Opuscula Math. 34, no. 1 (2014), 29–65
http://dx.doi.org/10.7494/OpMath.2014.34.1.29 Opuscula Mathematica

# $p$-ADIC BANACH SPACE OPERATORS AND ADELIC BANACH SPACE OPERATORS

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*Communicated by P.A. Cojuhari*

**Abstract.** In this paper, we study non-Archimedean Banach $*$-algebras $\mathfrak{M}_p$ over the $p$-adic number fields $\mathbb{Q}_p$, and $\mathfrak{M}_{\mathbb{Q}}$ over the adele ring $\mathbb{A}_{\mathbb{Q}}$. We call elements of $\mathfrak{M}_p$, *$p$-adic operators*, for all primes $p$, respectively, call those of $\mathfrak{M}_{\mathbb{Q}}$, *adelic operators*. We characterize $\mathfrak{M}_{\mathbb{Q}}$ in terms of $\mathfrak{M}_p$'s. Based on such a structure theorem of $\mathfrak{M}_{\mathbb{Q}}$, we introduce some interesting $p$-adic operators and adelic operators.

**Keywords:** prime fields, $p$-adic number fields, adele ring, $p$-adic Banach spaces, adelic Banach space, $p$-adic operators, adelic operators.

**Mathematics Subject Classification:** 05E15, 11G15, 11R47, 46L10, 47L30, 47L55.

## 1. INTRODUCTION

In this paper, we define Banach spaces $\mathcal{X}_p$ over *$p$-adic number fields* (or *$p$-prime fields*) $\mathbb{Q}_p$, and the Banach space $\mathcal{X}_{\mathbb{Q}}$ over the *adele ring* $\mathbb{A}_{\mathbb{Q}}$, and study Banach-space operators acting on $\mathcal{X}_p$, and those acting on $\mathcal{X}_{\mathbb{Q}}$, respectively. We call $\mathcal{X}_p$ and $\mathcal{X}_{\mathbb{Q}}$, the *$p$-prime Banach spaces* (over fields $\mathbb{Q}_p$) and the *adele-ring Banach space* (over a ring $\mathbb{A}_{\mathbb{Q}}$), respectively (see Section 3 below).

Matrices acting on $\mathbb{Q}_p^n$ and on $\mathbb{A}_{\mathbb{Q}}^n$ are considered in [3], and the structures of corresponding matricial algebras have been characterized. Remark here that the matrices are over $\mathbb{Q}_p$, respectively over $\mathbb{A}_{\mathbb{Q}}$.

In this paper, we study the case where $n = \infty$, under certain norm topologies. We define Banach spaces $\mathcal{X}_p$ and $\mathcal{X}_{\mathbb{Q}}$, consisting of sequences in $\mathbb{Q}_p$, respectively, in $\mathbb{A}_{\mathbb{Q}}$, and study *Banach-space operators* of $B(\mathcal{X}_p)$ (over $\mathbb{Q}_p$), called *$p$-adic operators*, and those of $B(\mathcal{X}_{\mathbb{Q}})$ (over $\mathbb{A}_{\mathbb{Q}}$), called *adelic operators*.

The main purpose of this paper is to study fundamental operator-theoretic properties of certain $p$-adic and adelic operators in terms of well-known number-theoretic results.

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In [3], we consider the relation between the matricial algebra $\mathfrak{M}_n = M_n(\mathbb{A}_{\mathbb{Q}})$, and the matricial algebras $\mathfrak{M}_{p:n} = M_n(\mathbb{Q}_p)$, motivated by [4] and [6]. We provide a way to study $\mathbb{A}_{\mathbb{Q}}$-matrices of $\mathfrak{M}_n$ in terms of its equivalent forms determined by $\mathbb{Q}_p$-matrices of $\mathfrak{M}_{p:n}$. In particular, $\mathfrak{M}_n$ is a Banach $*$-algebra, which is isomorphic to the *weak tensor product* (in the sense of Section 2.2 below) of $\mathfrak{M}_{p:n}$'s, for all $p \in \mathcal{P}$, i.e.,

$$\mathfrak{M}_n \overset{*\text{-iso}}{=} \underset{p\in\mathcal{P}}{\otimes_{\Theta}} \mathfrak{M}_{n:p},$$

induced by a system $\Theta = \{\Theta_p\}_{p\in\mathcal{P}}$ of certain morphisms

$$\Theta_p \colon \mathfrak{M}_{n:p} \to \mathfrak{M}_{n:p} \quad \text{for all} \quad p \in \mathcal{P},$$

where

$$\mathcal{P} \overset{def}{=} \{\infty\} \cup \{\text{all primes}\}.$$

In [5], we compute spectra of $\mathbb{Q}_p$-matrices of $\mathfrak{M}_{p:n}$, and those of $\mathbb{A}_{\mathbb{Q}}$-matrices of $\mathfrak{M}_n$. In particular, we showed that the $\mathbb{A}_{\mathbb{Q}}$-spectrum of an $\mathbb{A}_{\mathbb{Q}}$-matrix $A$ is computed by the $\mathbb{Q}_p$-spectra of $\mathbb{Q}_p$-matrices $A_p$, since

$$A = \Big[ (x_{p:ij})_{p\in\mathcal{P}} \Big]_{n\times n} \Longrightarrow A \overset{\text{equivalent}}{=} \underset{p\in\mathcal{P}}{\otimes} A_p,$$

where

$$A_p = [(x_{p:ij})]_{n\times n} \quad \text{for all} \quad p \in \mathcal{P}.$$

One may have a similar structure theorem for our case where $n = \infty$, under suitable topologies.

Analysis on $p$-adic number fields $\mathbb{Q}_p$ and that on the adele ring $\mathbb{A}_{\mathbb{Q}}$ is not only interesting, but also important in various mathematical fields and other scientific areas. In particular, *$p$-adic analysis* have been used for studying (non-Archimedean) structures with "small" distance (e.g., [1, 3, 4, 14] and [6]). *adelic analysis* on $\mathbb{A}_{\mathbb{Q}}$ is dictated by $p$-adic analysis by the very definition-and-construction of $\mathbb{A}_{\mathbb{Q}}$. Recall that the adele ring $\mathbb{A}_{\mathbb{Q}}$ is a *weak direct product* of prime fields $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e., in our sense (of Section 2.2), it is the weak tensor product $\underset{p\in\mathcal{P}}{\Pi_g} \mathbb{Q}_p$ induced by the system $g = \{g_p\}_{p\in\mathcal{P}}$ of the surjective functions $g_p$ from $\mathbb{Q}_p$ onto its unit disks $\mathbb{Z}_p$, traditionally denoted by

$$\mathbb{A}_{\mathbb{Q}} = \underset{p\in\mathcal{P}}{\Pi'} \mathbb{Q},$$

i.e., analysis on $\mathbb{Q}_p$ and $\mathbb{A}_{\mathbb{Q}}$ provides new paradigms and tools for studying non-Archimedean geometry of structures with small distances (e.g., [14]).

Mainly, the prime fields $\mathbb{Q}_p$ and the adele ring $\mathbb{A}_{\mathbb{Q}}$ are playing key roles in modern *number theory*, connected with *analytic number theory* and *algebraic geometry* (e.g., [6, 9, 10] and [15]). Recently, the author and Gillespie have shown that they are also closely related to operator algebraic structures via *free probability* (e.g., see [1] through [6]).

After submission, the author realized that there is a series of recent research of Kochubei, considering non-Archimedean operator theory (see [17–19] and [20]). In those papers, Kochubei study certain operators on non-Archimedean normed spaces, and define non-Archimedean version of normality, unitarity and shifting of operators.

Different from Kochubei's universal approach, here, we consider operators on non-Archimedean normed spaces as forms of infinite matrices, based on the author's recent interests; connecting number theory with operator theory; concentrated on $p$-adic analysis, adelic analysis (e.g., [1–3] and [6]), and free probability on arithmetic functions (e.g., [4,5,7] and [8]), purely motivated by modern number theory. In particular, the non-Archimedean normed spaces $\{\mathcal{X}_p\}_{p:primes}$ and $\mathcal{X}_{\mathbb{Q}}$ in this paper are constructed directly from $p$-adic number fields $\{\mathbb{Q}_p\}_{p:primes}$, respectively, the adele ring $\mathbb{A}_{\mathbb{Q}}$. The fundamental reason to handle such specific non-Archimedean normed spaces; $\{\mathcal{X}_p\}_{p:primes}$, $\mathcal{X}_{\mathbb{Q}}$; is to connect modern number-theoretic objects-and-results to operator theory via possible operator-algebraic tools, including representation theory and free probability, and vice versa.

In this paper, we focus on establishing backgrounds of such a study. One may/can apply these backgrounds to more deeper and developed researches to connect number theory and operator theory.

## 2. DEFINITIONS AND BACKGROUNDS

In this section, we introduce basic definitions and backgrounds of our study.

### 2.1. THE ADELE RING $\mathbb{A}_{\mathbb{Q}}$

*Fundamental theorem of arithmetic* says that every positive integer in the integer $\mathbb{Z}$ except 1 can be expressed as a usual multiplication of *primes* (or prime numbers), equivalently, all positive integers which are not 1 are *prime-factorized* under multiplication. And hence, all negative integers $n$, except $-1$, can be understood as products of $-1$ and prime-factorizations of $|n|$. Thus, primes are playing key roles in both classical and advanced *number theory*.

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is one of the main topics in advanced number theory connected with other mathematical fields like *algebraic geometry* and *L-function theory*, etc. Throughout this paper, we denote the set of all natural numbers (which are positive integers) by $\mathbb{N}$, and the set of all rational numbers by $\mathbb{Q}$.

Let us fix a prime $p$. Define *the p-norm* $|\cdot|_p$ on $\mathbb{Q}$ by

$$|q|_p = \left|p^r \frac{a}{b}\right|_p \overset{def}{=} \frac{1}{p^r},$$

whenever $q = p^r \frac{a}{b} \in \mathbb{Q}^{\times} = \mathbb{Q} \setminus \{0\}$, for some $r \in \mathbb{Z}$, with an additional identity:

$$|0|_p \overset{def}{=} 0 \quad \text{(for all primes } p\text{)}.$$

For example,

$$\left|-\frac{24}{5}\right|_2 = \left|2^3 \cdot \left(-\frac{3}{5}\right)\right|_2 = \frac{1}{2^3} = \frac{1}{8}$$

and

$$\left|\frac{1}{24}\right|_2 = \left|2^{-3} \cdot 3^{-1}\right| = \frac{1}{2^{-3}} = 8.$$

It is easy to check that:

(i) $|q|_p \geq 0$ for all $q \in \mathbb{Q}$,
(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$ for all $q_1,\, q_2 \in \mathbb{Q}$,
(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p\,,\, |q_2|_p\}$ for all $q_1,\, q_2 \in \mathbb{Q}$.

In particular, by (iii), we verify that

(iii)' $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p$ for all $q_1,\, q_2 \in \mathbb{Q}$.

Thus, by (i), (ii) and (iii)' the $p$-norm $|\cdot|_p$ is indeed a norm. However, by (iii), this norm is "*non-Archimedean*". Thus, the pair $(\mathbb{Q},\, |\cdot|_p)$ forms a normed space, for each prime $p$.

**Definition 2.1.** We define sets $\mathbb{Q}_p$ by the $p$-norm-closures of the normed spaces $(\mathbb{Q}, |\cdot|_p)$, for all primes $p$. We call it the $p$-prime field (or the $p$-adic number field).

For a fixed prime $p$, all elements of the $p$-prime field $\mathbb{Q}_p$ are formed by

$$p^r \left( \sum_{k=0}^{\infty} a_k p^k \right) \quad \text{for} \quad 0 \leq a_k < p, \tag{2.1}$$

for all $r \in \mathbb{Z}$, where $a_k \in \mathbb{N}_0 \overset{def}{=} \mathbb{N} \cup \{0\}$. For example,

$$-1 = (p-1)p^0 + (p-1)p + (p-1)p^2 + \dots.$$

The subset $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is the set consisting of all elements formed by

$$\sum_{k=0}^{\infty} a_k p^k \quad \text{for} \quad 0 \leq a_k < p \quad \text{in} \quad \mathbb{N}_0.$$

So, by definition, for any $x \in \mathbb{Q}_p$, there exist $r \in \mathbb{Z}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^r x_0.$$

Notice that if $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$, and vice versa, i.e.,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\}. \tag{2.2}$$

The subset $\mathbb{Z}_p$ of (2.2) is said to be the *unit disk of* $\mathbb{Q}_p$, for all primes $p$. Remark that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \dots,$$

since if $x \in p^k\, \mathbb{Z}_p$, then $|x|_p \leq \frac{1}{p^k}$ for all $k \in \mathbb{N}$, where $aY$ means $\{ay\colon y \in Y\}$ for all $a \in \mathbb{Q}_p$, and for all subsets $Y$ of $\mathbb{Q}_p$. Similarly, one can verify that

$$\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \dots,$$

and hence

$$\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k\mathbb{Z}_p, \quad \text{set-theoretically.} \tag{2.3}$$

Consider the boundary $U_p$ of $\mathbb{Z}_p$. By construction, the boundary $U_p$ of $\mathbb{Z}_p$ is identical to

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Z}_p\colon |x|_p = 1\}. \tag{2.4}$$

Similarly, the subsets $p^kU_p$ are the boundaries of $p^k\mathbb{Z}_p$ satisfying

$$p^kU_p = p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p \quad \text{for all} \quad k \in \mathbb{Z}.$$

We call the subset $U_p$ of $\mathbb{Z}_p$ in (2.4) the *unit circle of* $\mathbb{Q}_p$, and all elements of $U_p$ are said to be *units of* $\mathbb{Q}_p$.

Therefore, by (2.3) and (2.4), one obtains that

$$\mathbb{Q}_p = \bigsqcup_{k=-\infty}^{\infty} p^kU_p, \quad \text{set-theoretically,} \tag{2.5}$$

where $\sqcup$ means the disjoint union.

**Fact 2.2** ([14])**.** *The p-prime field $\mathbb{Q}_p$ is a Banach space and it is locally compact. In particular, the unit disk $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$.*

Define now the addition on $\mathbb{Q}_p$ by

$$\left(\sum_{n=-N_1}^{\infty} a_np^n\right) + \left(\sum_{n=-N_2}^{\infty} b_np^n\right) = \sum_{n=-\max\{N_1,N_2\}}^{\infty} c_np^n \tag{2.6}$$

for $N_1,\ N_2 \in \mathbb{N}$, where the summands $c_np^n$ satisfies that

$$c_np^n \overset{def}{=} \begin{cases} (a_n+b_n)p^n & \text{if } a_n+b_n < p, \\ p^{n+1} & \text{if } a_n+b_n = p, \\ s_np^{n+1} + r_np^n & \text{if } a_n+b_n = s_np + r_n, \end{cases}$$

for all $n \in \{-\max\{N_1,N_2\},\dots,0,1,2,\dots\}$.

Next, define the multiplication of two units "in $\mathbb{Q}_p$" by

$$\left(\sum_{k_1=0}^{\infty} a_{k_1}p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_2}p^{k_2}\right) = \sum_{n=-N}^{\infty} c_np^n, \tag{2.7}$$

where

$$c_n = \sum_{k_1+k_2=n} \left(r_{k_1,k_2}i_{k_1,k_2} + s_{k_1-1,k_2}i^c_{k_1-1,k_2} + s_{k_1,k_2-1}i^c_{k_1,k_2-1} + s_{k_1-1,k_2-1}i^c_{k_1-1,\ k_2-1}\right),$$

where

$$a_{k_1}b_{k_2} = s_{k_1,k_2}p + r_{k_1,k_2},$$

by the division algorithm, and

$$i_{k_1,k_2} = \begin{cases} 1 & \text{if } a_{k_1}b_{k_2} < p, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$i^c_{k_1,k_2} = 1 - i_{k_1,k_2}$$

for all $k_1, k_2 \in \mathbb{N}_0$. So, "on $\mathbb{Q}_p$", the multiplication is well-defined by

$$\begin{aligned} &\left(\sum_{k_1=-N_1}^{\infty} a_{k_1}p^{k_1}\right)\left(\sum_{k_2=-N_2}^{\infty} b_{k_2}p^{k_2}\right) = \\ &= \left(p^{-N_1}\right)\left(p^{-N_2}\right)\left(\sum_{k_1=0}^{\infty} a_{k_1-N_1}p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_1-N_2}p^{k_2}\right). \end{aligned} \tag{2.8}$$

Then, under the addition (2.6) and the multiplication (2.8), the algebraic triple $(\mathbb{Q}_p, +, \cdot)$ becomes a field for all primes $p$. Thus the $p$-prime fields $\mathbb{Q}_p$ are algebraically fields.

