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SOLVABILITY OF FUNCTIONAL QUADRATIC INTEGRAL EQUATIONS WITH PERTURBATION

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Abstract. We study the existence of solutions of the functional quadratic integral equation with a perturbation term in the space of Lebesgue integrable functions on an unbounded interval by using the Krasnoselskii fixed point theory and the measure of weak noncompactness.

Keywords: quadratic integral equation, measure of noncompactness, Krasnoselskii fixed point theorem, superposition operators.

Mathematics Subject Classification: 45G10, 47H30, 47N20.

1. INTRODUCTION

Nonlinear quadratic functional integral equations are often applicable for instance in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, in traffic theory and in numerous branches of mathematical physics (cf. [3, 7, 10, 11]). Additionally, some problems (motivated by some practical interests in plasma physics) was investigated in [28]. It is worthwhile mentioning that several problems considered in the theory of ordinary and partial differential equations lead us to integral or functional integral equations [15, 16, 23, 30].

We study the solvability of the following functional integral equation

$$x(t) = g(t, x(\varphi_3(t))) + f_1\left(t, f_2(t, x(\varphi_2(t))) \int_0^t u(t, s, x(\varphi_1(s)))ds\right), \quad t \in \mathbb{R}^+. \tag{1.1}$$

This equation has been studied for the non quadratic integral equation in [4] with $g = 0$, $f_2 = 1$ using the Schauder fixed point theorem and in [32] with a perturbation term. In [6], the existence of monotonic solutions was checked, where $g(t, x(t)) = h(t)$, $f_2 = 1$, this was also done by Emmanuele (cf. [22]). The authors used the general

Krasnoselskii fixed point theorem to obtain the existence result (cf. [20,29,32]). In [35] the author studied a special case of our equation in a general Banach space X by using the classical Krasnoselskii fixed point theorem. For quadratic integral equations a special case of (1.1) was considered in [12] with $g(t,x(t)) = h(t)$ and $f_2(t,x(t)) = x(t)$. If $g(t,x(t)) = h(t)$, $f_1(t,x(t)) = x(t)$ and $f_2(t,x(t)) = \lambda x(t)$ has been studied in Orlicz spaces in [13].

In the case of the sum of two sufficiently regular operators the contraction condition is easily verified. Nevertheless, a construction for the set M makes the above theorem more restrictive. The presence of the perturbation term $g(t,x(t))$ in the integral equation makes the Schauder fixed point theorem unavailable. Given operators A and B, it may be possible to find the bounded domains M_A and M_B in such a way that $A : M_A \to M_A$ and $B : M_B \to M_B$, but it is often impossible to arrange matters so that $M_A = M_B = M$ and $Ax + By \in M_A$ for $x, y \in M$. Actually, the Krasnoselskii fixed point theorem allows us to avoid these problems when obtaining the result of the solution.

The results presented in this paper are motivated by extending the recent results to the functional quadratic integral equation with a perturbation term by using the classical Krasnoselskii fixed point theorem and the measure of weak noncompactness.

2. PRELIMINARIES

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^+$ be the interval $[0,\infty)$. If A is a Lebesgue measurable subset of $\mathbb{R}$, then the symbol $meas(A)$ stands for the Lebesgue measure of A. Further, denote by $L_1(A)$ the space of all real functions defined and Lebesgue measurable on the set A. If $x \in L_1(A)$ then the norm of x is defined in the standard way by the formula

$$\|x\| = \|x\|_{L_1(A)} = \int_A |x(t)|dt.$$

Obviously $L_1(A)$ forms a Banach space under this norm. The space $L_1(A)$ will be called the Lebesgue space. In the case when $A = \mathbb{R}^+$ we will write L_1 instead of $L_1(\mathbb{R}^+)$.

Denote by $BC(\mathbb{R}^+)$ the Banach space of all real functions defined, continuous and bounded on $\mathbb{R}^+$. This space is furnished with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}^+\}$.

