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RANDOM INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we present the existence and uniqueness of random solution of a random integral equation of Volterra type on time scales. We also study the asymptotic properties of the unique random solution.

Keywords: random integral equations, time scale, existence, uniqueness, stability.

Mathematics Subject Classification: 34N05, 45D05, 45R99.

1. INTRODUCTION

The random integral equations of Volterra type, as a natural extension of deterministic ones, arise in many applications and have been investigated by many mathematicians. For details, the reader may see the monograph [22, 27], the papers [7, 12, 21, 26] and references therein. For the general theory of integral equations see, the monographs [8,11] and references therein. In recent years, it initiated the study of integral equations on time scales and obtained some significant results see [1, 16, 19, 25]. The stochastic differential equations on time scales was first studied by Sanyal in his Ph.D. Thesis [24]. For other results about stochastic processes see [23].

The aim of this paper is to obtain the general conditions which ensure the existence and uniqueness of a random solution of a random integral equation of Volterra type on time scales and to investigate the asymptotic behavior of such a random solution. The paper is organized as follows: in Section 2 we set up the appropriate framework on random processes on time scales. We also introduce some functional spaces within which the study of random integral equations can be developed. In Section 3 we present the existence and uniqueness of random solutions. Finally, we establish an asymptotic stability result.

2. PRELIMINARIES

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real number $\mathbb{R}$. Then the time scale $\mathbb{T}$ is a complete metric space with the usual metric on $\mathbb{R}$. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators. The *forward* (*backward*) *jump operator* $\sigma(t)$ at $t \in \mathbb{T}$ for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ for $t > \inf \mathbb{T}$) is given by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (respectively $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$) for all $t \in \mathbb{T}$. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say t is *right scattered*. If $\rho(t) < t$, $t \in \mathbb{T}$, we say t is *left scattered*. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say t is *right-dense*. If $\rho(t) = t$, $t \in \mathbb{T}$, we say t is *left-dense*. Also, define the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) := \sigma(t) - t$. We recall that a function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous function* if f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. We remark that every rd-continuous function is Lebesgue Δ-integrable (see [14]). A rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *positively regressive* if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$. We will denote by $\mathcal{R}^+$ the set of all positively regressive functions. In the following, assume that $\mathbb{T}$ is unbounded. Without lost the generality, assume that $0 \in \mathbb{T}$ and let $\mathbb{T}_0 = [0, \infty) \cap \mathbb{T}$. Also, assume that there exists a strictly increasing sequence $(t_n)_n$ of elements of $\mathbb{T}_0$ such that $t_n \to \infty$ as $n \to \infty$. Denote by $\mathcal{L}$ the σ-algebra of Δ-measurable subsets of $\mathbb{T}_0$ and by λ the Lebesgue Δ-measure of $\mathcal{L}$. Having the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ one can introduce the Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space by simply employing the standard procedure from measure theory (see [3, 18]). The Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space was introduced by Neidhart in [18] and the Henstock-Kurzweil-Pettis integral was introduced by Cichoń in [10]. For details on the construction of the Lebesgue integral for real functions defined on a time scale, see [2, 4, 5, 9, 14, 15]. Further, let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A function $x : \Omega \to \mathbb{R}$ is called a *random variable* if $\{\omega \in \Omega : x(\Omega) < a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Let $1 \leq p < \infty$. A random variable $x : \Omega \to \mathbb{R}$ is said to be *p-integrable* if $\int_\Omega |x(\omega)|^p \, dP(\omega) < \infty$. Let $\mathcal{L}^p(\Omega)$ be the space of all p-integrable random variables. Then $\mathcal{L}^p(\Omega)$ is a vector space and the function $x \mapsto \|x\|_{\mathcal{L}^p(\Omega)}$ defined by

$$\|x\|_{\mathcal{L}^p(\Omega)} = \left(\int_\Omega |x(\omega)|^p \, dP(\omega) \right)^{1/p}$$

is a seminorm on $\mathcal{L}^p(\Omega)$. If $x \in \mathcal{L}^1(\Omega)$, then

$$E[x] := \int_\Omega x(\omega) dP(\omega)$$

is called the *expected value* of random variable x. A random variable x is called a P-essentially *bounded* if there exists a $M > 0$ and $A \in \mathcal{A}$ with $P(A) = 0$ such that $|x(\omega)| \leq M$ for all $\omega \in \Omega \setminus A$. Let $\mathcal{L}^\infty(\Omega)$ be the space of all P-essentially bounded random variables. Then

$$\|x\|_{\mathcal{L}^\infty(\Omega)} = P\text{-}\operatorname*{ess\,sup}_{\omega \in \Omega} |x(\omega)|$$

is a seminorm on $\mathcal{L}^\infty(\Omega)$, where

$$P\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} |x(\omega)| := \inf\{M > 0 : |x(\omega)| \le M \quad P\text{-a.e. } \omega \in \Omega\}.$$