**Fact 2.3.** *Every $p$-prime field $\mathbb{Q}_p$ with the binary operations* (2.6) *and* (2.8) *is a field.*

Moreover, the Banach filed $\mathbb{Q}_p$ is also a (unbounded) Haar-measure space $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)$, for all primes $p$, where $\sigma(\mathbb{Q}_p)$ means the *$\sigma$-algebra* of $\mathbb{Q}_p$, consisting of all measurable subsets of $\mathbb{Q}_p$. Moreover, this measure $\rho_p$ satisfies that

$$\begin{aligned} \rho_p\left(a + p^k\mathbb{Z}_p\right) = \rho_p\left(p^k\mathbb{Z}_p\right) = \frac{1}{p^k} = \\ = \rho\left(p^k\mathbb{Z}_p^\times\right) = \rho\left(a + \mathbb{Z}_p^\times\right) \end{aligned} \tag{2.9}$$

for all $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$, where $\mathbb{Z}_p^\times \overset{def}{=} \mathbb{Z}_p \setminus \{0\}$. Also, one has

$$\begin{aligned} \rho_p(a + U_p) = \rho_p\left(U_p\right) = \rho_p\left(\mathbb{Z}_p \setminus p\mathbb{Z}_p\right) = \\ = \rho_p\left(\mathbb{Z}_p\right) - \rho_p\left(p\mathbb{Z}_p\right) = 1 - \frac{1}{p} \end{aligned}$$

for all $a \in \mathbb{Q}$. Similarly, we obtain that

$$\rho_p\left(a + p^kU_p\right) = \rho\left(p^kU_p\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}} \tag{2.10}$$

for all $a \in \mathbb{Q}$ and $k \in \mathbb{Z}$ (see Chapter IV of [14]).

**Fact 2.4.** *The Banach field $\mathbb{Q}_p$ is an unbounded Haar-measure space, where $\rho_p$ satisfies* (2.9) *and* (2.10)*, for all primes $p$.*

The above three facts show that $\mathbb{Q}_p$ is a unbounded Haar-measured, locally compact Banach field, for all primes $p$.

**Definition 2.5.** Let $\mathcal{P} = \{\text{all primes}\} \cup \{\infty\}$. The adele ring $\mathbb{A}_{\mathbb{Q}} = (\mathbb{A}_{\mathbb{Q}}, +, \cdot)$ is defined by the set

$$\{(x_p)_{p\in\mathcal{P}} : x_p \in \mathbb{Q}_p, \text{ almost all } x_p \in \mathbb{Z}_p, \text{ for } p \in \mathcal{P}\}, \tag{2.11}$$

with identification $\mathbb{Q}_\infty = \mathbb{R}$, and $\mathbb{Z}_\infty = [0,1]$, the closed unit interval in $\mathbb{R}$, equipped with the product topology of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, and with the product measure $\rho = \underset{p\in\mathcal{P}}{\times} \rho_p$, and with operations

$$(x_p)_p + (y_p)_p = (x_p + y_p)_p \tag{2.12}$$

and

$$(x_p)_p(y_p)_p = (x_p y_p)_p \tag{2.13}$$

for all $(x_p)_p, (y_p)_p \in \mathbb{A}_{\mathbb{Q}}$.

Indeed, the algebraic structure $\mathbb{A}_{\mathbb{Q}}$ is a ring. Also, under the product topology, the adele ring $\mathbb{A}_{\mathbb{Q}}$ is also a locally compact Banach space having its measure. Set-theoretically,

$$\mathbb{A}_{\mathbb{Q}} \subseteq \prod_{p\in\mathcal{P}} \mathbb{Q}_p = \mathbb{R} \times \left( \prod_{p:prime} \mathbb{Q}_p \right).$$

In fact, by the very definition of $\mathbb{A}_{\mathbb{Q}}$, it is a *weak direct product* $\underset{p\in\mathcal{P}}{\Pi'} \mathbb{Q}_p$ of prime fields $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathcal{P}}{}' \mathbb{Q}_p,$$

where $\prod'$ means the weak direct product, i.e., if

$$(x_\infty, x_2, x_3, x_5, x_7, x_{11}, \ldots) \in \mathbb{A}_{\mathbb{Q}},$$

then most of $x_p$ are contained in the unit disks $\mathbb{Z}_p$, but only finitely many $x_q$ are in $\mathbb{Q}_q$, for $p$, $q \in \mathcal{P}$.

The product measure $\rho = \underset{p\in\mathcal{P}}{\times} \rho_p$ of the adele ring $\mathbb{A}_{\mathbb{Q}}$ is well-defined on the $\sigma$-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$, with identification $\rho_\infty = \rho_{\mathbb{R}}$, the usual distance-measure on $\mathbb{R} = \mathbb{Q}_\infty$.

**Fact 2.6.** *The adele ring $\mathbb{A}_{\mathbb{Q}}$ is an unbounded-measured locally compact Banach ring.*

## 2.2. WEAK TENSOR PRODUCT STRUCTURES

Let $X_i$ be arbitrary sets, for $i \in \Lambda$, where $\Lambda$ means any countable index set. Let

$$g_i : X_i \to X_i \tag{2.14}$$

be well-defined functions for all $i \in \Lambda$.

Now, let $X$ be the *Cartesian product* $\underset{i\in\Lambda}{\Pi} X_i$ *of* $\{X_i\}_{i\in\Lambda}$. Define the subset $\mathcal{X}$ of $X$ by

$$\mathcal{X} = \left\{ (x_i)_i \in X \,\middle|\, \begin{array}{c} \text{finitely many } x_i \in X_i, \text{ and} \\ \text{almost of all } x_i \in g_i(X_i) \end{array} \right\}, \tag{2.15}$$

determined by a system $g = \{g_i\}_{i\in\Lambda}$ of (2.14). We denote this subset $\mathcal{X}$ by

$$\mathcal{X} = \prod_{i\in\Lambda}{}_{g} X_i.$$

It is clear that $\mathcal{X}$ is a subset of $X$, by the very definition (2.15). If $g_i$ are bijections for all $i \in \Lambda$, then $\mathcal{X}$ is *equipotent* (or *bijective*) to $X$. However, in general, $\mathcal{X}$ is a subset of $X$.

**Definition 2.7.** The subset $\mathcal{X} = \underset{i\in\Lambda}{\Pi_g} X_i$ of $X = \underset{i\in\Lambda}{\Pi} X_i$, in the sense of (2.15), is called the weak tensor product set of $\{X_i\}_{i\in\Lambda}$ induced by a system $g = \{g_i\}_{i\in\Lambda}$ of functions $g_i$.

Let $\mathbb{Q}_p$ be $p$-prime fields, for all $p \in \mathcal{P}$. Define a function

$$g_p \colon \mathbb{Q}_p \to \mathbb{Q}_p$$

by

$$g_p\left(p^{-N}\Big(\sum_{j=0}^{\infty} a_j p^j\Big)\right) \overset{def}{=} \sum_{j=0}^{\infty} a_j p^j \tag{2.16}$$

for all $p^{-N}\sum_{j=0}^{\infty} a_j p^j \in \mathbb{Q}_p$ (with $N \in \mathbb{N}\cup\{0\}$), for all $p \in \mathcal{P}$. Then the image $g_p(\mathbb{Q}_p)$ is identical to the compact subset $\mathbb{Z}_p$, the unit disk of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. Therefore, the adele ring $\mathbb{A}_{\mathbb{Q}} = \underset{p\in\mathcal{P}}{\Pi'} \mathbb{Q}_p$ is identified with

$$\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathcal{P}}{}_{g} \mathbb{Q}_p,$$

in the sense of (2.15), where $g = \{g_p\}_{p\in\mathcal{P}}$ is the system of functions $g_p$ of (2.16).

Remark here that, for example, if we have real number $r$ in $\mathbb{R} = \mathbb{Q}_\infty$, with its decimal notation

$$|r| = \sum_{k\in\mathbb{Z}} t_k \cdot 10^{-k} = \dots t_{-2}t_{-1}t_0.t_1t_2t_3\dots$$

with $0 \leq t_k < 10$ in $\mathbb{N}$, then

$$g_\infty(r) = 0.t_1t_2t_3\dots, \tag{2.17}$$

with identification $g_\infty(\pm 1) = 1$. Traditionally, we simply write $\mathbb{A}_{\mathbb{Q}} = \underset{p\in\mathcal{P}}{\Pi'} \mathbb{Q}_p$ as before if there is no confusion.

Remark also that $X_i$'s of (2.14) and (2.15) may/can be algebraic structures (e.g., semigroups, or groups, or monoids, or groupoids, or vector spaces, etc.), or topological spaces (e.g., Hilbert spaces, or Banach spaces, etc.). One may put product topology on the weak tensor product, with continuity on $\{g_i\}_{i\in\Lambda}$. Similarly, if $X_i$'s are topological algebras (e.g., Banach algebras, or $C^*$-algebras, or von Neumann algebras, etc.), then we may have suitable product topology, with bounded (or continuous) linearity on $\{g_i\}_{i\in\Lambda}$.

In topological-$*$-algebraic case, to distinguish with other situations, we use the notation $\underset{i\in\Lambda}{\otimes_\Phi}$, instead of using $\underset{i\in\Lambda}{\Pi_\Phi}$, for any system $\Phi$ of functions.

**Remark 2.8.** Let $X_i$ be algebras (or topological algebras, or topological $*$-algebras etc.), for $i \in \Lambda$. Then the weak tensor product $\underset{i\in\Lambda}{\otimes_\Phi}$ induced by a system $\Phi$ becomes a conditional sub-structure of the usual tensor product $\underset{i\in\Lambda}{\otimes_\mathbb{C}}$, whenever functions in the system $\Phi$ are algebraic (resp., continuous-algebraic, resp., continuous-$*$-algebraic) homomorphisms. In such a case, our weak tensor product algebras (resp., topological algebras, or topological $*$-algebras) are subalgebras (resp., topological subalgebras, resp., topological $*$-subalgebras) of the usual tensor product algebras (resp., topological algebras, resp., topological $*$-algebras), i.e., $\underset{i\in\Lambda}{\otimes_\Phi} X_i$ are well-determined sub-structures of $\underset{i\in\Lambda}{\otimes_\mathbb{C}} X_i$, whenever functions $\Phi_i$ in $\Phi$ preserve the structures of $X_i$'s to those of $\Phi_i(X_i)$ for all $i \in \Lambda$.

## 3. BANACH SPACES $\mathcal{X}_p$ AND $\mathcal{X}_\mathbb{Q}$

In this section, we define normed spaces where our operators act. Over prime fields $\mathbb{Q}_p$, we introduce Banach spaces $\mathcal{X}_p$, for all primes $p$, and similarly, over the adele ring $\mathbb{A}_\mathbb{Q}$, we define a Banach space $\mathcal{X}_\mathbb{Q}$.

### 3.1. BANACH SPACES $\mathcal{X}_p$ OVER $\mathbb{Q}_p$

Recall that, in [3] and [5], we defined *$n$-products $\mathbb{Q}_p^n$ of $\mathbb{Q}_p$* for $n \in \mathbb{N}$. Here, $\mathbb{Q}_p^n$ is the Cartesian product of $n$-copies of $\mathbb{Q}_p$, as a set. So, all elements of $\mathbb{Q}_p^n$ have their forms, $n$-tuples of $p$-adic numbers. By defining a norm $|\cdot|_{p:n}$ on $\mathbb{Q}_p^n$,

$$|(x_1,\ \ldots,\ x_n)|_{p:n} \overset{def}{=} \max\{|x_j|_p : j = 1, \ldots, n\},$$

for all $(x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, one can understand $\mathbb{Q}_p^n$ as a normed space. Moreover, it is a Banach space because $\mathbb{Q}_p$ is a Banach space.

We are interested only in the case where $n = \infty$.

Define now a set $X_p$ by a collection of all $\mathbb{Q}_p$-sequences, i.e.,

$$X_p \overset{def}{=} \{(x_n)_{n=1}^\infty : x_n \in \mathbb{Q}_p \text{ for all } n \in \mathbb{N}\}. \tag{3.1}$$

The set $X_p$ of (3.1) is identical to $\mathbb{Q}_p^\infty$ (briefly mentioned in [3]) as "sets". Define a vector addition on $X_p$ by

$$(x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty = (x_n + y_n)_{n=1}^\infty \tag{3.2}$$

for all $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \in X_p$, where $x_n + y_n$ of the right-hand side of (3.2) is in the sense of (2.6).

Also, define a $\mathbb{Q}_p$-scalar multiplication on $X_p$ by

$$x(x_n)_{n=1}^{\infty} = (xx_n)_{n=1}^{\infty} \tag{3.3}$$

for all $x \in \mathbb{Q}_p$, and $(x_n)_{n=1}^{\infty} \in X_p$, where $xx_n$ of the right-hand side of (3.3) is in the sense of (2.8).

**Proposition 3.1.** *The set $X_p$ of* (3.1) *is a well-defined vector space over a field $\mathbb{Q}_p$ equipped with* (3.2) *and* (3.3)*.*

From now on, we understand $X_p$ as a vector space over a field $\mathbb{Q}_p$.

Define now a *norm* $\|\cdot\|_p$ *on the vector space* $X_p$ by

$$\|(x_n)_{n=1}^{\infty}\|_p \overset{def}{=} \sup\{|x_j|_p : j \in \mathbb{N}\} \tag{3.4}$$

for all $(x_n)_{n=1}^{\infty} \in X_p$.