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}^+)$ and a positive number T. For $x \in X$ and $\varepsilon \geq 0$ let us denote by $\omega^T(x,\varepsilon)$ the modulus of continuity of the function x, on the closed and bounded interval $[0,T]$ (cf. [9]) defined by

$$\omega^T(x,\varepsilon) = \sup\{|x(t_2) - x(t_1)| : t_1, t_2 \in [0,T], \quad |t_2 - t_1| \leq \varepsilon\}.$$

One of the most important operators studied in nonlinear functional analysis is the so-called superposition operator [1]. Now, let us assume that $I \subset \mathbb{R}$ is a given interval, bounded or not.

Definition 2.1. Assume that a function $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function x being measurable on I we may assign the function

$$F_f(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F_f defined in such a way is called the *superposition* (*Nemytskii*) *operator* generated by the function f.

Furthermore, for every $f \in L_1$ and every $\phi : I \to I$ we define the superposition operator generated by the functions f and ϕ, $F_{\phi,f} : L_1(I) \to L_1(I)$ as

$$F_{\phi,f}(t) = f(t, x(\phi(t))), \quad t \in I.$$

We have the following theorem.

Theorem 2.1. *Suppose that f satisfies the Carathéodory conditions. The superposition operator F maps the space $L_1(I)$ into $L_1(I)$ if and only if*

$$|f(t,x)| \leq a(t) + b|x|, \tag{2.2}$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L_1(I)$ and $b \geq 0$. Moreover, this operator is continuous.

This theorem was proved by Krasnoselskii [25] in the case when I is a bounded interval. The generalization to the case of an unbounded interval I was given by Appell and Zabrejko [1].

Let I be an interval in the following two theorems "Lusin and Dragoni" [18, 31], which explain the structure of measurable functions and functions satisfying Carathéodory conditions, where D^c denotes the complement of D.

Theorem 2.2. *Let $m : I \to \mathbb{R}$ be a measurable function. For any $\varepsilon > 0$ there exists a closed subset D_ε of the interval I such that meas$(D_\varepsilon^c) \leq \varepsilon$ and $m|_{D_\varepsilon}$ is continuous.*

Theorem 2.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the Carathéodory conditions. Then for each $\varepsilon > 0$ there exists a closed subset D_ε of the interval I such that meas$(D_\varepsilon^c) \leq \varepsilon$ and $f|_{D_\varepsilon \times \mathbb{R}}$ is continuous.*

We need to recall some basic facts about Urysohn operators. The continuity for Urysohn operators is not "automatic" as in the case of superposition operators. We will use also the majorant principle for Urysohn operators (cf. [27, Theorem 10.1.11]). The following theorem which is a particular case of a much more general result ([27, Theorem 10.1.16]), will be very useful in the proof of the main result for operators in $L_\infty(\mathbb{R}^+)$.

Theorem 2.4 ([27]). *Let $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t,s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t,s). Assume that*

$$|u(t,s,x)| \leq k(t,s),$$

where the nonnegative function k is measurable in (t,s) such that the linear integral operator K_0 with the kernel $k(t,s)$ maps $L_1(\mathbb{R}^+)$ into $L_\infty(\mathbb{R}^+)$. Then the operator U maps $L_1(\mathbb{R}^+)$ into $L_\infty(\mathbb{R}^+)$. Moreover, if for arbitrary $h > 0$

$$\lim_{\delta \to 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t,s,x_1) - u(t,s,x_2)| ds \right\|_{L_\infty(I)} = 0,$$

then U is a continuous operator.

We mention also that some particular conditions guaranteeing the continuity of the operator U may be found in [33, 34].

For the continuous case the situation is simpler (cf. [9, 21], for instance) and [12] for more information about the multiplication of the operators.

Theorem 2.5 ([26]). *Let M be a nonempty, closed, and convex subset of E. Suppose, that A, B be two operators such that:*

(i) $A(M) + B(M) \subseteq M$,
(ii) *A is a contraction mapping,*
(iii) *$B(M)$ is relatively compact and B is continuous.*

Then there exists a $y \in M$ with $Ay + By = y$.

Now we present the concept of measure of weak non compactness. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E^W$ its subfamily consisting of all relatively weakly compact sets. The symbol $\bar{X}^W$ stands for the weak closure of a set X and the symbol $Conv\, X$ will denote the convex closed hull of a set X.

