^2.

Proof. Straightforward calculations lead to (5.13) because $J^* = J((\overline{d_n}), (\overline{c_n}), (\overline{c_n}))$. □

Let $f^{(n)}$, $n \geq 1$, be given by (5.12). Define

$$F = \left(f^{(1)}; f^{(2)}; \ldots\right), \tag{5.14}$$

i.e. F is an infinite matrix, in which n-th column is given by the sequence $f^{(n)}$. We construct also an infinite matrix

$$R = \left(R^{(1)}; R^{(2)}; R^{(3)}; \ldots\right), \tag{5.15}$$

where $R^{(n)}$ is treated as an n-th column of R.

By S we denote the shift operator on l^2 given on the basis vectors as follows

$$Se_k = e_{k+1}, \quad k \geq 1.$$

Then S^* stands for the adjoint operator to S.

Notice that the structure of F has a band diagonal shape

$$F = I + G,$$

where $G = \sum_{j=1}^{p}(S^j W_j + V_j S^{*j})$ and W_j, V_j are diagonal operators

$$W_j = \mathrm{Diag}\left(\left(\overline{f_{n+j}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}\right), \ V_j = \mathrm{Diag}\left(\left(\overline{f_n^{(n+j)}(\lambda_{n+j})}\right)_{n=1}^{\infty}\right).$$

Lemma 5.2 yields G which is a compact operator because sequences $\left(\overline{f_{n+j}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}$ and $\left(\overline{f_n^{(n+j)}(\lambda_{n+j})}\right)_{n=1}^{\infty}$ $(j \in \{1, 2, \ldots, p\})$ converge to 0.

Moreover, the matrix R may also be written in a 4-diagonal form

$$R = S^{p-1}A_1 + S^{p+1}A_2 + B_1 S^{*p-1} + B_2^{*p+1},$$

where A_1, A_2, B_1, B_2 are diagonal operators

$$A_1 = \mathrm{Diag}\left(\left(\overline{c_{n+p-1}f_{n+p}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}\right), \quad \mathrm{diag}A_2 = \mathrm{Diag}\left(\left(\overline{c_{n+p}f_{n+p}^{(n)}(\lambda_n)}\right)_{n=1}^{\infty}\right).$$

$$B_1 = \mathrm{Diag}\left(\left(\overline{c_{n-1}f_{n-1}^{(n-1+p)}(\lambda_{n-1+p})}\right)_{n=1}^{\infty}\right), \quad (c_0 = 0),$$

$$B_2 = \mathrm{Diag}\left(\left(\overline{c_n f_{n+1}^{(n+p+1)}(\lambda_{n+p+1})}\right)_{n=1}^{\infty}\right).$$

Then we observe that R is a compact operator in l^2. Indeed, from Lemma 5.2 we derive

$$c_{n\pm p}f_{n\pm p}^{(n)}(\lambda_n) = O\left(\frac{n^{\beta}}{n^{p(\alpha-\beta-1)}}\right) = O\left(n^{-p(\alpha-\frac{p+1}{p}\beta-1)}\right)$$

and

$$c_{n\pm p-1}f_{n\pm p}^{(n)}(\lambda_n) = O\left(n^{-p(\alpha-\frac{p+1}{p}\beta-1)}\right), \; n \to \infty,$$

so the sequences above converge to 0, because of the choice of the constant p in (C1) and (C2). Moreover,

$$\|Re_n\| = \|R^{(n)}\| = O\left(n^{-p(\alpha-\frac{p+1}{p}\beta-1)}\right), \; n \to \infty. \tag{5.16}$$

Let

$$\Lambda = \mathrm{Diag}\left((\lambda_n)_{n=1}^{\infty}\right) \tag{5.17}$$

be a diagonal operator such that the sequence (λ_n) satisfies (E).

Proposition 5.4. *If* (C1) *and* (C2) *hold then J is similar to $\Lambda + K$, where K is a compact operator such that $\|K^* e_n\| = O\left(n^{-p(\alpha-\frac{p+1}{p}\beta-1)}\right)$, $n \to \infty$.*