**Proposition 3.2.** *Let $X_p$ be the vector space* (3.1) *over $\mathbb{Q}_p$, and let $\|\cdot\|_p$ be a morphism* (3.4)*. Then it is a well-defined norm on $X_p$, i.e., $(X_p, \|\cdot\|_p)$ is a normed space over $\mathbb{Q}_p$. Moreover, this norm $\|\cdot\|_p$ is non-Archimedean in the sense that*

$$\|\alpha + \beta\|_p \leq \max\{\|\alpha\|_p, \|\beta\|_p\}$$

*for all $\alpha$, $\beta \in X_p$.*

*Proof.* Let $X_p$ be given as above, and $\|\cdot\|_p$ be as in (3.4). By definition,

$$\|\alpha\|_p \geq 0 \quad \text{for all} \quad \alpha \in X_p. \tag{3.5}$$

Now, let $x \in \mathbb{Q}_p$, and $\alpha = (x_n)_{n=1}^{\infty} \in X_p$. If $\|\alpha\|_p = |x_o|_p$, for some $x_o \in \{x_1, x_2, \ldots\}$ (by (3.4)), then

$$\begin{aligned} \|x\alpha_p\|_p &= |xx_o|_p = \\ &= \left|\left(p^{-N_x}\beta_x\right)\left(p^{-N_{x_o}}\beta_{x_o}\right)\right|_p = \left|p^{-N_x}p^{-N_{x_o}}(\beta_x\beta_{x_o})\right|_p = \end{aligned}$$

whenever $x = p^{-N_x}\beta_x$ and $x_o = p^{-N_{x_o}}\beta_{x_o}$, with $N_x$, $N_{x_o} \in \mathbb{N} \cup \{0\}$, and $\beta_x$, $\beta_{x_o} \in \mathbb{Z}_p$ (see Section 2.1)

$$\begin{aligned} &= \left|p^{-(N_x+N_{x_o})}(\beta_x\beta_{x_o})\right| = p^{N_x+N_{x_o}} = \\ &= \left(p^{N_x}\right)\left(p^{N_{x_o}}\right) = |x|_p\,|x_o|_p = |x|_p\,\|(x_n)_{n=1}^{\infty}\|_p = |x|_p\,\|\alpha\|_p\,, \end{aligned}$$

and hence

$$\|x\alpha\|_p = |x|_p\|\alpha\|_p \quad \text{for all} \quad x \in \mathbb{Q}_p \quad \text{and} \quad \alpha \in X_p. \tag{3.6}$$

Now, let $\alpha = (x_n)_{n=1}^{\infty}$, $\beta = (y_n)_{n=1}^{\infty} \in X_p$. Then

$$\begin{aligned} \|\alpha + \beta\|_p &= \|(x_n + y_n)_{n=1}^{\infty}\|_p = \\ &= \sup\{|x_j + y_j|_p : j \in \mathbb{N}\} = \\ &= |x_o + y_o|_p \leq \qquad\qquad \text{(for some } o \in \mathbb{N}) \\ &\leq \max\{|x_o|_p, |y_o|_p\} \leq \end{aligned}$$

by the non-Archimedean property of $|\cdot|_p$

$$\leq \max\{\|\alpha\|_p, \|\beta\|_p\},$$

i.e.,

$$\|\alpha + \beta\|_p \leq \max\{\|\alpha\|_p, \|\beta\|_p\} \quad \text{for all} \quad \alpha, \beta \in X_p. \tag{3.7}$$

Therefore, by (3.7), one can obtain that

$$\|\alpha + \beta\|_p \leq \|\alpha\|_p + \|\beta\|_p \quad \text{for all} \quad \alpha, \beta \in X_p. \tag{3.8}$$

Therefore, by (3.5), (3.6) and (3.8), the morphism $\|\cdot\|_p$ is a well-defined norm on $X_p$. Also, by (3.7), this norm is non-Archimedean. Equivalently, the pair $(X_p, \|\cdot\|_p)$ is a normed space over a field $\mathbb{Q}_p$. □

By the above proposition, the pair $(X_p, \|\cdot\|_p)$ is a non-Archimedean normed vector space over the $p$-prime field $\mathbb{Q}_p$. We denote this pair simply by $X_p$.

**Definition 3.3.** Let $X_p$ be a normed vector space as above. Under the $\|\cdot\|_p$-norm topology, let $\mathcal{X}_p$ be the completion of $X_p$. We call $\mathcal{X}_p$, the $p$-adic Banach space over $\mathbb{Q}_p$.

### 3.2. A BANACH SPACE $\mathcal{X}_{\mathbb{Q}}$ OVER $\mathbb{A}_{\mathbb{Q}}$

Similar to Section 3.1, we define a set $X_{\mathbb{Q}}$ by a set of all $\mathbb{A}_{\mathbb{Q}}$-sequences over the adele ring $\mathbb{A}_{\mathbb{Q}}$, i.e.,

$$X_{\mathbb{Q}} \stackrel{def}{=} \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{A}_{\mathbb{Q}} \text{ for all } n \in \mathbb{N}\}.$$

Define now a vector addition on $X_{\mathbb{Q}}$ by

$$(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty} = (a_n + b_n)_{n=1}^{\infty} \tag{3.9}$$

for all $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \in X_{\mathbb{Q}}$, where $a_n + b_n$ in the right-hand side of (3.9) is in the sense of (2.12) for all $n \in \mathbb{N}$.

Also, define a scalar multiplication by

$$a(a_n)_{n=1}^{\infty} = (aa_n)_{n=1}^{\infty} \tag{3.10}$$

for all $a \in \mathbb{A}_{\mathbb{Q}}$ and $(a_n)_{n=1}^{\infty} \in X_{\mathbb{Q}}$, where the entries $aa_n$ in the right-hand side of (3.10) is in the sense of (2.13).

The operations (3.9) and (3.10) are well-defined in $X_{\mathbb{Q}}$.

**Proposition 3.4.** *The set* $X_{\mathbb{Q}}$*, equipped with* (3.9) *and* (3.10)*, is a vector space over a ring* $\mathbb{A}_{\mathbb{Q}}$.

From now on, we understand $X_{\mathbb{Q}}$ as a vector space over $\mathbb{A}_{\mathbb{Q}}$.

Recall that as a weak direct product $\underset{p\in\mathcal{P}}{\Pi'}\ \mathbb{Q}_p$ (equivalently, the weak tensor product $\underset{p\in\mathcal{P}}{\Pi_g}\ \mathbb{Q}_p$ in the sense of Section 2.1) of prime fields $\mathbb{Q}_p$, the adele ring $\mathbb{A}_{\mathbb{Q}}$ has its *norm* $|\cdot|_{\mathbb{Q}}\colon \mathbb{A}_{\mathbb{Q}} \to \mathbb{R}_0^+$,

$$|(x_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} \overset{def}{=} \prod_{p\in\mathcal{P}} |x_p|_p \quad \text{for all} \quad (x_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}, \tag{3.11}$$

with identity: $|x_\infty|_\infty = |x_\infty|$, the absolute value of $x_\infty$, for all $x_\infty \in \mathbb{Q}_\infty = \mathbb{R}$.

Remark that, since almost of all entries $x_q$ of $(x_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$ are contained in $\mathbb{Z}_q$, for $q \in \mathcal{P}$,

$$|(x_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} < \infty.$$

Furthermore, one can have

$$\begin{aligned} |(x_p)_{p\in\mathcal{P}} + (y_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} &= |(x_p + y_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} = \\ &= \prod_{p\in\mathcal{P}} |x_p + y_p|_p = \\ &= |x_\infty + y_\infty| + \prod_{p:prime} |x_p + y_p|_p \leq \\ &\leq |x_\infty| + |y_\infty| + \prod_{p:prime} \max\{|x_p|_p, |y_p|_p\} \leq \end{aligned} \tag{3.12}$$

by the non-Archimedean property of $|\cdot|_p$, for all primes $p$

$$\leq \prod_{p\in\mathcal{P}} |x_p|_p + \prod_{p\in\mathcal{P}} |y_p|_p = |(x_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} + |(y_p)_{p\in\mathcal{P}}|_{\mathbb{Q}}$$

for all $(x_p)_{p\in\mathcal{P}}$, $(y_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$. So, indeed, the morphism (3.11) is a well-defined norm on $\mathbb{A}_{\mathbb{Q}}$, satisfying (3.12).

Now, define a morphism $\|\cdot\|_{\mathbb{Q}}\colon X_{\mathbb{Q}} \to \mathbb{R}_0^+$ on $X_{\mathbb{Q}}$ by

$$\|(a_n)_{n=1}^\infty\|_{\mathbb{Q}} \overset{def}{=} \sup\{|a_j|_{\mathbb{Q}} : j \in \mathbb{N}\} \tag{3.13}$$

for all $(a_n)_{n=1}^\infty \in X_{\mathbb{Q}}$, where $\mathbb{R}_0^+ = \{r \in \mathbb{R}\colon r \geq 0\}$.

**Proposition 3.5.** *The vector space* $X_{\mathbb{Q}}$ *equipped with the morphism* $\|\cdot\|_{\mathbb{Q}}$ *of* (3.13) *is a normed space over a ring* $\mathbb{A}_{\mathbb{Q}}$.

*Proof.* By definition, it is clear that

$$\|\alpha\|_{\mathbb{Q}} \geq 0 \quad \text{for all} \quad \alpha \in X_{\mathbb{Q}}.$$

Now, let $a = (x_p)_{p \in \mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$ and $\alpha = ((x_{n:p})_{p \in \mathcal{P}})_{n=1}^{\infty} \in X_{\mathbb{Q}}$. If

$$\|\alpha\|_{\mathbb{Q}} = |(x_{o:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} \quad \text{for some} \quad o \in \mathbb{N}$$

in $\mathbb{R}_0^+ \cup \{\infty\}$, then one can have that

$$\begin{aligned}
\|a\alpha\|_{\mathbb{Q}} &= \left\|((x_p)_{p \in \mathcal{P}})\left((x_{n:p})_{p \in \mathcal{P}}\right)_{n=1}^{\infty}\right\|_{\mathbb{Q}} = \\
&= \left\|((x_p)_{p \in \mathcal{P}}(x_{n:p})_{p \in \mathcal{P}})_{n=1}^{\infty}\right\|_{\mathbb{Q}} = \\
&= \left\|\Big((x_p x_{n:p})_{p \in \mathcal{P}}\Big)_{n=1}^{\infty}\right\|_{\mathbb{Q}} = \\
&= \sup\Big\{|(x_p x_{n:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} : n \in \mathbb{N}\Big\} = \\
&= \sup\left\{\prod_{p \in \mathcal{P}} |x_p x_{n:p}|_p : n \in \mathbb{N}\right\} = \\
&= \sup\left\{\prod_{p \in \mathcal{P}} |x_p|_p \, |x_{n:p}|_p : n \in \mathbb{N}\right\} = \\
&= \prod_{p \in \mathcal{P}} |x_p|_p \, |x_{o:p}|_p = \qquad \text{(for some } o \in \mathbb{N}) \\
&= \left(\prod_{p \in \mathcal{P}} |x_p|_p\right)\left(\prod_{p \in \mathcal{P}} |x_{o:p}|_p\right) =
\end{aligned}$$

by (2.13)

$$= |a|_{\mathbb{Q}} \, \|\alpha\|_{\mathbb{Q}} \, .$$

So, for $a \in \mathbb{A}_{\mathbb{Q}}$ and $\alpha \in X_{\mathbb{Q}}$,

$$\|a\alpha\|_{\mathbb{Q}} = |a|_{\mathbb{Q}} \, \|\alpha\|_{\mathbb{Q}} \, .$$

Also, with help of (3.12), one can obtain

$$\|\alpha + \beta\|_{\mathbb{Q}} \leq \|\alpha\|_{\mathbb{Q}} + \|\beta\|_{\mathbb{Q}} \quad \text{for all} \quad \alpha, \beta \in X_{\mathbb{Q}}.$$

Indeed, if $\alpha = ((x_{n:p})_{p \in \mathcal{P}})_{n=1}^{\infty}$, and $\beta = ((y_{n:p})_{p \in \mathcal{P}})_{n=1}^{\infty}$ in $X_{\mathbb{Q}}$, then

$$\begin{aligned}
\|\alpha + \beta\|_{\mathbb{Q}} &= \left\|((x_{n:p})_{p \in \mathcal{P}} + (y_{n:p})_{p \in \mathcal{P}})_{n=1}^{\infty}\right\|_{\mathbb{Q}} = \\
&= \left\|\Big((x_{n:p} + y_{n:p})_{p \in \mathcal{P}}\Big)_{n=1}^{\infty}\right\|_{\mathbb{Q}} = \\
&= \sup\Big\{|(x_{n:p} + y_{n:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} : n \in \mathbb{N}\Big\} = \\
&= |(x_{o:p} + y_{o:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} \leq \qquad \text{(for some } o \in \mathbb{N}) \\
&\leq |(x_{o:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} + |(y_{o:p})_{p \in \mathcal{P}}|_{\mathbb{Q}} \leq \quad \text{(by (3.12))} \\
&\leq \|\alpha\|_{\mathbb{Q}} + \|\beta\|_{\mathbb{Q}} \, .
\end{aligned}$$

Therefore, the pair $(X_{\mathbb{Q}}, \|\cdot\|_{\mathbb{Q}})$ is a normed space over a ring $\mathbb{A}_{\mathbb{Q}}$. $\square$

The above proposition shows that the vector space $X_{\mathbb{Q}}$ with $\|\cdot\|_{\mathbb{Q}}$ is a normed space over the adele ring $\mathbb{A}_{\mathbb{Q}}$.

**Definition 3.6.** Let $\mathcal{X}_{\mathbb{Q}}$ be the $\|\cdot\|_{\mathbb{Q}}$-norm-topology completion of the normed space $X_{\mathbb{Q}}$. We call $\mathcal{X}_{\mathbb{Q}}$ the adelic Banach space over $\mathbb{A}_{\mathbb{Q}}$.

### 3.3. BANACH SPACES $\mathcal{X}_{\mathbb{Q}}$ AND $\mathcal{X}_p$

Construct a topological product vector space $\mathcal{X} = \underset{p\in\mathcal{P}}{\Pi} \mathcal{X}_p$ of $p$-adic Banach spaces $\mathcal{X}_p$, equipped with the norm

$$\|(v_p)_{p\in\mathcal{P}}\|_{\otimes} \overset{def}{=} \prod_{p\in\mathcal{P}} \|v_p\|_p\,, \tag{3.14}$$

where $\|\cdot\|_p$ are in the sense of (3.4), for all $p \in \mathcal{P}$. It is not difficult to check that this vector space $\mathcal{X}$ is over the adele ring $\mathbb{A}_{\mathbb{Q}}$. Indeed, since each $p$-adic Banach space $\mathcal{X}_p$ is over $\mathbb{Q}_p$, for $p \in \mathcal{P}$, the space $\mathcal{X}$ is over $\underset{p\in\mathcal{P}}{\Pi} \mathbb{Q}_p$ (containing $\underset{p\in\mathcal{P}}{\Pi'} \mathbb{Q}_p$). Thus, $\mathcal{X}$ is over $\mathbb{A}_{\mathbb{Q}}$.

Naturally, we have the vector addition

$$(v_p)_{p\in\mathcal{P}} + (w_p)_{p\in\mathcal{P}} = (v_p + w_p)_{p\in\mathcal{P}} \quad \text{on} \quad \mathcal{X}$$

for all $(v_p)_{p\in\mathcal{P}}, (w_p)_{p\in\mathcal{P}} \in \mathcal{X}$, and the $\mathbb{A}_{\mathbb{Q}}$-scalar product

$$(t_p)_{p\in\mathcal{P}}(v_p)_{p\in\mathcal{P}} = (t_p v_p)_{p\in\mathcal{P}} \quad \text{on} \quad \mathcal{X}$$

for all $(t_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$ and $(v_p)_{p\in\mathcal{P}} \in \mathcal{X}$.

In the rest of this section, understand $\mathcal{X}$ as a normed vector space with its norm $\|\cdot\|_{\otimes}$ of (3.14).

Recall now functions $g_p$ on $\mathbb{Q}_p$ as in (2.16) and (2.17), for all $p \in \mathcal{P}$, i.e.,

$$g_p\left(p^{-N}(\sum_{n=0}^{\infty} a_n p^n)\right) \overset{def}{=} \sum_{n=0}^{\infty} a_n p^n$$

for all $p^{-N}\left(\sum_{n=0}^{\infty} a_n p^n\right) \in \mathbb{Q}_p$, with $N \in \mathbb{N} \cup \{0\}$, and $0 \le a_n < p$.

Then they are kind of normalization maps, compressing elements of $\mathbb{Q}_p$ to those of $\mathbb{Z}_p$, the unit disks of $\mathbb{Q}_p$. We call $g_p$ the *$p$-normalizations on* $\mathbb{Q}_p$ for all $p \in \mathcal{P}$.

Define now a function $\varphi_p \colon \mathcal{X}_p \to \mathcal{X}_p$ on the $p$-adic Banach space $\mathcal{X}_p$ by

$$\varphi_p\left((x_n)_{n=1}^{\infty}\right) \overset{def}{=} \left(g_p(x_n)\right)_{n=1}^{\infty} \tag{3.15}$$

for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$, for all $p \in \mathcal{P}$. The functions $\varphi_p$ are well-defined continuous function on $\mathcal{X}_p$ for all $p \in \mathcal{P}$.