Now we present the following definition [8].

Definition 2.2. A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a *measure of weak noncompactness* in E if it satisfies the following conditions:

(i) The family Ker $\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and Ker $\mu \subset \mathcal{N}_E^W$,
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
(iii) $\mu(Conv X) = \mu(X)$,
(iv) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$,
(v) If $X_n \in \mathcal{M}_E$ and $X_n = \bar{X_n}^W$, and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.

It is worthwhile mentioning that the first important example of measure of weak noncompactness has been defined by De Blasi [14] in the following way:

$$\beta(X) = \inf\{r > 0 : \text{ there exists a weakly compact subset } W \text{ of } E \\ \text{ such that } x \subset W + B_r\}.$$

Also, we recall the following criterion for weak noncompactness due to Dieudonné [17, 19], which is of fundamental importance in our subsequent analysis.

Theorem 2.6. *A bounded set X is relatively weakly compact in L_1 if and only if the following two conditions are satisfied:*

(a) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $meas(D) < \delta$ then $\int\limits_D |x(t)|dt \leq \varepsilon$ for all $x \in X$,*

(b) *for any $\varepsilon > 0$ there is $T > 0$ such that $\int\limits_T^\infty |x(t)|dt \leq \varepsilon$ for any $x \in X$.*

Now, for a nonempty and bounded subset X of the space L_1 let us define:

$$c(X) = \lim_{\epsilon \to 0} \sup_{x \in X} \Big\{ \sup \Big\{ \int\limits_D |x(t)|dt \colon D \subset \mathbb{R}^+, meas(D) \leq \varepsilon \Big\} \Big\}, \tag{2.3}$$

and

$$d(X) = \lim_{T \to \infty} \sup \Big\{ \int\limits_T^\infty |x(t)|dt \colon x \in X \Big\}. \tag{2.4}$$

Put

$$\mu(X) = c(X) \ + \ d(X). \tag{2.5}$$

It can be shown [5] that the function μ is a measure of weak noncompactness in the space L_1.

Remark 2.1. Suitable choices for E are the space $\mathcal{C}$ of continuous function, the Hölder spaces C^α, the Lebesgue spaces L_p, the Orlicz spaces L_φ, or more generally, ideal spaces (cf. [2]). If $x \in E$ and $y \in L_\infty$ implies that $xy \in E$ and $\|xy\|_E \leq \|x\|_E \|y\|_{L_\infty}$, i.e. the elements from L_∞ are pointwise multipliers for E. For more details in L_1 space see [12].

3. MAIN RESULT

Equation (1.1) takes the following form

$$x = Ax + Bx,$$

where

$$Ax(t) = F_{\varphi_3, g} x(t) - g(t, 0),$$

$$(Bx)(t) = f_1\left(t, f_2(t, x(\varphi_2(t))) \int_0^t u(t, s, x(\varphi_1(s)))ds\right) + $$
$$+ g(t, 0) = F_{f_1}(Kx)(t) + g(t, 0),$$

$$Kx(t) = F_{\varphi_2, f_2}x \cdot Ux(\varphi_1)(t) \quad \text{and} \quad Ux(t) = \int_0^t u(t, s, x(s))\, ds.$$

We shall treat the equation (1.1) under the following assumptions listed below.

(i) $g, f_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and there are positive functions $a_i \in L_1$ and constants $b_i \geq 0$ for $i = 1, 2, 3$ such that

$$|f_i(t, x)| \leq a_i(t) + b_i\, |x|, i = 1, 2, \quad \text{and} \quad |g(t, 0)| \leq a_3(t),$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, the function g is assumed to satisfy the Lipschitz condition with constant b_3 for almost all t:

$$|g(t, x) - g(t, y)| \leq b_3|x - y|.$$

(ii) $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s). Moreover, for arbitrary fixed $s \in \mathbb{R}^+$ and $x \in \mathbb{R}$ the function $t \mapsto u(t, s, x(s))$ is integrable.