Proof. We can rewrite (5.13) with use of an operator form

$$J^*F - F\Lambda^* = R + R', \tag{5.18}$$

where F is given by (5.14), R by (5.15) and R' is a finite dimensional operator, which is represented in the canonical basis by a matrix with a finite number of non-zero entries only. There exists an invertible in l^2 operator $\tilde{F}$ such that the matrix representation of $F - \tilde{F}$ has a finite number of non-zero entries. Therefore, (5.18) implies

$$J^*\tilde{F} = \tilde{F}\Lambda^* + R + M, \tag{5.19}$$

where $M = R' + J^*(\tilde{F} - F) - (\tilde{F} - F)\Lambda^*$ is such that the entries of the matrix M, except for a finite number, are equal to zero. From (5.19) we derive

$$\tilde{F}^*J = \Lambda\tilde{F}^* + R^* + M^*.$$

$\tilde{F}^*$ is invertible, so

$$J = (\tilde{F}^*)^{-1} \left(\Lambda + (R^* + M^*)(\tilde{F}^*)^{-1} \right) \tilde{F}^*,$$

i.e., J is a similar operator to $\Lambda + K$, where $K = (R^* + M^*)(\tilde{F}^*)^{-1}$. Obviously K is a compact operator because R^* and M^* are compact and $(\tilde{F}^*)^{-1}$ is bounded. Moreover,

$$\|K^* e_n\| = \|\tilde{F}^{-1}(R + M)e_n\| \leq C\|Re_n\|,$$

where $C > 0$ is a constant independent on n. Then we apply equation (5.16). □

Remark 5.5. The condition of similarity of operators preserves the structure of spectra, so

$$\lambda_n(J) = \lambda_n(\Lambda + K), \ \ n \geq 1.$$

Remark 5.6. If (d_n) satisfies one of the conditions given in Remark 4.4 and $\lambda_n = d_n + O(n^\beta)$ according to (E), then Remark 4.4 can be applied also to (λ_n). Therefore, from Theorem 4.1, we derive that the sequence of the eigenvalues of $\Lambda + K$ has the asymptotic

$$\lambda_n(\Lambda + K) = \lambda_n + O(\|K^* e_n\|) = \lambda_n + O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$$

as $n \to \infty$.

To complete the diagonalization procedure for the complex Jacobi matrices we show that under (C1) and (C2) the explicit asymptotic formulae for the sequence (λ_n), chosen by (E), can be found. We refer also to [18]. If $p \geq 2$, from the Laplace formula we derive

$$\begin{aligned} D^{n-p+1}_{n+p-1}(\lambda) = (d_n - \lambda) D^{n-p+1}_{n-1}(\lambda) D^{n+1}_{n+p-1}(\lambda) + \\ - c^2_{n-1} D^{n-p+1}_{n-2}(\lambda) D^{n+1}_{n+p-1}(\lambda) - c^2_n D^{n-p+1}_{n-1}(\lambda) D^{n+2}_{n+p-1}(\lambda). \end{aligned}$$

Notice that if n is large enough and $|\lambda - d_n| \leq R_n$, where R_n is fixed by (5.4), then the main diagonals of the matrices $J^{n-p+1}_{n-1} - \lambda$ and $J^{n+1}_{n+p-1} - \lambda$ dominate, so $D^{n-p+1}_{n-1}(\lambda) \neq 0$ and $D^{n+1}_{n+p-1}(\lambda) \neq 0$. Therefore, λ_n satisfies

$$\lambda_n = d_n - c^2_{n-1} \frac{D^{n-p+1}_{n-2}(\lambda_n)}{D^{n-p+1}_{n-1}(\lambda_n)} - c^2_n \frac{D^{n+2}_{n+p-1}(\lambda_n)}{D^{n+1}_{n+p-1}(\lambda_n)}. \tag{5.20}$$

For $n \geq 1$, denote

$$w_{p,n}(\lambda) = d_n - c^2_{n-1} \frac{D^{n-p+1}_{n-2}(\lambda)}{D^{n-p+1}_{n-1}(\lambda)} - c^2_n \frac{D^{n+2}_{n+p-1}(\lambda)}{D^{n+1}_{n+p-1}(\lambda)}. \tag{5.21}$$

Next denote

$$\lambda^{(1)}_n = d_n, \quad n \geq 1, \tag{5.22}$$

$$\lambda^{(j)}_n = w_{p,n}(\lambda^{(j-1)}_n), \quad n \geq 1, \quad j \geq 2. \tag{5.23}$$

(For $p = 1$ we simply take $\lambda_n = d_n$ and $w_{1,n}(\lambda) = d_n$.)

Proposition 5.7. *Let* λ_n *satisfy* (E) *and* $\lambda_n^{(k)}$ *for* $n \geq 1$ *be given by* (5.22) *and* (5.23), *where* $k = \min\{j \in \mathbb{N} : \ j \geq \frac{p+1}{2}\}$. *Under conditions* (C1) *and* (C2),

$$\lambda_n = \lambda_n^{(k)} + O\left(n^{-p(\alpha - \frac{p+1}{p}\beta - 1)}\right)$$

as $n \to \infty$.

Proof. The proof of the similar result can be found in [18]. □

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