Notice however that each $\varphi_p$ is "not" $\mathbb{Q}_p$-linear, because

$$\varphi_p\left((x_n)_{n=1}^{\infty} + (y_n)_{n=1}^{\infty}\right) = \varphi_p\left((x_n + y_n)_{n=1}^{\infty}\right) = \left(g_p(x_n + y_n)\right)_{n=1}^{\infty}$$

and

$$g_p(x_n + y_n) \neq g_p(x_n) + g_p(y_n),$$

in general, for $n \in \mathbb{N}$. For example, let

$$x_n = p^{-3} + p^{-2} + 2p^{-1} + p^0 + 0 \cdot p + p^2 + p^3 + \dots,$$

and

$$y_n = p^{-2} + p^{-1} + p^0 + p + p^2 + p^3 + \dots$$

in $\mathbb{Q}_p$, for some $n \in \mathbb{N}$. If $p = 3$, then

$$\begin{aligned} g_3(x_n + y_n) &= g_3\left(p^{-3} + 2p^{-2} + \boxed{3p^{-1} + 2p^0} + p + 2p^2 + 2p^3 + \dots\right) = \\ &= g_3\left(p^{-3} + 2p^{-2} + \boxed{p^0 + 2p^0} + p + 2p^2 + 2p^3 + \dots\right) = \\ &= g_3\left(p^{-3} + 2p^{-2} + \boxed{3p^0} + p + 2p^2 + 2p^3 + \dots\right) = \\ &= g_3\left(p^{-3} + 2p^{-2} + \boxed{p} + p + 2p^2 + 2p^3 + \dots\right) = \\ &= g_3\left(p^{-3} + 2p^{-2} + 2p + 2p^2 + 2p^3 + \cdots\right) = \\ &= 2p + 2p^2 + 2p^3 + \dots, \end{aligned}$$

but

$$\begin{aligned} g_3(x_n) + g_p(y_n) &= (p^0 + 0 \cdot p + p^2 + p^3 + \cdots) + (p^0 + p + p^2 + p^3 + \dots) = \\ &= 2p^0 + p + 2p^2 + 2p^3 + 2p^4 + \dots. \end{aligned}$$

So, in general, $g_p$ are not $\mathbb{Q}_p$-linear, and hence $\varphi_p$ are "not" $\mathbb{Q}_p$-linear, i.e.,

$$\varphi_p\left((x_n)_{n=1}^\infty + (y_n)_{n=1}^\infty\right) \neq \varphi_p\left((x_n)_{n=1}^\infty\right) + \varphi_p\left((y_n)_{n=1}^\infty\right),$$

in general. However, it is a well-defined (topological continuous) function on the topological space $\mathcal{X}_p$.

These functions $\varphi_p$ satisfy that

$$\begin{aligned} \left\|\varphi_p\left((x_n)_{n=1}^\infty\right)\right\|_p &= \left\|\left(g_p(x_n)\right)_{n=1}^\infty\right\|_p = \\ &= \sup\left\{|g_p(x_n)|_p : n \in \mathbb{N}\right\} \leq 1 \end{aligned}$$

for all $(x_n)_{n=1}^\infty \in \mathcal{X}_p$, for $p \in \mathcal{P}$, i.e., this map is understood as a normalization on $\mathcal{X}_p$, for $p \in \mathcal{P}$.

**Definition 3.7.** We call the functions $\varphi_p$ of (3.14) the $p$-normalization on the $p$-adic Banach space $\mathcal{X}_p$ for all $p \in \mathcal{P}$. Also, we denote the system $\{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations simply by $\varphi$.

Let $\mathcal{X} = \prod_{p\in\mathcal{P}} \mathcal{X}_p$ be given as above, equipped with its norm $\|\cdot\|_\otimes$ in the sense of (3.14). As a "subset" of $\mathcal{X}$, define $\mathfrak{X}_o$ by

$$\mathfrak{X}_o \overset{def}{=} \left\{ (v_p)_{p\in\mathcal{P}} \in \mathcal{X} \;\middle|\; \begin{array}{c} v_p \in \mathcal{X}_p \text{ for finitely many } p, \\ \text{and all other } a_q \text{ are} \\ \text{contained in } \varphi_q(\mathcal{X}_q) \end{array} \right\}, \tag{3.16}$$

i.e., $\mathfrak{X}_o$ is the weak tensor product $\underset{p\in\mathcal{P}}{\Pi_\varphi} \mathcal{X}_p$ of $\{\mathcal{X}_p\}_{p\in\mathcal{P}}$, induced by the system $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations $\varphi_p$ (3.15) on $\mathcal{X}_p$. Under the inherited operations and norm, also denoted by $\|\cdot\|_\otimes$, from $\mathcal{X}$, this set $\mathfrak{X}_o$ is a normed vector space over the adele ring $\mathbb{A}_\mathbb{Q}$, too.

**Definition 3.8.** Let $\mathfrak{X}_o$ be the normed space (3.16) over $\mathbb{A}_\mathbb{Q}$. Denote the $\|\cdot\|_\otimes$-norm-topology closure of $\mathfrak{X}_o$ by $\mathfrak{X}$.

The following proposition is the summary of the above discussion.

**Proposition 3.9.** *The weak tensor product space* $\mathfrak{X} = \underset{p\in\mathcal{P}}{\Pi_\varphi} \mathcal{X}_p$ *of* $\mathcal{X} = \underset{p\in\mathcal{P}}{\Pi} \mathcal{X}_p$ *is a Banach space over the adele ring* $\mathbb{A}_\mathbb{Q}$.

We now show the adelic Banach space $\mathcal{X}_\mathbb{Q}$ is Banach-space isomorphic to the weak tensor product Banach space $\mathfrak{X}$ of $p$-adic Banach spaces $\{\mathcal{X}_p\}_{p\in\mathcal{P}}$ induced by $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations.

**Theorem 3.10.** *Let* $\mathcal{X}_\mathbb{Q}$ *be the adelic Banach space over the adele ring* $\mathbb{A}_\mathbb{Q}$, *and let* $\mathfrak{X}$ *be the weak tensor product Banach space* $\underset{p\in\mathcal{P}}{\Pi_\varphi} \mathcal{X}_p$ *of $p$-adic Banach spaces* $\{\mathcal{X}_p\}_{p\in\mathcal{P}}$, *induced by the system* $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ *of $p$-normalizations* $\varphi_p$ (3.15) *on* $\mathcal{X}_p$. *Then* $\mathcal{X}_\mathbb{Q}$ *and* $\mathfrak{X}$ *are Banach-space isomorphic over* $\mathbb{A}_\mathbb{Q}$, *i.e.,*

$$\mathcal{X}_\mathbb{Q} \overset{Banach}{=} \prod_{p\in\mathcal{P}}{}_{\varphi} \mathcal{X}_p, \tag{3.17}$$

*where* "$\overset{Banach}{=}$" *means "being Banach-space isomorphic".*

*Proof.* Define now a morphism

$$\Phi\colon \mathcal{X}_\mathbb{Q} \to \mathfrak{X}$$

by the function satisfying

$$\Phi\left(\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^\infty\right) \overset{def}{=} \prod_{p\in\mathcal{P}} \left((x_{n:p})_{n=1}^\infty\right) \quad \text{in} \quad \mathfrak{X} \tag{3.18}$$

for all $((x_{n:p})_{p\in\mathcal{P}})_{n=1}^\infty = \prod_{n=1}^\infty (x_{n:p})_{p\in\mathcal{P}} \in \mathcal{X}_\mathbb{Q}$. In other notations,

$$\Phi\left(\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^\infty\right) \overset{def}{=} \left((x_{n:p})_{n=1}^\infty\right)_{p\in\mathcal{P}}.$$

Then it is a well-defined "injective" map from the adelic Banach space $\mathcal{X}_\mathbb{Q}$ to the weak direct product Banach space $\mathfrak{X}$.

Similarly, define a morphism

$$\Psi\colon \mathfrak{X} \to \mathcal{X}_\mathbb{Q}$$

by

$$\Psi\left(\left((x_{n:p})_{n=1}^{\infty}\right)_{p\in\mathcal{P}}\right) \overset{def}{=} \left(\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}\right) \quad \text{in} \quad \mathcal{X}_{\mathbb{Q}} \tag{3.19}$$

for all $\left((x_{n:p})_{n=1}^{\infty}\right)_{p\in\mathcal{P}} \in \mathfrak{X}$. Then one can verify that $\Psi$ is a well-defined injective map, too. Especially, the well-definedness of $\Psi$ is guaranteed by the weak tensor product structure of $\mathfrak{X}$. By the injectivity and (3.19), we have

$$\Phi^{-1} = \Psi,$$

i.e., the morphism $\Phi$ of (3.18) is a bijective function from $\mathcal{X}_{\mathbb{Q}}$ onto $\mathfrak{X}$, with its inverse $\Psi$.

Now, let $((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}$, $((y_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty} \in \mathcal{X}_{\mathbb{Q}}$. Then

$$\begin{aligned}
&\Phi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty} + ((y_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right) = \\
&= \Phi\left(\left((x_{n:p}+y_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}\right) = \\
&= \prod_{p\in\mathcal{P}}\left((x_{n:p}+y_{n:p})_{n=1}^{\infty}\right) = \left((x_{n:p}+y_{n:p})_{n=1}^{\infty}\right)_{p\in\mathcal{P}} = \\
&= \left(\prod_{p\in\mathcal{P}}((x_{n:p})_{n=1}^{\infty})\right) + \left(\Pi_{p\in\mathcal{P}}((y_{n:p})_{n=1}^{\infty})\right) = \\
&= \Phi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right) + \Phi\left(((y_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right).
\end{aligned}$$

Thus, we obtain that

$$\Phi(\alpha+\beta) = \Phi(\alpha) + \Phi(\beta) \quad \text{in} \quad \mathfrak{X} \tag{3.20}$$

for all $\alpha$, $\beta \in \mathcal{X}_{\mathbb{Q}}$.

Also, let $(a_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$, and let $((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty} \in \mathcal{X}_{\mathbb{Q}}$. Then

$$\begin{aligned}
\Phi\left(((a_p)_{p\in\mathcal{P}})\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right)\right) &= \Phi\left(((a_p x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right) = \\
&= \prod_{p\in\mathcal{P}}(a_p x_{n:p})_{n=1}^{\infty} = \\
&= \left(\Pi_{p\in\mathcal{P}} a_p\right)\left(\prod_{p\in\mathcal{P}}(x_{n:p})_{n=1}^{\infty}\right) = \\
&= ((a_p)_{p\in\mathcal{P}})\left(\Phi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right)\right),
\end{aligned}$$

and hence

$$\Phi(a\alpha) = a\Phi(\alpha) \quad \text{in} \quad \mathfrak{X} \tag{3.21}$$

for all $a \in \mathbb{A}_{\mathbb{Q}}$ and $\alpha \in \mathcal{X}_{\mathbb{Q}}$.

Thus, by (3.20) and (3.21), this bijective map $\Phi$ is a $\mathbb{A}_{\mathbb{Q}}$-vector-space isomorphism.

Also, one can check that

$$\left\|\Phi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right)\right\|_{\otimes} = \left\|((x_{n:p})_{n=1}^{\infty})_{p\in\mathcal{P}}\right\|_{\otimes} =$$

where $\|\cdot\|_{\otimes}$ is the norm (3.14) on $\mathfrak{X}$

$$= \prod_{p\in\mathcal{P}} \|(x_{n:p})_{n=1}^{\infty}\|_p =$$

where $\|\cdot\|_p$ means the $\mathcal{X}_p$-norm in the sense of (3.4)

$$= \prod_{p\in\mathcal{P}} \left(\sup\{|x_{n:p}|_p : n \in \mathbb{N}\}\right) \leq$$

$$\leq \prod_{p\in\mathcal{P}} |x_{o_p:p}|_p = \qquad \text{(for some } o_p \in \mathbb{N}, \text{ for } p \in \mathcal{P})$$

where $|\cdot|_p$ is the $\mathbb{Q}_p$-norm in the sense of Section 2.1

$$= |(x_{o_p:p})_{p\in\mathcal{P}}|_{\mathbb{Q}} =$$

where $|\cdot|_{\mathbb{Q}}$ is the $\mathbb{A}_{\mathbb{Q}}$-norm (3.11)

$$= \left\|((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right\|_{\mathbb{Q}},$$

where $\|\cdot\|_{\mathbb{Q}}$ is the norm (3.13) on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$. So,

$$\left\|\Phi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right)\right\|_{\otimes} \leq \left\|((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right\|_{\mathbb{Q}}.$$

So, this $\mathbb{A}_{\mathbb{Q}}$-vector-space isomorphism $\Phi$ is bounded. Similarly, one can find that

$$\left\|\Psi\left(((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right)\right\|_{\mathbb{Q}} \leq \left\|((x_{n:p})_{n=1}^{\infty})_{p\in\mathcal{P}}\right\|_{\otimes}.$$

Since $\Psi = \Phi^{-1}$, we obtain that

$$\left\|\Phi\left(\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}\right)\right\|_{\otimes} = \left\|((x_{n:p})_{p\in\mathcal{P}})_{n=1}^{\infty}\right\|_{\mathbb{Q}}.$$

Therefore, the $\mathbb{A}_{\mathbb{Q}}$-vector-space isomorphism $\Phi$ is isometric. And hence, this morphism $\Phi$ is a Banach-space isomorphism, equivalently, the Banach spaces $\mathcal{X}_{\mathbb{Q}}$ and $\mathfrak{X}$ are isomorphic over $\mathbb{A}_{\mathbb{Q}}$. $\square$

The above characterization (3.17) shows that our adelic Banach space $\mathcal{X}_{\mathbb{Q}}$ is Banach-space isomorphic to the weak tensor product Banach space $\underset{p\in\mathcal{P}}{\Pi_{\varphi}} \mathcal{X}_p$ of $p$-adic

Banach spaces $\mathcal{X}_p$ induced by the system $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations $\varphi_p$. In particular, one has an isometric isomorphism

$$\Phi\left(\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}\right) = \left((x_{n:p})_{n=1}^{\infty}\right)_{p\in\mathcal{P}} \quad \text{in} \quad \prod_{p\in\mathcal{P}}{}_{\varphi}\, \mathcal{X}_p$$

for all $\left((x_{n:p})_{p\in\mathcal{P}}\right)_{n=1}^{\infty} \in \mathcal{X}_{\mathbb{Q}}$.

In the rest of this paper, we use $\mathcal{X}_{\mathbb{Q}}$ and $\Pi_{p\in\mathcal{P}}^{\varphi}\, \mathcal{X}_p$, alternatively, as same Banach spaces.

Also, by (3.17), if we define an operator $T$ acting on the Banach space $\mathcal{X}_{\mathbb{Q}}$, then one may understand $T$ as an (equivalent form of) operator acting on $\Pi_{p\in\mathcal{P}}^{\varphi}\, \mathcal{X}_p$ over $\mathbb{A}_{\mathbb{Q}}$. We will consider it in the following sections.

## 4. $p$-ADIC OPERATORS ON $\mathcal{X}_p$

In this section, we study *$p$-adic operators* acting on $\mathcal{X}_p$, for primes $p$. Throughout this section, let's fix a prime $p$, and let $\mathcal{X}_p$ be the corresponding $p$-adic Banach space consisting of all $\mathbb{Q}_p$-sequences $(x_n)_{n=1}^{\infty}$ over the $p$-prime field $\mathbb{Q}_p$.