(iii) There exists a function $k(t, s) = k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ which satisfies the Carathéodory conditions such that

$$|u(t, s, x)| \leq k(t, s)$$

for all $t, s \geq 0$ and $x \in \mathbb{R}$, such that the linear integral operator K_0 with kernel $k(t, s)$ maps L_1 into L_∞. Moreover, assume that for arbitrary $h > 0$ $(i = 1, 2)$

$$\lim_{\delta \to 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L^\infty} = 0.$$

(iv) $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing absolutely continuous function and there are positive constants M_i such that $\varphi_i' \geq M_i$ a.e. on $\mathbb{R}^+$ for $i = 1, 2, 3$.

(v) $q = \left(\frac{b_3}{M_3} + \frac{b_2 b_1}{M_2} \|K_0\|_\infty\right) < 1$.

(vi) $p = \frac{b_3}{M_3} < 1$.

Then we can prove the following theorem.

Theorem 3.1. *Assume assumptions* (i)–(vi) *are satisfied, then equation* (1.1) *has at least one integrable solution on* $\mathbb{R}^+$.

Proof. The proof will be given in six steps.

Step 1. The operator $A : L_1 \to L_1$ is a contraction mapping.
Step 2. We will construct the ball B_r such that $A(B_r) + B(B_r) \subseteq B_r$, where r will be determined later.
Step 3. We will prove that $\mu(A(Q) + B(Q)) \le q\mu(Q)$ for all bounded subset Q of B_r.
Step 4. We will construct a nonempty closed convex weakly compact set M in on which we will apply a fixed point theorem to prove the existence of solutions.
Step 5. $B(M)$ is relatively strongly compact in L_1.
Step 6. We will check out the conditions needed in Krasonselskii's fixed point theorem are fulfilled.

Step 1. From assumption (i), we have

$$\| g(t,x)| - |g(t,0)\| \le |g(t,x) - g(t,0)| \le b_3|x|, \tag{3.6}$$

$$|g(t,x)| - a_3(t) \le \ b_3|x| \quad \Rightarrow \quad |g(t,x)| \le a_3(t) + b_3|x|. \tag{3.7}$$

The inequality obtained above with Theorem 2.1 permits us to deduce that the operator A maps L_1 into itself. Now,

$$\begin{aligned}\int_0^\infty |(Ax)(t) - (Ay)(t)|dt &= \int_0^\infty |g(t, x(\varphi_3(t))) - g(t, y(\varphi_3(t)))|dt \le \\ &\le b_3 \int_0^\infty |x(\varphi_3(t)) - y(\varphi_3(t))|dt \le \\ &\le \frac{b_3}{M_3} \int_0^\infty |x(\varphi_3(t)) - y(\varphi_3(t))|\varphi_3'(t)dt = \\ &= \frac{b_3}{M_3} \int_{\varphi_3(0)}^{\varphi_3(\infty)} |x(v) - y(v)|dv \le \\ &\le \frac{b_3}{M_3} \int_0^\infty |x(v) - y(v)|dv,\end{aligned}$$

which implies that

$$\|Ax - Ay\|_{L_1} \le \frac{b_3}{M_3}\|x - y\|_{L_1}. \tag{3.8}$$

Assumption (vi) permits us to deduce that the operator A is a contraction mapping.

Step 2. Let x and y be arbitrary functions in $B_r \subset L_1(\mathbb{R}^+)$. In view of our assumptions we get a priori estimation

$$
\begin{aligned}
\|Ax + By\|_{L_1} &= \int_0^\infty \left| g(t, x(\varphi_3(t))) + f_1\Big(t, f_2(t, y(\varphi_2(t))) \int_0^t u(t,s,y(\varphi_1(s)))ds\Big)\right| dt \le \\
&\le \int_0^\infty |g(t, x(\varphi_3(t)))| dt + \int_0^\infty \left| f_1\Big(t, f_2(t, y(\varphi_2(t))) \int_0^t u(t,s,y(\varphi_1(s)))ds\Big)\right| dt \le \\
&\le \int_0^\infty \big[a_3(t) + b_3|x(\varphi_3(t))|\big] dt + \\
&\quad + \int_0^\infty \Big[a_1(t) + b_1|f_2(t, y(\varphi_2(t)))| \int_0^t |u(t,s,y(\varphi_1(s)))|\, ds\Big] dt \le \\
&\le \int_0^\infty \big[a_3(t) + b_3|x(\varphi_3(t))|\big] dt + \int_0^\infty \Big[a_1(t) + b_1(a_2(t) + b_2|y(\varphi_2(t))|) \int_0^t k(t,s)\, ds\Big] dt \le \\
&\le \|a_1\|_{L_1} + \|a_3\|_{L_1} + \\
&\quad + \frac{b_3}{M_3}\int_0^\infty |x(\varphi_3(t))|\varphi_3'(t) dt + b_1\|K_0\|_{L_\infty}\int_0^\infty (a_2(t) + b_2|y(\varphi_2(t))|) dt \le \\
&\le \|a_1\|_{L_1} + \|a_3\|_{L_1} + \\
&\quad + \frac{b_3}{M_3}\int_{\varphi_3(0)}^{\varphi_3(\infty)} |x(v)| dv + b_1\|K_0\|_{L_\infty}\Big[\|a_2\|_{L_1} + \frac{b_2}{M_2}\int_0^\infty |y(\varphi_2(t))|\varphi_2'(t) dt\Big] \le \\
&\le \|a_1\|_{L_1} + \|a_3\|_{L_1} + \\
&\quad + \frac{b_3}{M_3}\int_0^\infty |x(t)| dt + b_1\|K_0\|_{L_\infty}\Big[\|a_2\|_{L_1} + \frac{b_2}{M_2}\int_0^\infty |y(\varphi_2(t))|\varphi_2'(t) dt\Big] \le \\
&\le \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3}\|x(t)\|_{L_1} + b_1\|a_2\|_{L_1}\|K_0\|_{L_\infty} + \frac{b_1 b_2}{M_2}\|K_0\|_{L_\infty}\|y\|_{L_1} \le \\
&\le \|a_1\|_{L_1} + \|a_3\|_{L_1} + \frac{b_3}{M_3} r + b_1\|a_2\|_{L_1}\|K_0\|_{L_\infty} + \frac{b_1 b_2}{M_2}\|K_0\|_{L_\infty} r \le r.
\end{aligned}
$$

From the above estimate, we have that $A(B_r) + B(B_r) \subseteq B_r$ provided

$$
r = \frac{\|a_1\|_{L_1} + \|a_3\|_{L_1} + b_1\|a_2\|_{L_1}\|K_0\|_{L_\infty}}{1 - (\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2}\|K_0\|_{L_\infty})} > 0.
$$

Step 3. Take an arbitrary number $\varepsilon > 0$ and a set $D \subset \mathbb{R}^+$ such that $meas(D) \leq \varepsilon$. For any $x, y \in Q$, we have

$$
\begin{aligned}
&\int_D |Ax(t) + By(t)|dt \leq \int_D |Ax(t)|dt + \int_D |By(t)|dt = \\
&= \int_D |F_{g,\varphi_3}x(t)|dt + \int_D |F_{f_1}Ky(t)|dt \leq \\
&\leq \int_D \big[a_3(t) + b_3|x(\varphi_3(t))|\big]dt + \\
&\quad + \int_D \Big[a_1(t) + b_1|f_2(t, y(\varphi_2(t)))| \int_0^t |u(t,s,y(\varphi_1(s)))|\, ds\Big]dt \leq \\
&\leq \int_D \big[a_3(t) + b_3|x(\varphi_3(t))|\big]dt + \int_D \Big[a_1(t) + b_1(a_2(t) + b_2|y(\varphi_2(t))|) \int_0^t k(t,s)\, ds\Big]dt \leq \\
&\leq \int_D a_1(t)dt + \int_D a_3(t)dt + \frac{b_3}{M_3}\int_D |x(\varphi_3(t))|\varphi_3'(t)dt + \\
&\quad + b_1\|K_0\|_{L_\infty} \int_D \Big[a_2(t) + b_2|y(\varphi_2(t))|\Big]dt \leq \\
&\leq \int_D a_1(t)dt + \int_D a_3(t)dt + \frac{b_3}{M_3}\int_{\varphi_3(D)} |x(v)|dv + \\
&\quad + b_1\|K_0\|_{L_\infty} \int_D \Big[a_2(t) + \frac{b_2}{M_2}|y(\varphi_2(t))|\varphi_2'(t)\Big]dt \leq \\
&\leq \int_D a_1(t)dt + \int_D a_3(t)dt + \frac{b_3}{M_3}\int_{\varphi_3(D)} |x(v)|dv + \\
&\quad + b_1\|K_0\|_{L_\infty} \Big[\int_D a_2(t)dt + \frac{b_2}{M_2}\int_{\varphi_2(D)} |y(v)|dv\Big],
\end{aligned}
$$