As we have seen in Section 3.1, the $p$-adic Banach space is well-defined Banach space $\mathcal{X}_p$ equipped with its norm $\|\cdot\|_p$, satisfying

$$\|(x_n)_{n=1}^{\infty}\|_p = \sup\{|x_n|_p : n \in \mathbb{N}\} \tag{4.1}$$

for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$. Define the *unit ball $\mathcal{B}_p$ of $\mathcal{X}_p$*, and the *unit circle $\mathcal{U}_p$ of $\mathcal{X}_p$* by

$$\mathcal{B}_p \overset{def}{=} \{(x_n)_{n=1}^{\infty} \in \mathcal{X}_p \colon \|(x_n)_{n=1}^{\infty}\|_p \leq 1\} \tag{4.2}$$

and

$$\mathcal{U}_p \overset{def}{=} \{(x_n)_{n=1}^{\infty} \in \mathcal{X}_p \colon \|(x_n)_{n=1}^{\infty}\|_p = 1\}.$$

**Lemma 4.1.** *Let $\mathcal{B}_p$ and $\mathcal{U}_p$ be in the sense of (4.2), for a prime $p$. Then:*

$$(x_n)_{n=1}^{\infty} \in \mathcal{B}_p \text{ in } \mathcal{X}_p \text{ if and only if } x_n \in \mathbb{Z}_p \text{ in } \mathbb{Q}_p \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

$$(x_n)_{n=1}^{\infty} \in \mathcal{U}_p \quad \text{in} \quad \mathcal{X}_p, \tag{4.4}$$

*if and only if*

(i) $(x_n)_{n=1}^{\infty} \in \mathcal{B}_p$, *and*
(ii) *there exists at least one entry $x_o$ in $(x_n)_{n=1}^{\infty}$ such that $x_o \in U_p$ in $\mathbb{Q}_p$.*

*Proof.* The proof of (4.3) is by the very definition (4.1) of $\mathcal{X}_p$-norm $\|\cdot\|_p$. If we assume that there exists at least one $j$ in $\mathbb{N}$ such that $x_j = p^{-N}x_1$, with "$N \in \mathbb{N}$", in $(x_n)_{n=1}^{\infty}$, (equivalently, if $x_j \in \mathbb{Q}_p \setminus \mathbb{Z}_p$), then

$$\|(x_n)_{n=1}^{\infty}\|_p \geq |x_j|_p = p^N > 1.$$

Conversely, if all entries $x_j$ of $(x_n)_{n=1}^{\infty}$ are in $\mathbb{Z}_p$, then $\|(x_n)_{n=1}^{\infty}\|_p \leq 1$, since $|x_n|_p \leq 1$ for all $n \in \mathbb{N}$.

So, the statement (4.3) holds.

Now, suppose $(x_n)_{n=1}^{\infty} \in \mathcal{U}_p$, and assume that either a condition (i) or (ii) does not hold. First, suppose $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p \setminus \mathcal{B}_p$. Then, by the proof of (4.3), $\|(x_n)_{n=1}^{\infty}\| > 1$. It contradicts our assumption that $(x_n)_{n=1}^{\infty} \in \mathcal{U}_p$. Now, suppose there does not exist entry $x_n$ in $(x_n)_{n=1}^{\infty} \in \mathcal{B}_p$, such that $x_n \in U_p$. This means that all entries of $(x_n)_{n=1}^{\infty}$ are contained in $p^k\mathbb{Z}_p$, for some $k \in \mathbb{N}$, i.e.,

$$\|(x_n)_{n=1}^{\infty}\|_p \leq \frac{1}{p^k} - \frac{1}{p^{k+1}} < 1 \quad \text{for some} \quad k \in \mathbb{N},$$

and hence $(x_n)_{n=1}^{\infty} \notin \mathcal{U}_p$. It contradicts our assumption.

Conversely, an element $(x_n)_{n=1}^{\infty}$ of $\mathcal{X}_p$ satisfies both conditions (i) and (ii). Then

$$\|(x_n)_{n=1}^{\infty}\|_p = \sup\{|x_n|_p : n \in \mathbb{N}\} = |x_o|_p = 1,$$

where $x_o \in U_p$ in $(x_n)_{n=1}^{\infty}$ for some $o \in \mathbb{N}$. Therefore, the statement (4.4) holds. $\square$

Consider certain operators (continuous or bounded $\mathbb{Q}_p$-linear transformations) acting on $\mathcal{X}_p$.

Define naturally $(\infty \times \infty)$-$\mathbb{Q}_p$-matrices $T$ by the rectangular arrays of $p$-adic numbers,

$$T = [x_{ij}]_{\infty\times\infty} \overset{denote}{=} [x_{ij}] \quad \text{with} \quad x_{ij} \in \mathbb{Q}_p.$$

Denote the collection of all such $\mathbb{Q}$-matrices by $\mathcal{M}_p$.

On $\mathcal{M}_p$, define the $\mathbb{Q}_p$-matrix addition, $\mathbb{Q}_p$-scalar multiplication, and the $\mathbb{Q}_p$-matrix multiplication just like in operator theory. Then $\mathcal{M}_p$ becomes an (pure algebraic) algebra over $\mathbb{Q}_p$.

Act $T = [x_{ij}] \in \mathcal{M}_p$ on $\mathcal{X}_p$ by the rule:

$$T\left((x_n)_{n=1}^{\infty}\right) = [x_{ij}]\left((x_n)_{n=1}^{\infty}\right) = \left(\sum_{j=1}^{\infty} x_{ij}x_j\right)_{i=1}^{\infty}, \tag{4.5}$$

where the addition $\sum$ and the multiplication $x_{ij}x_j$ at the far right-side of (4.5) are in the sense of (2.6), and (2.8), respectively.

Then, the $\mathbb{Q}_p$-matrices of $\mathcal{M}_p$ are $\mathbb{Q}_p$-linear transformations on $\mathcal{X}_p$. Indeed, if $T = [x_{ij}]$ in $\mathcal{M}_p$, then

$$\begin{aligned}
T\left((x_n)_{n=1}^{\infty} + (y_n)_{n=1}^{\infty}\right) = T\left((x_n + y_n)_{n=1}^{\infty}\right) = \\
&= [x_{ij}]\left((x_n + y_n)_{n=1}^{\infty}\right) = \left(\sum_{j=1}^{\infty} x_{ij}(x_j + y_j)\right)_{i=1}^{\infty} = \\
&= \left(\sum_{j=1}^{\infty} (x_{ij}x_j + x_{ij}y_j)\right)_{i=1}^{\infty} = \\
&= \left(\sum_{j=1}^{\infty} x_{ij}x_j + \sum_{j=1}^{\infty} x_{ij}y_j\right)_{i=1}^{\infty} = \\
&= \left(\sum_{j=1}^{\infty} x_{ij}x_j\right)_{i=1}^{\infty} + \left(\sum_{j=1}^{\infty} x_{ij}y_j\right)_{i=1}^{\infty} = \\
&= T\left((x_n)_{n=1}^{\infty}\right) + T\left((y_n)_{n=1}^{\infty}\right)
\end{aligned}$$

for all $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \in \mathcal{X}_p$. Thus, we have

$$T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \text{for all} \quad \alpha, \beta \in \mathcal{X}_p. \tag{4.6}$$

Also, for $x \in \mathbb{Q}_p$ and $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$,

$$\begin{aligned}
T\left(x(x_n)_{n=1}^{\infty}\right) = T\left((xx_n)_{n=1}^{\infty}\right) = [x_{ij}]\left((xx_n)_{n=1}^{\infty}\right) = \left(\sum_{j=1}^{\infty} x_{ij}xx_j\right)_{i=1}^{\infty} = \\
= \left(x\sum_{j=1}^{\infty} x_{ij}x_j\right)_{i=1}^{\infty} = x\left(\sum_{j=1}^{\infty} x_{ij}x_j\right)_{i=1}^{\infty} = xT\left((x_n)_{n=1}^{\infty}\right).
\end{aligned}$$

So, we get that

$$T(x\alpha) = xT(\alpha) \quad \text{for all} \quad x \in \mathbb{Q}_p, \alpha \in \mathcal{X}_p. \tag{4.7}$$

Therefore, by (4.6) and (4.7), one obtains the following lemma.

**Lemma 4.2.** *The $\mathbb{Q}_p$-algebra $\mathcal{M}_p$ of all $\mathbb{Q}_p$-infinite-matrices is realized on the p-adic Banach space $\mathcal{X}_p$ with its representation* (4.5).

However, the continuity (or boundedness) of elements of $\mathcal{M}_p$ is not guaranteed. Like in *operator theory*, define the *operator-norm* $\|\cdot\|$ on $\mathcal{M}_p$ by

$$\|T\| \overset{def}{=} \sup\{\|T(\alpha)\|_p \colon \|\alpha\|_p = 1\}. \tag{4.8}$$

By (4.2), one can re-define the operator-norm $\|\cdot\|$ of (4.8) by

$$\|T\| \overset{def}{=} \sup\{\|T(\alpha)\|_p \colon \alpha \in \mathcal{U}_p\}. \tag{4.9}$$

Remark here that the norm $\|\cdot\|$ of (4.8) is unbounded in general on $\mathcal{M}_p$. So, we construct the maximal subalgebra $\mathfrak{M}_p$ of $\mathcal{M}_p$, where $\|\cdot\|$ is complete on $\mathfrak{M}_p$.

**Definition 4.3.** Let $\mathcal{M}_p$ be the $\mathbb{Q}_p$-algebra of $\mathbb{Q}_p$-infinite-matrices, acting on the $p$-adic Banach space $\mathcal{X}_p$. Let $\|\cdot\|$ be the operator norm (4.8) or (4.9) on $\mathcal{M}_p$. Let $\mathfrak{M}_p$ be the maximal subalgebra of $\mathcal{M}_p$ (consisting of $\mathbb{Q}_p$-infinite-matrices), where $\|\cdot\|$ is complete on $\mathfrak{M}_p$. All elements of $\mathfrak{M}_p$ are called $p$-adic operators. We call $\mathfrak{M}_p$, the $p$-adic operator algebra (on the $p$-adic Banach space $\mathcal{X}_p$).

The following theorem characterizes an operator-algebraic property of the $p$-adic operator algebra $\mathfrak{M}_p$.

**Theorem 4.4.** *Let $\mathfrak{M}_p$ be the $p$-adic operator algebra on the $p$-adic Banach space $\mathcal{X}_p$. Then $\mathfrak{M}_p$ is a Banach $*$-algebra over the $p$-prime field $\mathbb{Q}_p$.*

*Proof.* Recall that $\mathfrak{M}_p$ is the $\|\cdot\|_p$-norm completion of an (pure algebraic) algebra $\mathcal{M}_p$ of all infinite-$\mathbb{Q}_p$-matrices acting on $\mathcal{X}_p$, over $\mathbb{Q}_p$. Thus, it is a Banach algebra over $\mathbb{Q}_p$ acting on $\mathcal{X}_p$.

Define now a unary operation $(*)$ by

$$[x_{ij}]^* \overset{def}{=} [x_{ji}] \quad \text{for all} \quad [x_{ij}] \in \mathfrak{M}_p. \tag{4.10}$$

Then the operation (4.10) satisfies

$$([x_{ij}]^*)^* = [x_{ji}]^* = [x_{ij}] \quad \text{for all} \quad [x_{ij}] \in \mathfrak{M}_p. \tag{4.11}$$

Also, it holds

$$([x_{ij}] + [y_{ij}])^* = [x_{ij} + y_{ij}]^* = [x_{ji} + y_{ji}] = [x_{ji}] + [y_{ji}] = [x_{ij}]^* + [y_{ij}]^* \tag{4.12}$$

and

$$([x_{ij}][y_{ij}])^* = \left( \left[ \sum_{k=1}^{\infty} x_{ik} y_{kj} \right]_{ij} \right)^* = \left[ \sum_{k=1}^{\infty} y_{jk} x_{ki} \right] = [y_{ji}][x_{ji}] = [y_{ij}]^*[x_{ij}]^* \tag{4.13}$$

for all $[x_{ij}]$, $[y_{ij}] \in \mathfrak{M}_p$.

So, by (4.11), (4.12) and (4.13), the $p$-adic operator algebra $\mathfrak{M}_p$ is a Banach algebra equipped with adjoint $(*)$ of (4.10). Equivalently, $\mathfrak{M}_p$ is a Banach $*$-algebra over $\mathbb{Q}_p$. □

The above theorem shows that our $p$-adic operators are elements of the Banach $*$-algebra $\mathfrak{M}_p$ over the $p$-prime field $\mathbb{Q}_p$ for all primes $p$.

## 4.1. $p$-ADIC DIAGONAL OPERATORS

As a starting point, we consider diagonal operators in the $p$-adic operator algebra $\mathfrak{M}_p$ acting on the $p$-adic Banach space $\mathcal{X}_p$. Remark that $T \in \mathfrak{M}_p$ if and only if $\|T\| < \infty$. Assume a $p$-adic operator $D$ has its form

$$D = \begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & x_3 & \\ 0 & & & \ddots \end{pmatrix} \in \mathfrak{M}_p \quad \text{with} \quad x_k \in \mathbb{Q}_p.$$

We call such operators $D$, *$p$-adic diagonal operators*. Denote $D$ by $diag(x_n)_{n=1}^{\infty}$, whenever one wants to emphasize the diagonal entries.

**Proposition 4.5.** *Let $D = diag(x_n)_{n=1}^{\infty}$ be a $p$-adic diagonal operator in $\mathfrak{M}_p$. Then*

$$\|D\| = \|(x_n)_{n=1}^{\infty}\|_p \,. \tag{4.14}$$

*Proof.* Observe that

$$\|D\| = \sup\{\|D((y_n)_{n=1}^{\infty})\|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\} = \sup\{\|(x_n y_n)_{n=1}^{\infty}\|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\} =$$
$$= \sup\left\{\sup\{|x_n y_n|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\right\} = \sup\left\{\sup\{|x_n|_p : n \in \mathbb{N}\}\right\} =$$

by (4.3) and (4.4)

$$= \sup\{|x_n|_p : n \in \mathbb{N}\} = \|(x_n)_{n=1}^{\infty}\|_p \,.$$

Therefore, we have

$$\|diag(x_n)_{n=1}^{\infty}\| = \|(x_n)_{n=1}^{\infty}\|_p \,.$$

□

Now, consider a special type of $p$-adic diagonal operators of $\mathfrak{M}_p$. Let $D_p = diag(p)_{n=1}^{\infty}$ be a $p$-adic diagonal operator of $\mathfrak{M}_p$, i.e.,

$$\begin{pmatrix} p & & \\ & p & \\ & & \ddots \end{pmatrix}$$

with

$$\|D_p\| = \left\|(p)_{n=1}^{\theta}\right\|_p = |p|_p = \frac{1}{p}.$$

By Section 2.1, the $p$-prime field $\mathbb{Q}_p$ has an embedded lattice

$$\ldots \subset p^2\mathbb{Z}_p \subset p\mathbb{Z}_p \subset \mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset \ldots. \tag{4.15}$$

If $x \in \mathbb{Q}_p$, then there exists $N \in \mathbb{N} \cup \{0\}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^{-N}x_0 \in p^{-N}\mathbb{Z}_p.$$

So, the element $px$ makes

$$px = p^{-N+1}x_0 \in p^{-N+1}\mathbb{Z}_p.$$

So, the $p$-adic diagonal operator $D_p$ acts like a shift (or a shift operator) on the filtering (4.15).

**Lemma 4.6.** *Let $D_p$ be a $p$-adic diagonal operator $diag(p)_{n=1}^{\infty}$ in the $p$-adic operator algebra $\mathfrak{M}_p$. If $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$, with $x_n \in p^{-N_n}\mathbb{Z}_p$ for $N_n \in \mathbb{N} \cup \{0\}$, for all $n \in \mathbb{N}$, then the image $(y_n)_{n=1}^{\infty} = D_p\left((x_n)_{n=1}^{\infty}\right)$ satisfies that $y_n \in p^{-N_n+1}\mathbb{Z}_p$ for all $n \in \mathbb{N}$.*

Similar to $D_p$, one can define $p$-adic diagonal operators

$$D_{p^k} = diag(p^k)_{n=1}^{\infty} \quad \text{in} \quad \mathfrak{M}_p \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.16}$$

It is trivial that the $p$-adic operators $D_{p^k}$ of (4.16) satisfy that

$$D_{p^k} = D_p^k \quad \text{in} \quad \mathfrak{M}_p \quad \text{for all} \quad k \in \mathbb{N}.$$

Then, by the modification of the above lemma, we obtain the following corollary.