where the symbol $\|K_0\|_{L_\infty(D)}$ denotes the norm of the operator K_0 acting from the space $L_1(D)$ into $L_\infty(D)$.

Now, using the fact that

$$\lim_{\varepsilon \to \infty} \sup\Big\{\int_D a_i(t)dt : D \subset \mathbb{R}^+, m(D) \leq \varepsilon\Big\} = 0, \quad \text{for } i = 1, 2, 3.$$

From Definition 2.3 it follows that

$$c(A(Q) + B(Q)) \leq \Big[q = \Big(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2}\|K_0\|_{L_\infty}\Big)\Big]c(Q). \tag{3.9}$$

For $T > 0$ and any $x, y \in Q$, we have

$$\int_T^\infty |Ax(t) + By(t)|dt \le \int_T^\infty a_1(t)dt + \int_T^\infty a_3(t)dt + \frac{b_3}{M_3} \int_{\varphi_3(T)}^\infty |x(v)|dv +$$
$$+ b_1 \|K_0\|_{L_\infty} \left[\int_T^\infty a_2(t)dt + \frac{b_2}{M_2} \int_{\varphi_2(T)}^\infty |y(v)|)dv \right],$$

where $\varphi_i(T) \to \infty$ as $T \to \infty$ for $i = 2, 3$. Then as $T \to \infty$ by the Definition 2.4 we get

$$d(A(Q) + B(Q)) \le \Big[q = \Big(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty}\Big)\Big] d(Q). \tag{3.10}$$

By combining equations (3.9) and (3.10) and Definition 2.5, we have

$$\mu(A(Q) + B(Q)) \le \Big[q = \Big(\frac{b_3}{M_3} + \frac{b_1 b_2}{M_2} \|K_0\|_{L_\infty}\Big)\Big] \mu(Q).$$

Step 4. Let $B_r^1 = Conv(A(B_r) + B(B_r))$, where B_r is defined in Step 1, $B_r^2 = Conv(A(B_r^1) + B(B_r^1))$ and so on. We then get a decreasing sequence $\{B_r^n\}$, that is $B_r^{n+1} \subset B_r^n$ for $n = 1, 2, \ldots$ Obviously all sets belonging to this sequence are closed and convex, so weakly closed. By the fact proved in Step 2 that $\mu(A(Q) + B(Q)) \le q\mu(Q)$ for all bounded subsets Q of B_r, we have

$$\mu(B_r^n) \le q^n \mu(B_r),$$

which yields that $\lim_{n\to\infty} \mu(B_r^n) = 0$. Denote $M = \bigcap_{n=1}^\infty B_r^n$, and then $\mu(M) = 0$. By the definition of the measure of weak noncompactness, we know that M is nonempty. From the definition of the operator A, we can deduce that $B(M) \subset M$. M is just a nonempty closed convex weakly compact set which we need in the following steps.

Step 5. Let $\{x_n\} \subset M$ be an arbitrary sequence. Since there exists T such that for all n, the following inequality is satisfied:

$$\int_T^\infty |x_n(t)|dt \le \frac{\varepsilon}{4}. \tag{3.11}$$

Considering the function $f_i(t, x)$ on $[0, T]$ $(i = 1, 2)$, $u(t, s, x)$ on $[0, T] \times [0, T] \times \mathbb{R}$, and $k(t, s)$ on $[0, T] \times [0, T]$, in view of Theorem (2.3) we can find a closed subset D_ε of the interval $[0, T]$, such that $meas(D_\varepsilon^c) \le \varepsilon$, and such that $f_i \mid_{D_\varepsilon \times \mathbb{R}}$ $(i = 1, 2)$, $u \mid_{D_\varepsilon \times D_\varepsilon \times \mathbb{R}}$, and $k \mid_{D_\varepsilon \times [0,T]}$ are continuous. Especially $k \mid_{D_\varepsilon \times [0,T]}$ is uniformly continuous.

Let us take arbitrary $t_1, t_2 \in D_\varepsilon$ and assume $t_1 < t_2$ without loss of generality. For an arbitrary fixed $n \in \mathbb{N}$ and denoting $H_n(t) = (F_{\varphi_2,f_2}x_n)\cdot(Ux_n)(t)$ we obtain

$$
\begin{aligned}
|H_n(t_2) - H_n(t_1)| &= \Big| f_2(t_2, x_n(\varphi_2(t_2))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s)))ds - f_2(t_1, x_n(\varphi_2(t_1)))\times \\
&\quad \times \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s)))ds \Big| \le \big| f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1))) \big| \times \\
&\quad \times \int_0^{t_2} |u(t_2, s, x_n(\varphi_1(s)))|ds + \Big| f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s)))ds - \\
&\quad - f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s)))ds \Big| \le \\
&\le |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} |u(t_2, s, x_n(\varphi_1(s)))|ds + \\
&\quad + \Big| f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_2} u(t_2, s, x_n(\varphi_1(s)))ds - f_2(t_1, x_n(\varphi_2(t_1)))\times \\
&\quad \times \int_0^{t_1} u(t_2, s, x_n(\varphi_1(s)))ds \Big| + \Big| f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_2, s, x_n(\varphi_1(s)))ds - \\
&\quad - f_2(t_1, x_n(\varphi_2(t_1))) \int_0^{t_1} u(t_1, s, x_n(\varphi_1(s)))ds \Big| \le \\
&\le |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} k(t_2, s)ds + \\
&\quad + |f_2(t_1, x_n(\varphi_2(t_1)))| \int_{t_1}^{t_2} |u(t_2, s, x_n(\varphi_1(s)))|ds + \\
&\quad + |f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_1} |u(t_2, s, x_n(\varphi_1(s))) - u(t_1, s, x_n(\varphi_1(s)))|ds \le \\
&\le |f_2(t_2, x_n(\varphi_2(t_2))) - f_2(t_1, x_n(\varphi_2(t_1)))| \int_0^{t_2} k(t_2, s)ds + \\
&\quad + \big[a_2(t_1) + b_2|x_n(\varphi_2(t_1))|\big] \int_{t_1}^{t_2} k(t_2, s)ds + \\
&\quad + \big[a_2(t_1) + b_2|x_n(\varphi_2(t_1))|\big] \int_0^{t_1} |u(t_2, s, x_n(\varphi_1(s))) - u(t_1, s, x_n(\varphi_1(s)))|ds.
\end{aligned}
$$

Then we have

$$\begin{aligned} |H_n(t_2) - H_n(t_1)| \le \omega^T(f_2, |t_2 - t_1|)T\tilde{k} + \\ + \big[a_2(t_1) + b_2|x_n(\varphi_2(t_1))|\big](t_2 - t_1)\tilde{k} + \\ + \big[a_2(t_1) + b_2|x_n(\varphi_2(t_1))|\big]T\omega^T(u, |t_2 - t_1|), \end{aligned} \tag{3.12}$$

where $\omega^T(f_2, \cdot)$ and $\omega^T(u, \cdot)$ denotes the modulus continuity of the functions f_2 and u on the sets $D_\varepsilon \times \mathbb{R}$ and $D_\varepsilon \times D_\varepsilon \times \mathbb{R}$ respectively and

$$\tilde{k} = \max\{|k(t, s)| : (t, s) \in D_\varepsilon \times [0, T]\}.$$

The last inequality (3.12) is obtained since $M \subset B_r$.