**Corollary 4.7.** *Let $D_{p^k}$ be a $p$-adic diagonal operator in the sense of* (4.16) *in the $p$-adic Banach algebra $\mathfrak{M}_p$. If $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$ with $x_n \in p^{-N_n}\mathbb{Z}_p$ for $N_n \in \mathbb{N} \cup \{0\}$, for all $n \in \mathbb{N}$, then the image $(y_n)_{n=1}^{\infty} = D_p\left((x_n)_{n=1}^{\infty}\right)$ satisfies that $y_n \in p^{-N_n+k}\mathbb{Z}_p$ for all $n \in \mathbb{N}$.*

By the above lemma and corollary, we obtain the following theorem, which provides a normalization process among the $p$-adic diagonal operators.

**Theorem 4.8.** *Let $D = diag(x_n)_{n=1}^{\infty}$ be an arbitrary $p$-adic diagonal operator in $\mathfrak{M}_p$. Then there exist $k \in \mathbb{N}$ and the corresponding $p$-adic diagonal operator $D_{p^k}$ in the sense of* (4.16) *such that $\left\|D_{p^k}D\right\| = 1$.*

*Proof.* Let $D = diag(x_n)_{n=1}^{\infty}$ be a $p$-adic diagonal operator in $\mathfrak{M}_p$. Then, by (4.14), we have

$$\|D\| = \|(x_n)_{n=1}^{\infty}\|_p = \sup\{|x_n|_p : x_n \in \mathbb{Q}_p\},$$

and assume there exists $x_o$ in $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$ such that

$$\|D\| = |x_o|_p = p^{n_o} \quad \text{for some} \quad n_o \in \mathbb{Z}.$$

Assume that $N_o \in \mathbb{Z}$ such that $N_o + n_o = 0$, in $\mathbb{Z}$, i.e., $N_o = -n_o$. Define the $p$-adic diagonal operator $D_{p^{N_o}}$ as in (4.16). Then

$$D_{p^{N_o}}D = diag(p^{N_o}x_n)_{n=1}^{\infty} \quad \text{in} \quad \mathfrak{M}_p,$$

with $p^{N_o}x_o \in \mathbb{Z}_p$, by the above theorem. Hence,

$$\left\|D_{p^{N_0}}D\right\| = 1.$$

□

The above theorem shows that every $p$-adic diagonal operator $D$ in the $p$-adic operator algebra $\mathfrak{M}_p$ is normalized to a certain $p$-adic diagonal operator $D_0$, with $\|D_0\| = 1$.

## 4.2. $p$-ADIC WEIGHTED SHIFTS

As a continuation for studying some nice examples for $p$-adic operators of the $p$-adic Banach algebra $\mathfrak{M}_p$, acting on the $p$-adic Banach space $\mathcal{X}_p$, we introduce natural *shifts* (or *shift operators*) like in the usual *operator theory* (e.g., [16]).

Consider a function $U\colon \mathcal{X}_p \to \mathcal{X}_p$ defined by

$$U\left((x_1, x_2, \ldots)\right) = (0, x_1, x_2, \ldots) \quad \text{in} \quad \mathcal{X}_p \tag{4.17}$$

for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$. Then, as in [16], such an operator $U$ is expressed by a $p$-adic operator

$$\begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix} \quad \text{in} \quad \mathfrak{M}_p.$$

**Definition 4.9.** We call the $p$-adic operator $U$ of $\mathfrak{M}_p$, in the sense of (4.17), the $p$-adic (unilateral) shift.

Clearly, the $p$-adic operator $U^n = \underbrace{U \ldots U}_{n\text{-times}}$ of the $p$-adic shift $U$, satisfies that:

$$U^n\left((x_1, x_2, \ldots)\right) = \Big(\underbrace{0, \ldots, 0}_{n\text{-times}}, x_1, x_2, \ldots\Big), \tag{4.18}$$

on $\mathcal{X}_p$ for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$, for all $n \in \mathbb{N}$. We say the $p$-adic operator $U^n$ are the *$p$-adic $n$-shifts*, for all $n \in \mathbb{N}$. By definition, the $p$-adic 1-shift is nothing but the $p$-adic shift $U$ of (4.17).

It is not difficult to check that

$$\|U^n\| = 1 \quad \text{for all} \quad n \in \mathbb{N},$$

because

$$\left\|U^n((x_n)_{n=1}^{\infty})\right\|_p = \left\|\Big(\underbrace{0, \ \ldots\ldots.., 0}_{n\text{-times}}, \ x_1, x_2, \ \ldots\Big)\right\|_p = \|(x_n)_{n=1}^{\infty}\|_p$$

for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$.

**Proposition 4.10.** *If $U^n$ are the $p$-adic $n$-shifts, then*

$$\|U^n\| = 1 \quad \textit{for all} \quad n \in \mathbb{N}. \tag{4.19}$$

Let $U^*$ be the adjoint of the $p$-adic unilateral shift $U$. Then it has its $\mathbb{Q}_p$-matricial form

$$\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & \ddots \\ 0 & & & & \ddots \end{pmatrix} \quad \text{in} \quad \mathfrak{M}_p,$$

i.e., it satisfies

$$U^*\left((x_1, x_2, x_3, x_4, \ \ldots)\right) = (x_2, x_3, x_4, \ \ldots)$$

on $\mathcal{X}_p$ for all $(x_n)_{n=1}^{\infty} \in \mathcal{X}_p$.

**Proposition 4.11.** *Let $U^n$ be the $p$-adic $n$-shift. Then*

$$U^{n*}U^n = diag(1, 1, 1, \ldots)$$

*and*

$$U^n U^{n\,*} = diag\Big(\underbrace{0, \ldots, 0}_{n\text{-}times}, 1, 1, 1, \ \ldots\Big)$$

*for all $n \in \mathbb{N}$.*

Let $D$ be a $p$-adic diagonal operator $diag(x_n)_{n=1}^{\infty}$ in $\mathfrak{M}_p$, and let $U$ be the $p$-adic unilateral shift. Then the product $UD$ is equivalent to

$$\begin{pmatrix} 0 & & & & 0 \\ x_1 & 0 & & & \\ & x_2 & 0 & & \\ & & x_3 & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix} \quad \text{in} \quad \mathfrak{M}_p. \tag{4.20}$$

**Definition 4.12.** The above new $p$-adic operator $UD$ of (4.20) is called the $p$-adic weighted shift with weights $(x_n)_{n=1}^{\infty}$ in $\mathfrak{M}_p$. The $p$-adic operators $U^n D$ are called the $p$-adic weighted $n$-shifts, for all $n \in \mathbb{N}$.

With help of (4.14) and (4.19), we obtain the following operator-norm computation. It is a $p$-adic version of the usual weighted-shift-norm computation (e.g., [16]).

**Proposition 4.13.** *Let $W = U^n D$ be a $p$-adic weighted $n$-shift in $\mathfrak{M}_p$, for $n \in \mathbb{N}$, where $U^n$ is the $p$-adic $n$-shift and $D = diag(x_n)_{n=1}^{\infty}$ be a $p$-adic diagonal operator. Then*

$$\|W\| = \left\|(x_n)_{n=1}^{\infty}\right\|_p. \tag{4.21}$$

*Proof.* By definition, if $W = U^n D$ is a $p$-adic weighted $n$-shift, where $D = diag(x_n)_{n=1}^{\infty}$ in $\mathfrak{M}_p$, then

$$\begin{aligned} \|W\| &= \sup\Big\{\left\|W\left((y_n)_{n=1}^{\infty}\right)\right\|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\Big\} = \\ &= \sup\Big\{\Big\|\Big(\underbrace{0, \ \ldots, 0}_{n\text{-times}}, \ x_1y_1, x_2y_2, \ \ldots\Big)\Big\|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\Big\} = \\ &= \sup\Big\{\left\|(x_n y_n)_{n=1}^{\infty}\right\|_p : (y_n)_{n=1}^{\infty} \in \mathcal{U}_p\Big\} = \left\|(x_n)_{n=1}^{\infty}\right\|_p = \|D\|, \end{aligned}$$

by (4.14). □

## 4.3. $p$-ADIC TOEPLITZ OPERATORS

Let $U$ be the $p$-adic unilateral shift in the sense of (4.17) in the $p$-adic Banach algebra $\mathfrak{M}_p$, acting on the $p$-adic Banach space $\mathcal{X}_p$, for a fixed prime $p$. Now, we are interested in the Banach $*$-subalgebra

$$\mathfrak{T}_p = \overline{\mathbb{Q}_p[U, U^*]}^{\|\cdot\|}$$

of $\mathfrak{M}_p$, where $\mathbb{Q}_p[U,\, U^*]$ is a module over $\{U,\, U^*\}$ in the following sense, and where $\overline{Y}^{\|\cdot\|}$ mean the $\|\cdot\|$-operator-norm closures of all subsets $Y$ of $\mathfrak{M}_p$.

Define first a *polynomial ring* $\mathbb{Q}_p[t_1, t_2]$ with two $\mathbb{Q}_p$-*variables* (or two $\mathbb{Q}_p$-*indeterminents*) $t_1$ and $t_2$ over the $p$-prime field $\mathbb{Q}_p$ by

$$\mathbb{Q}_p[t_1, t_2] \overset{def}{=} \left\{ x_0 + \sum_{j=1}^{n_1} x_j t_1^j + \sum_{i=1}^{n_2} x_i t_2^i \;\middle|\; \begin{array}{c} x_k \in \mathbb{Q}_p \text{ for all } k, \\ \text{and for all } n_1, n_2 \in \mathbb{N} \end{array} \right\}.$$

Then it is a well-defined algebraic ring over a field $\mathbb{Q}_p$. Construct the module $\mathbb{Q}_p\,[U, U^*]$ by

$$\mathbb{Q}_p\,[U, U^*] = \{f(U, U^*) \colon f(t_1, t_2) \in \mathbb{Q}_p[t_1, t_2]\}, \tag{4.22}$$

i.e., if $T \in \mathbb{Q}_p\,[U, U^*]$, then

$$T = x_0 I_p + \sum_{j=1}^{n_1} x_j U^j + \sum_{i=1}^{n_2} x_i U^{*i}$$

for some $n_1,\, n_2 \in \mathbb{N}$, where $x_k \in \mathbb{Q}_p$, with identity:

$$I_p \overset{def}{=} U^* U = diag(1)_{n=1}^{\infty} \in \mathfrak{M}_p.$$

Then one can obtain the following proposition.

**Proposition 4.14.** *Let $\mathcal{M}_p$ be the* (*pure algebraic*) *algebra consisting of all $\mathbb{Q}_p$-infinite-matrices over $\mathbb{Q}_p$. Let $\mathcal{T}_p = \mathbb{Q}_p\,[U, U^*]$ be in the sense of* (4.22)*. Then $\mathcal{T}_p$ is a* (*pure algebraic*) *$*$-subalgebra of $\mathcal{M}_p$ over $\mathbb{Q}_p$.*

The proof is trivial by construction and definition.

**Definition 4.15.** Define a closed $*$-subalgebra

$$\mathfrak{T}_p \overset{def}{=} \overline{\mathbb{Q}_p[U, U^*]}^{\|\cdot\|}$$

of the $p$-adic Banach $*$-algebra $\mathfrak{M}_p$. Then it is called the $p$-adic Toeplitz algebra. All elements of $\mathfrak{T}_p$ are said to be $p$-adic Toeplitz operators.

**Proposition 4.16.** *If $T \in \mathfrak{T}_p$, if and only if*

$$T = \begin{pmatrix} x_0 & x_{-1} & x_{-2} & & & * \\ x_1 & x_0 & x_{-1} & x_{-2} & & \\ x_2 & x_1 & x_0 & x_{-1} & x_{-2} & \\ & x_2 & x_1 & x_0 & x_{-1} & \ddots \\ & & x_2 & x_1 & x_0 & \ddots \\ * & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.23}$$

*in* $\mathfrak{M}_p$ *over* $\mathbb{Q}_p$, *whenever*

$$T = x_0 I_p + \sum_{i=1}^{\infty} x_i U^i + \sum_{j=1}^{\infty} x_{-j} U^{*j}.$$

*Proof.* Since all elements of $\mathfrak{T}_p$ are generated by the $p$-adic unilateral shift $U$ and its adjoint $U^*$, the expression (4.23) is shown by the very definition of $p$-adic Toeplitz operators, and by Section 4.2. □

## 5. ADELIC OPERATORS ON $\mathcal{X}_{\mathbb{Q}}$

In this section, we study operators acting on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$. Recall that, by (3.16), this Banach space $\mathcal{X}_{\mathbb{Q}}$ is Banach-space isomorphic to the weak tensor product Banach space $\underset{p\in\mathcal{P}}{\Pi_{\varphi}} \mathcal{X}_p$ of $p$-adic Banach spaces $\mathcal{X}_p$ induced by the system $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations $\varphi_p$, over the adele ring $\mathbb{A}_{\mathbb{Q}}$, i.e., there exists a Banach space isomorphism $\Phi\colon \mathcal{X}_{\mathbb{Q}} \to \underset{p\in\mathcal{P}}{\Pi_{\varphi}} \mathcal{X}_p$ such that

$$\Phi\left(\left((x_{p:n})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}\right) = \left((x_{p:n})_{n=1}^{\infty}\right)_{p\in\mathcal{P}} \quad \text{in} \quad \underset{p\in\mathcal{P}}{\Pi_{\varphi}} \mathcal{X}_p \tag{5.1}$$

for all $((x_{p:n})_{p\in\mathcal{P}})_{n=1}^{\infty} \in \mathcal{X}_{\mathbb{Q}}$.

Define now a set $\mathcal{M}_{\mathbb{Q}}$ of all $\mathbb{A}_{\mathbb{Q}}$-infinite-matrices $[X_{ij}]$, acting on $\mathcal{X}_{\mathbb{Q}}$, with entries $X_{ij} \in \mathbb{A}_{\mathbb{Q}}$, i.e.,

$$[X_{ij}] = [(x_{p:ij})_{p\in\mathcal{P}}].$$

As in Section 4, the $\mathbb{A}_{\mathbb{Q}}$-infinite-matrix set $\mathcal{M}_{\mathbb{Q}}$ is equipped with matrix addition, $\mathbb{A}_{\mathbb{Q}}$-scalar product, and matrix multiplication as in operator theory.

Define the *operator norm* $\|\cdot\|$ *on* $\mathcal{M}_{\mathbb{Q}}$ by

$$\|[X_{ij}]\| = \sup\left\{\|[X_{ij}]\left((X_n)_{n=1}^{\infty}\right)\|_{\mathbb{Q}} : (X_n)_{n=1}^{\infty} \in \mathcal{U}_{\mathbb{Q}}\right\}, \tag{5.2}$$

where

$$\mathcal{U}_{\mathbb{Q}} \overset{def}{=} \{(X_n)_{n=1}^{\infty} \in \mathcal{X}_{\mathbb{Q}} \colon \|(X_n)_{n=1}^{\infty}\|_{\mathbb{Q}} = 1\},$$

where $\|\cdot\|_{\mathbb{Q}}$ is the norm on $\mathcal{X}_{\mathbb{Q}}$ in the sense of (3.13).