Taking into account the fact that $\mu(\{x_n\}) \le \mu(M) = 0$, we infer that the number $t_2 - t_1$ is small enough, then the right hand side of (3.12) tends to zero independently of x_n as $t_2 - t_1$ tends to zero. We have $\{H_n\}$ is equicontinuous in the space $C(D_\varepsilon)$.

Moreover,

$$|H_n(t)| \le |f_2(t, x_n(\varphi_2(t)))| \int_0^t |u(t, s, x_n(\varphi_1(s)))|ds \le$$

$$\le \big[|a_1(t)| + b_2|x_n(\varphi_2(t))|\big] \int_0^t k(t, s)ds \le \tilde{k}T[d_1 + b_2d_2],$$

where $|a_1(t)| \le d_1, |x_n(\varphi_2(t))| \le d_2$ for $t \in D_\varepsilon$. From the above, we have that $\{H_n\}$ is equibounded in the space $C(D_\varepsilon)$. Next, let us put

$$Y = \sup\{|H_n(t)| : t \in D_\varepsilon, n \in \mathbb{N}\}.$$

Obviously Y is finite in view of the choice of D_ε. Assumption (i) concludes that the function $f_1|_{D_\varepsilon \times [-Y,Y]}$ is uniformly continuous. So $\{B(x_n)\} = \{F_{f_1}H_n + g(t, 0)\}$ is equibounded and equicontinuous in the space $C(D_\varepsilon)$. Hence, by the Ascoli-Arzéla theorem [24], we obtain that the sequence $\{B(x_n)\}$ forms a relatively compact set in the space $C(D_\varepsilon)$.

Further observe that the above reasoning does not depend on the choice of ε. Thus we can construct a sequence D_l of closed subsets of the interval $[0, T]$ such that $meas(D_l^c) \to 0$ as $l \to \infty$ and such that the sequence $\{B(x_n)\}$ is relatively compact in every space $C(D_l)$. Passing to subsequences if necessary we can assume that $\{B(x_n)\}$ is a Cauchy sequence in each space $C(D_l)$ for $l = 1, 2, \ldots$.

In what follows, utilizing the fact that the set $B(M)$ is weakly compact, let us choose a number $\delta > 0$ such that for each closed subset D_δ of the interval $[0, T]$ such that $meas(D_\delta^c) \le \delta$, we have

$$\int_{D_\delta^c} |Bx(t)|dt \le \frac{\varepsilon}{4} \tag{3.13}$$

for any $x \in M$. Keeping in mind the fact that the sequence $\{Bx_n\}$ is a Cauchy sequence in each space $C(D_l)$ we can choose a natural number l_0 such that $meas(D_{l_0}^c) \leq \delta$ and $meas(D_{l_0}) > 0$, and for arbitrary natural numbers $n, m \geq l_0$ the following inequality holds

$$|(B(x_n))(t) - (B(x_m))(t)| \leq \frac{\varepsilon}{4meas(D_{l_0})} \tag{3.14}$$

for any $t \in D_{l_0}$. Now using the above facts together with (3.11), (3.13), (3.14) we obtain

$$\begin{aligned}\int_0^\infty |(Bx_n)(t) - (Bx_m)(t)|dt = &\int_T^\infty |(Bx_n)(t) - (Bx_m)(t)|dt + \\ &+ \int_{D_{l_0}} |(Bx_n)(t) - (Bx_m)(t)|dt + \\ &+ \int_{D_{l_0}^c} |(Bx_n)(t) - (Bx_m)(t)|dt \leq \varepsilon,\end{aligned}$$

which means that $\{B(x_n)\}$ is a Cauchy sequence in the space $L_1(\mathbb{R}^+)$. Hence we conclude that the set $B(M)$ is relatively strongly compact in the space $L_1(\mathbb{R}^+)$.

Step 6. We now can show that:

(1) From Step 4 we obtain that $A(M) + B(M) \subseteq M$, where M is the set constructed in Step 3.
(2) Step 1 allows us to know that A is a contraction mapping.
(3) By Step 5, $B(M)$ is relatively compact and by assumptions (i), (iii) B is continuous.

We can apply Theorem 2.5, and have that equation (1.1) has at least one integrable solution in $\mathbb{R}^+$. □

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