**Definition 5.1.** Let $\mathfrak{M}_{\mathbb{Q}}$ be the operator-norm closure of $\mathcal{M}_{\mathbb{Q}}$. Then we call $\mathfrak{M}_{\mathbb{Q}}$ the adelic operator set acting on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$ (over the adele ring $\mathbb{A}_{\mathbb{Q}}$). All elements of $\mathfrak{M}_{\mathbb{Q}}$ are said to be adelic operators (over $\mathbb{A}_{\mathbb{Q}}$).

Then one can have a following proposition.

**Proposition 5.2.** *The adelic operator set* $\mathfrak{M}_{\mathbb{Q}}$ *is a Banach* $*$*-algebra over the adele ring* $\mathbb{A}_{\mathbb{Q}}$.

*Proof.* By the very definition, the adelic operator set $\mathfrak{M}_{\mathbb{Q}}$ is complete under operator-norm topology inherited from that of $\mathcal{M}_{\mathbb{Q}}$.

Let $T_1 = [X_{ij}]$ and $T_2 = [Y_{ij}]$ be in $\mathfrak{M}_{\mathbb{Q}}$. Then

$$T_1 + T_2 = [X_{ij} + Y_{ij}] \quad \text{is in } \mathfrak{M}_{\mathbb{Q}}, \text{ too,}$$

where the entry-wise addition in right-hand side is in the sense of (2.12). Also, for any fixed $X \in \mathbb{A}_{\mathbb{Q}}$ and $T = [X_{ij}] \in \mathfrak{M}_{\mathbb{Q}}$,

$$XT = X[X_{ij}] = [XX_{ij}] \quad \text{is in } \mathfrak{M}_{\mathbb{Q}},$$

where the multiplication $XX_{ij}$ in the right-hand side is in the sense of (2.13). Therefore, the adelic operator set $\mathfrak{M}_{\mathbb{Q}}$ is a vector space over a ring $\mathbb{A}_{\mathbb{Q}}$, and hence it is a Banach space over $\mathbb{A}_{\mathbb{Q}}$.

Under (2.12) and (2.13),

$$T_1 T_2 = \left[ \sum_{k=1}^{\infty} X_{ik} Y_{kj} \right] \quad \text{is in } \mathfrak{M}_{\mathbb{Q}}, \text{ too.}$$

Furthermore,

$$T_1(T_2 + T_3) = T_1T_2 + T_1T_2,$$

and

$$(T_1 + T_2)T_3 = T_1T_3 + T_2T_3,$$

in $\mathfrak{M}_{\mathbb{Q}}$ for all $T_1$, $T_2$, $T_3 \in \mathfrak{M}_{\mathbb{Q}}$. It means that the Banach space $\mathfrak{M}_{\mathbb{Q}}$ is a Banach algebra over $\mathbb{A}_{\mathbb{Q}}$.

Define the unary operation $(*)$, called the adjoint, by

$$[X_{ij}]^* = [X_{ji}] \quad \text{for all} \quad [X_{ij}] \in \mathfrak{M}_{\mathbb{Q}}.$$

Then the adjoint $(*)$ is well-defined on $\mathfrak{M}_{\mathbb{Q}}$. Also, it satisfies that

$$(T^*)^* = T \ \text{ for all } \ T \in \mathfrak{M}_{\mathbb{Q}}, \quad (T_1 + T_2)^* = T_1^* + T_2^* \ \text{ for all } \ T_1, T_2 \in \mathfrak{M}_{\mathbb{Q}}$$

and

$$(T_1T_2)^* = T_2^* T_1^* \ \text{ for all } \ T_1, T_2 \in \mathfrak{M}_{\mathbb{Q}}.$$

So, $\mathfrak{M}_{\mathbb{Q}}$ is a Banach $*$-algebra over the adele ring $\mathbb{A}_{\mathbb{Q}}$. □

The above proposition shows that the adelic operator set $\mathfrak{M}_{\mathbb{Q}}$ is a Banach $*$-algebra over $\mathbb{A}_{\mathbb{Q}}$. From now on, we call $\mathfrak{M}_{\mathbb{Q}}$, the *adelic operator algebra* (over $\mathbb{A}_{\mathbb{Q}}$).

Let us consider detailed structure theorem of the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$.

Let $\mathfrak{M}_p$ be the $p$-adic operator algebras over the $p$-prime fields $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. Construct the product (topological) space $\prod_{p \in \mathcal{P}} \mathfrak{M}_p$ of them under the product topology of the $\| \cdot \|_p$-topologies, i.e., the topological space $\underset{p \in \mathcal{P}}{\Pi} \mathfrak{M}_p$, itself, is a Banach space. Furthermore, it is over the adele ring $\mathbb{A}_{\mathbb{Q}}$. Since each direct summand $\mathfrak{M}_p$ is over $\mathbb{Q}_p$, this Banach space $\prod_{p \in \mathcal{P}} \mathfrak{M}_p$ is over $\underset{p \in \mathcal{P}}{\Pi} \mathbb{Q}_p$. Since $\mathbb{A}_{\mathbb{Q}} \subset \prod_{p \in \mathcal{P}} \mathbb{Q}_p$, it is over $\mathbb{A}_{\mathbb{Q}}$.

Define now a binary operation $(+)$ on $\underset{p\in\mathcal{P}}{\Pi}\mathfrak{M}_p$ by

$$(T_p)_{p\in\mathcal{P}} + (S_p)_{p\in\mathcal{P}} \overset{def}{=} (T_p + S_p)_{p\in\mathcal{P}},$$

and another binary operation $(\cdot)$ on it by

$$((T_p)_{p\in\mathcal{P}})\,((S_p)_{p\in\mathcal{P}}) \overset{def}{=} (T_p S_p)_{p\in\mathcal{P}}$$

for all $(T_p)_{p\in\mathcal{P}}$, $(S_p)_{p\in\mathcal{P}} \in \prod_{p\in\mathcal{P}} \mathfrak{M}_p$, with a $\mathbb{A}_{\mathbb{Q}}$-scalar product,

$$(x_p)_{p\in\mathcal{P}}\,(T_p)_{p\in\mathcal{P}} \overset{def}{=} (x_p T_p)_{p\in\mathcal{P}}$$

for all $(x_p)_{p\in\mathcal{P}} \in \mathbb{A}_{\mathbb{Q}}$.

Then it is not difficult to check that the Banach space $\underset{p\in\mathcal{P}}{\Pi}\mathfrak{M}_p$ becomes a Banach algebra $\underset{p\in\mathcal{P}}{\otimes_{\mathbb{A}_{\mathbb{Q}}}}\mathfrak{M}_p$, which is the tensor product algebra over the adele ring $\mathbb{A}_{\mathbb{Q}}$. Furthermore, one may define the adjoint on $\underset{p\in\mathcal{P}}{\Pi}\mathfrak{M}_p$ by

$$((T_p)_{p\in\mathcal{P}})^* \overset{def}{=} \left(T_p^*\right)_{p\in\mathcal{P}}$$

for all $(T_p)_{p\in\mathcal{P}} \in \underset{p\in\mathcal{P}}{\Pi}\mathfrak{M}_p$.

**Proposition 5.3.** *Let $\prod_{p\in\mathcal{P}}\mathfrak{M}_p$ be the Banach space over the adele ring $\mathbb{A}_{\mathbb{Q}}$ introduced as above, where $\mathfrak{M}_p$ are $p$-adic operator algebras over $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. Then it is a Banach $*$-algebra over $\mathbb{A}_{\mathbb{Q}}$, and it is Banach-$*$-algebra-isomorphic to the tensor product algebra $\underset{p\in\mathcal{P}}{\otimes_{\mathbb{A}_{\mathbb{Q}}}}\mathfrak{M}_p$ over $\mathbb{A}_{\mathbb{Q}}$.*

Again, notice that the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$ is Banach-space isomorphic to the weak tensor product Banach space $\underset{p\in\mathcal{P}}{\Pi_{\varphi}}\mathcal{X}_p$ of $p$-adic Banach spaces $\mathcal{X}_p$, induced by the system $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ of $p$-normalizations $\varphi_p$.

Thus, one may define a morphism $\Omega\colon \mathfrak{M}_{\mathbb{Q}} \to \underset{p\in\mathcal{P}}{\otimes_{\mathbb{A}_{\mathbb{Q}}}}\mathfrak{M}_p$ by

$$\Omega\left([(x_{p:ij})_{p\in\mathcal{P}}]\right) \overset{def}{=} \underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}] \quad \text{in} \quad \underset{p\in\mathcal{P}}{\otimes_{\mathbb{A}_{\mathbb{Q}}}}\mathfrak{M}_p \tag{5.3}$$

for all $[(x_{p:ij})_{p\in\mathcal{P}}] \in \mathfrak{M}_{\mathbb{Q}}$. This morphism let us have equivalent forms $\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]$, acting on $\underset{p\in\mathcal{P}}{\Pi}\mathcal{X}_p$, of $\left[(x_{p:ij})_{p\in\mathcal{P}}\right]$, acting on $\mathcal{X}_{\mathbb{Q}}$.

Define now the functions $\Theta_p\colon \mathfrak{M}_p \to \mathfrak{M}_p$ by

$$\Theta_p\left([x_{ij}]\right) = [\varphi_p(x_{ij})] \quad \text{for all} \quad [x_{ij}] \in \mathfrak{M}_p, \tag{5.4}$$

for all $p \in \mathcal{P}$, where $\varphi_p$ are the $p$-normalizations in the sense of Section 3.

Define now the weak tensor product Banach $*$-algebra

$$\underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$$

induced by the system $\Theta = \{\Theta_p\}_{p\in\mathcal{P}}$ of the functions $\Theta_p$ in the sense of (5.4). Remark that it is a Banach $*$-subalgebra of $\underset{p\in\mathcal{P}}{\otimes_{\mathbb{A}_\mathbb{Q}}}\mathfrak{M}_p$.

**Theorem 5.4.** *Let $\mathfrak{M}_\mathbb{Q}$ be the adelic operator algebra, and let $\mathfrak{M}$ be the weak tensor product Banach $*$-algebra $\underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$ of the p-adic operator algebras $\mathfrak{M}_p$, induced by the system $\Theta = \{\Theta_p\}_{p\in\mathcal{P}}$ of $\Theta_p$ in the sense of* (5.4)*. Then the Banach $*$-algebras $\mathfrak{M}_\mathbb{Q}$ and $\mathfrak{M}$ are $*$-isomorphic over $\mathbb{A}_\mathbb{Q}$, i.e.,*

$$\mathfrak{M}_\mathbb{Q} \overset{*\text{-}iso}{=} \underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p. \tag{5.5}$$

*Proof.* We show that the morphism $\Omega$ of (5.3) is a $*$-isomorphism from $\mathfrak{M}_\mathbb{Q}$ onto $\mathfrak{M} = \underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$. We already checked at the above paragraphs that this morphism $\Omega$ let us have equivalent forms $\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]$ on $\underset{p\in\mathcal{P}}{\Pi}\mathcal{X}_p$, of $\left[(x_{p:ij})_{p\in\mathcal{P}}\right]$ on $\mathcal{X}_\mathbb{Q}$. So, this morphism $\Omega$ let us have equivalent forms $\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]$ on the weak tensor product Banach space $\underset{p\in\mathcal{P}}{\Pi_\varphi}\mathcal{X}_p$, of $\left[(x_{p:ij})_{p\in\mathcal{P}}\right]$ acting on $\mathcal{X}_\mathbb{Q}$.

One can check that $\Omega$ is injective, by the very definition. Also, one may have that the inverse morphism $\Omega^{-1}$ has its domain $\mathfrak{M}$, and hence, $\Omega$ is surjective onto $\underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$, too. Therefore,

$$\Omega\colon \mathfrak{M}_\mathbb{Q} \to \mathfrak{M}$$

is bijective.

Let $[(x_{p:ij})_{p\in\mathcal{P}}], [(y_{p:ij})_{p\in\mathcal{P}}] \in \mathfrak{M}_\mathbb{Q}$. Then

$$\begin{aligned}
\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right] + \left[(y_{p:ij})_{p\in\mathcal{P}}\right]\right) &= \Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}} + (y_{p:ij})_{p\in\mathcal{P}}\right]\right) = \\
&= \Omega\left(\left[(x_{p:ij} + y_{p:ij})_{p\in\mathcal{P}}\right]\right) = \underset{p\in\mathcal{P}}{\otimes}[x_{p:ij} + y_{p:ij}] = \\
&= \left(\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]\right) + \left(\underset{p\in\mathcal{P}}{\otimes}[y_{ij}]\right) = \\
&= \Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\right) + \Omega\left(\left[(y_{p:ij})_{p\in\mathcal{P}}\right]\right)
\end{aligned}$$

in $\mathfrak{M}$. For all $(x_p)_{p\in\mathcal{P}} \in \mathbb{A}_\mathbb{Q}$, we have

$$\begin{aligned}
\Omega\left((x_p)_{p\in\mathcal{P}}\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\right) &= \Omega\left(\left[(x_p)_{p\in\mathcal{P}}\,(x_{p:ij})_{p\in\mathcal{P}}\right]\right) = \\
&= \Omega\left(\left[(x_p x_{p:ij})_{p\in\mathcal{P}}\right]\right) = \underset{p\in\mathcal{P}}{\otimes}[x_p x_{p:ij}] = \\
&= (x_p)_{p\in\mathcal{P}}\left(\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]\right) = (x_p)_{p\in\mathcal{P}}\left(\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\right)\right).
\end{aligned}$$

Therefore, the morphism $\Omega$ is a Banach-space isomorphism over $\mathbb{A}_{\mathbb{Q}}$. Moreover,

$$\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\left[(y_{p:ij})_{p\in\mathcal{P}}\right]\right) = \Omega\left(\left[\sum_{k=1}^{\infty}(x_{p:ik})_{p\in\mathcal{P}}\,(y_{p:kj})_{p\in\mathcal{P}}\right]\right) =$$

$$= \Omega\left(\left[\sum_{k=1}^{\infty}(x_{p:ik}y_{kj})_{p\in\mathcal{P}}\right]\right) = \underset{p\in\mathcal{P}}{\otimes}\left[\sum_{k=1}^{\infty}x_{p:ik}y_{p:kj}\right] =$$

$$= \underset{p\in\mathcal{P}}{\otimes}([x_{p:ij}][y_{p:ij}]) = \left(\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]\right)\left(\underset{p\in\mathcal{P}}{\otimes}[y_{p:ij}]\right) =$$

$$= \left(\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\right)\right)\left(\Omega\left(\left[(y_{p:ij})_{p\in\mathcal{P}}\right]\right)\right).$$

Therefore, $\Omega$ is a Banach-algebra isomorphism over $\mathbb{A}_{\mathbb{Q}}$. Also, one can check that

$$\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]^{*}\right) = \Omega\left(\left[(x_{p:ji})_{p\in\mathcal{P}}\right]\right) = \underset{p\in\mathcal{P}}{\otimes}[x_{p:ji}] = \underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]^{*} = \left(\underset{p\in\mathcal{P}}{\otimes}[x_{p:ij}]\right)^{*} =$$

$$= \left(\Omega\left(\left[(x_{p:ij})_{p\in\mathcal{P}}\right]\right)\right)^{*}.$$

Since $\Omega$ is a $*$-isomorphism, and since $\mathcal{X}_{\mathbb{Q}}$ and $\underset{p\in\mathcal{P}}{\Pi_{\varphi}}\mathcal{X}_p$ are isomorphic Banach spaces, this morphism $\Omega$ is isometric, or norm-preserving. Therefore, two Banach $*$-algebras $\mathfrak{M}_{\mathbb{Q}}$ and $\underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$ are isomorphic. □

The above theorem characterize the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$ acting on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$ in terms of $p$-adic operator algebras $\mathfrak{M}_p$ acting on $p$-adic Banach spaces $\mathcal{X}_p$, under weak tensor product with product topology.

By the structure theorem (5.5), and by Sections 4.1, 4.2 and 4.3, one may consider the following adelic operators.

## 5.1. ADELIC DIAGONAL OPERATORS

Recall that the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$ is $*$-isomorphic to the weak tensor product Banach $*$-algebra $\mathfrak{M} = \underset{p\in\mathcal{P}}{\otimes_{\Theta}}\mathfrak{M}_p$ of $p$-adic operator algebras $\mathfrak{M}_p$, induced by the system $\Theta = \{\Theta_p\}_{p\in\mathcal{P}}$ of functions $\Theta_p$ on $\mathfrak{M}_p$, over the adele ring $\mathbb{A}_{\mathbb{Q}}$, by (5.5).

Now, let $D(p) = diag(x_{p:n})_{n=1}^{\infty}$ be $p$-adic diagonal operators of $\mathfrak{M}_p$ for $p \in \mathcal{P}$. The operator

$$D_o = \underset{p\in\mathcal{P}}{\otimes} D(p)$$

is very well-defined on $\underset{p\in\mathcal{P}}{\Pi_{\varphi}}\mathcal{X}_p$ (because each $\|D(p)\| < \infty$), isomorphic to $\mathcal{X}_{\mathbb{Q}}$. And hence, $D_o$ is in $\mathfrak{M}$. Therefore, by the $*$-isomorphism $\Omega^{-1}$ from $\mathfrak{M}$ onto $\mathfrak{M}_{\mathbb{Q}}$, where $\Omega$ is a $*$-isomorphism of (5.3).

One can define an element $D$ by

$$D = \Omega^{-1}(D_o) = diag\left((x_{p:n})_{p\in\mathcal{P}}\right)_{n=1}^{\infty} \quad \text{in} \quad \mathfrak{M}_{\mathbb{Q}}. \tag{5.6}$$

**Definition 5.5.** The elements $D$ of the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$ with their forms (5.6) are called the adelic diagonal operators on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$.

By (5.6), we obtain the following proposition.

**Proposition 5.6.** *Let $D = diag\left(\left(x_{p:n}\right)_{p\in\mathcal{P}}\right)_{n=1}^{\infty}$ be an adelic diagonal operator in $\mathfrak{M}_{\mathbb{Q}}$ with $\Omega(D) = \underset{p\in\mathcal{P}}{\otimes} D(p)$ in $\mathfrak{M}$. Then*

$$\|D\| = \sup\{|(x_{p:n})_{n=1}^{\infty}|_p : p \in \mathcal{P}\}. \tag{5.7}$$

*Proof.* The proof of (5.7) is straightforward, i.e., if $D$ is given as above, then

$$\|D\| = \sup\{\|D(p)\|_p : p \in \mathcal{P}\} = \sup\{|(x_{p:n})_{n=1}^{\infty}|_p : p \in \mathcal{P}\}.$$ □

Remember that, in Section 4.1, we showed that if $D(p)$ is a $p$-adic diagonal operator in $\mathfrak{M}_p$, then there exists a $p$-adic diagonal operator $D_o(p)$ in $\mathfrak{M}_p$ such that

$$\|D_o(p)D(p)\|_p = 1 \tag{5.8}$$

for $p \in \mathcal{P}$. So, by (5.7), we obtain the following theorem.

**Theorem 5.7.** *Let $D$ be an adelic diagonal operator in $\mathfrak{M}_{\mathbb{Q}}$. Then there exists an adelic diagonal operator $D_o$ such that $\|D_oD\| = 1$.*

*Proof.* Let $D$ be an adelic diagonal operator in $\mathfrak{M}_{\mathbb{Q}}$. Then it is uniquely equivalent to an operator $\underset{p\in\mathcal{P}}{\oplus} D(p)$ in the weak direct product Banach $*$-algebra $\mathfrak{M}$ (over $\mathbb{A}_{\mathbb{Q}}$), where each summand $D(p)$ is a $p$-adic diagonal operator in $\mathfrak{M}_p$. By the existence of $D_o(p)$ satisfying (5.8), there exists an operator $\underset{p\in\mathcal{P}}{\oplus} D_o(p)$ in $\mathfrak{M}$ such that

$$\left\|\left(\underset{p\in\mathcal{P}}{\oplus} D_o(p)\right)\left(\underset{p\in\mathcal{P}}{\oplus} D(p)\right)\right\|_{\oplus} = \left\|\underset{p\in\mathcal{P}}{\oplus}\left(D_o(p)D(p)\right)\right\|_{\oplus} = 1. \tag{5.9}$$

So, by the Banach $*$-algebra isomorphism $\Omega$, there exists

$$D_o = \Omega^{-1}\left(\underset{p\in\mathcal{P}}{\oplus} D_o(p)\right) \quad \text{in} \quad \mathfrak{M}_{\mathbb{Q}}$$

such that $\|D_oD\| = 1$, by (5.9). □

## 5.2. ADELIC WEIGHTED SHIFTS

In this section, we consider weighted shifts of the adelic Banach $*$-algebra $\mathfrak{M}_{\mathbb{Q}}$. Let $U_p$ be the $p$-adic unilateral shifts of the $p$-adic Banach $*$-algebras $\mathfrak{M}_p$, for all $p \in \mathcal{P}$. Since $\mathfrak{M}_{\mathbb{Q}}$ is $*$-isomorphic to the weak tensor product Banach $*$-algebra $\underset{p\in\mathcal{P}}{\otimes_{\Theta}} \mathfrak{M}_p$, induced by $\Theta$, one can define an element $U$ of $\mathfrak{M}_{\mathbb{Q}}$ by the equivalent form $\underset{p\in\mathcal{P}}{\otimes} U_p$.

Then the operator $U$ is a well-determined element of $\mathfrak{M}_{\mathbb{Q}}$, which is expressed by

$$\begin{pmatrix} (0_p)_p & & & \\ (1_p)_p & (0_p)_p & & \\ & (1_p)_p & (0_p)_p & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{in} \quad \mathfrak{M}_{\mathbb{Q}}, \tag{5.10}$$

by $\Omega^{-1}$, where

$$(0_p)_p = (0, 0, 0, \dots) \in \mathbb{A}_{\mathbb{Q}}$$

and

$$(1_p)_p = (1, 1, 1, \dots) \in \mathbb{A}_{\mathbb{Q}}.$$

**Definition 5.8.** Let $U$ be an adelic operator (5.10) in $\mathfrak{M}_{\mathbb{Q}}$, equivalent to $\underset{p\in\mathcal{P}}{\otimes} U_p$ in $\underset{p\in\mathcal{P}}{\otimes_\Theta} \mathfrak{M}_p$. Then it is called the adelic unilateral shift on $\mathcal{X}_{\mathbb{Q}}$.

It is not difficult to check that

$$\Omega\left(U^n\right) = \underset{p\in\mathcal{P}}{\otimes} U_p^n \quad \text{for all} \quad n \in \mathbb{N}, \tag{5.11}$$

where $U_p^n$ are the $p$-adic $n$-shifts for all $p \in \mathcal{P}$ and $n \in \mathbb{N}$, and

$$\Omega\left(U^*\right) = \underset{p\in\mathcal{P}}{\oplus} U_p^*,$$

and hence

$$\Omega(U^{*\ n}) = \Omega(U^{n\ *}) = \underset{p\in\mathcal{P}}{\oplus} U_p^{n*} \tag{5.12}$$

for all $n \in \mathbb{N}$. The adelic operators $U^n$ of (5.11) are called the *adelic $n$-shifts of* $\mathfrak{M}_{\mathbb{Q}}$ for all $n \in \mathbb{N}$.

Similar to Section 4.2, if $D$ is an adelic diagonal operator, and if $U^n$ is an adelic $n$-shift, for $n \in \mathbb{N}$, then we define the adelic weighted $n$-shift $U^n D$ in $\mathfrak{M}_{\mathbb{Q}}$.

**Definition 5.9.** Let $D$ be an adelic diagonal operator, and let $U^n$ be the adelic $n$-shift in the adelic Banach $*$-algebra $\mathfrak{M}_{\mathbb{Q}}$, for $n \in \mathbb{N}$. The element $U^n D$ of $\mathfrak{M}_{\mathbb{Q}}$ is called the weighted $n$-shift on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$ for $n \in \mathbb{N}$.

Adelic weighted $n$-shifts are characterized by the following proposition.

**Proposition 5.10.** *Let $D = diag\left((x_{p:n})_{p\in\mathcal{P}}\right)_{n=1}^{\infty}$ be an adelic diagonal operator and let $U^n$ be the adelic $n$-shift of $\mathfrak{M}_{\mathbb{Q}}$ for $n \in \mathbb{N}$. Let $W = U^n D$ be the corresponding adelic weighted $n$-shift in $\mathfrak{M}_{\mathbb{Q}}$. Then $W$ is equivalent to $\underset{p\in\mathcal{P}}{\otimes} U_p^n D_p$ in $\underset{p\in\mathcal{P}}{\otimes_\Theta} \mathfrak{M}_p$, where $U_p^n D_p$ are the $p$-adic weighted $n$-shift of the $p$-adic Banach $*$-algebra $\mathfrak{M}_p$, in the sense of Section 4.2, for $p \in \mathcal{P}$. In particular,*

$$D_p = diag(x_{p:n})_{n=1}^{\infty} \in \mathfrak{M}_p.$$

*Proof.* Let $W = U^n D$ be an adelic weighted $n$-shift in $\mathfrak{M}_{\mathbb{Q}}$, for $D = diag((x_{p:n})_{p\in\mathcal{P}})_{n=1}^{\infty}$, and $n \in \mathbb{N}$. We considered that the adelic $n$-shift $U^n$ is unitarily equivalent to $\underset{p\in\mathcal{P}}{\oplus} U_p^n$ in the weak direct product Banach $*$-algebra $\mathfrak{M} = \underset{p\in\mathcal{P}}{\oplus_{\Theta}} \mathfrak{M}_p$, where $U_p$ are the $p$-adic unilateral shifts in the $p$-adic Banach $*$-algebras $\mathfrak{M}_p$. Similarly, the adelic diagonal operator $D$ is unitarily equivalent to $\underset{p\in\mathcal{P}}{\oplus} D_p$ in $\mathfrak{M}$, where $D_p$ are the $p$-adic diagonal operators $diag(x_{p:n})_{n=1}^{\infty}$, by the $*$-isomorphism $\Omega$ from $\mathfrak{M}_{\mathbb{Q}}$ onto $\mathfrak{M}$. Therefore,

$$\Omega(W) = \Omega(U^n D) = \left(\underset{p\in\mathcal{P}}{\oplus} U_p^n\right)\left(\underset{p\in\mathcal{P}}{\oplus} D_p\right) = \underset{p\in\mathcal{P}}{\oplus} U_p^n D_p,$$

where $U_p^n D_p$ are the $p$-adic weighted $n$-shifts in the sense of Section 4.2, for all $p \in \mathcal{P}$. In particular, if $p = \infty$ in $\mathcal{P}$, the $\infty$-unilateral shift $U_\infty$ is the usual shift on $\mathbb{R}^\infty (= (\mathbb{Q}_\infty)^\infty)$, and $D_\infty$ is a diagonal operator on $\mathbb{R}^\infty$. □

## 5.3. ADELIC TOEPLITZ OPERATORS

In this section, we introduce adelic Toeplitz operators acting on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$, contained in the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$. Let $U$ be the adelic unilateral shift (5.10), satisfying (5.11) and (5.12). Define a (closed) $*$-subalgebra $\mathfrak{T}_{\mathbb{Q}}$ of $\mathfrak{M}_{\mathbb{Q}}$ by

$$\mathfrak{T}_{\mathbb{Q}} \overset{def}{=} \overline{\mathbb{A}_{\mathbb{Q}}[U, U^*]}^{\|\cdot\|} \quad \text{in} \quad \mathfrak{M}_{\mathbb{Q}}, \tag{5.13}$$

where $\mathbb{A}_{\mathbb{Q}}[t_1, t_2]$ is the polynomial ring over the adele ring $\mathbb{A}_{\mathbb{Q}}$ with two $\mathbb{A}_{\mathbb{Q}}$-*variables* $t_1$ and $t_2$ (as in Section 4.3). Then the (pure algebraic) algebra $\mathbb{A}_{\mathbb{Q}}[U, U^*]$ is a well-defined (non-closed) $*$-subalgebra of the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$. By completing $\mathbb{A}_{\mathbb{Q}}[U, U^*]$ under the operator-norm-topology for $\mathfrak{M}_{\mathbb{Q}}$, the $*$-algebra $\mathfrak{T}_{\mathbb{Q}}$ of (5.13) becomes a closed $*$-subalgebra of the adelic operator algebra $\mathfrak{M}_{\mathbb{Q}}$, i.e., it is a Banach $*$-algebra over $\mathbb{A}_{\mathbb{Q}}$, too.

**Definition 5.11.** The closed $*$-subalgebra $\mathfrak{T}_{\mathbb{Q}}$ (5.13) of $\mathfrak{M}_{\mathbb{Q}}$ is called the adelic Toeplitz algebra (over the adele ring $\mathbb{A}_{\mathbb{Q}}$). And all elements of $\mathfrak{T}_{\mathbb{Q}}$ are said to be adelic Toeplitz operators on the adelic Banach space $\mathcal{X}_{\mathbb{Q}}$.

By the structure theorem (5.5) of $\mathfrak{M}_{\mathbb{Q}}$, we obtain the following structure theorem for the adelic Toeplitz algebra $\mathfrak{T}_{\mathbb{Q}}$.

**Theorem 5.12.** *Let $\mathfrak{T}_{\mathbb{Q}}$ be the adelic Toeplitz algebra over $\mathbb{A}_{\mathbb{Q}}$, and let $\mathfrak{T}_p$ be p-adic Toeplitz algebras over $\mathbb{Q}_p$, for $p \in \mathcal{P}$. Then*

$$\mathfrak{T}_{\mathbb{Q}} \overset{*\text{-}iso}{=} \underset{p\in\mathcal{P}}{\otimes_{\Theta}} \mathfrak{T}_p. \tag{5.14}$$

*Proof.* By constructions and definitions, one can determine the $*$-isomorphism $\Omega_0$ from the adelic Toeplitz algebra $\mathfrak{T}_{\mathbb{Q}}$ onto the $*$-subalgebra $\mathfrak{T} = \underset{p\in\mathcal{P}}{\otimes_{\Theta}} \mathfrak{T}_p$ of $\mathfrak{M} = \underset{p\in\mathcal{P}}{\oplus_{\Theta}} \mathfrak{M}_p$. Indeed, if $\Omega$ is in the sense of (5.3), then one can get the $*$-isomorphism,

$$\Omega_0 \overset{def}{=} \Omega \mid_{\mathfrak{T}_{\mathbb{Q}}}, \quad \text{the restriction of } \Omega \text{ on } \mathfrak{T}_{\mathbb{Q}}, \tag{5.15}$$

is a well-determined $*$-isomorphism from $\mathfrak{T}_{\mathbb{Q}}$ onto $\mathfrak{T}$. □

The above characterization (5.14) of the adelic Toeplitz algebra $\mathfrak{T}_{\mathbb{Q}}$ also shows that the properties of adelic Toeplitz operators are fully determined by those of $p$-adic Toeplitz operators. More precisely, one can get the following corollary.

**Corollary 5.13.** *Let $T$ be an adelic Toeplitz operator of the adelic Toeplitz algebra $\mathfrak{T}_{\mathbb{Q}}$. Then there exist $p$-adic Toeplitz operators $T_p$ of $p$-adic Toeplitz algebras $\mathfrak{T}_p$ such that $T$ is equivalent to $\underset{p\in\mathcal{P}}{\otimes} T_p$ in $\underset{p\in\mathcal{P}}{\otimes_\Theta} \mathfrak{T}_p$.*

**Acknowledgments**
*The author specially thanks to editors and reviewers of the journal Opuscula Math. for valuable advices and warm kindness.*

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*Received: April 11, 2013.*
*Revised: June 21, 2013.*
*Accepted: July 22, 2013.*