When a random variable x is p-integrable or P-essentially bounded it is convenient to use notation $\widehat{x}$ to denote the equivalent class of random variables which coincide with x for P-a.e. $\omega \in \Omega$. Let us denote by $L^p(\Omega)$ the space of all equivalence classes of random variables that are p-integrable and by $L^\infty(\Omega)$ the space of all equivalence classes of random variables that are P-essentially bounded. If $x \in \mathcal{L}^p(\Omega)$, $1 \le p \le \infty$, we denote by $\widehat{x}$ its equivalence class, that is, $y \in \widehat{x}$ if and only if $y(\omega) = x(\omega)$ for P-a.e. $\omega \in \Omega$. Moreover, we have that $\|y\|_{\mathcal{L}^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$. Thus we can define a norm $\|\cdot\|_{L^p(\Omega)}$ on $L^p(\Omega)$ by means of the formula $\|\widehat{x}\|_{L^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$, $1 \le p \le \infty$. Then $L^p(\Omega)$, $1 \le p \le \infty$, is a Banach space with respect to the norm $\|\cdot\|_{L^p(\Omega)}$.

Since, for $1 \le p \le \infty$, $L^p(\Omega)$ is a Banach space, then all elementary properties of the calculus (such as continuity, differentiability, and integrability) for abstract functions defined on a subset of $\mathbb{T}$ with values into a Banach space remain also true for the functions defined a subset of $\mathbb{T}$ with values into $L^p(\Omega)$, $1 \le p \le \infty$.

Thereby, if $X : \mathbb{T}_0 \to L^p(\Omega)$ is strongly measurable then the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue measurable on $\mathbb{T}_0$. Also, a strongly measurable function $X : \mathbb{T}_0 \to L^p(\Omega)$ is Bochner Δ-integrable on $\mathbb{T}_0$ if and only if the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue Δ-integrable on $\mathbb{T}_0$ (see [3]).

Let $1 \le p \le \infty$. A function $X : \mathbb{T}_0 \to L^p(\Omega)$ is called *rd-continuous function* if X is continuous at every right-dense point $t \in \mathbb{T}_0$, and $\lim_{s\to t^-} X(s)$ exists in $L^p(\Omega)$ at every left-dense point $t \in \mathbb{T}_0$.

Of particular importance is the fact that every rd-continuous function $X : \mathbb{T}_0 \to L^p(\Omega)$ is Bochner Δ-integrable on $\mathbb{T}_0$ (see [3, Theorem 6.3]).

If $X : \mathbb{T}_0 \to L^p(\Omega)$ is a strongly measurable function then for each fixed $t \in \mathbb{T}_0$, $X(t) \in L^p(\Omega)$ is an equivalence class. If for each $t \in \mathbb{T}_0$ we select a particular function $x(t, \cdot) \in X(t)$ then we obtain a function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ such that $\omega \mapsto x(t, \omega)$ is a random variable for each $t \in \mathbb{T}_0$. This resulting function is called *a representation* of X. In fact, such a representation is so called a *random process*. However, is not immediate that this representation function is even a $\mathcal{L} \times \mathcal{A}$-measurable function. In this sense, we have the following result.

Lemma 2.1. (a) ([13*, Theorem III.11.17*]). *Let $(\mathbb{T}_0 \times \Omega, \mathcal{L} \times \mathcal{A}, \lambda \times P)$ be the product space of the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ and $(\Omega, \mathcal{A}, P)$. Let $1 \le p \le \infty$ and let $X : \mathbb{T}_0 \to L^p(\Omega)$ be a Bochner Δ-integrable function. Then there exists a $\mathcal{L} \times \mathcal{A}$-measurable function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ which is uniquely determined except a set of $\lambda \times P$-measure zero, such that $\widehat{x}(t, \cdot) = X(t)$ for λ-a.e. $t \in \mathbb{T}_0$. Moreover, $x(\cdot, \omega)$ is Lebesgue Δ-integrable on $\mathbb{T}_0$ for P- a.e. $\omega \in \Omega$ and integral $\int_{\mathbb{T}_0} x(t, \omega)\Delta t$, as a function of ω, is equal to the element $\int_{\mathbb{T}_0} X(t)\Delta t$ of $L^p(\Omega)$, that is,*

$$\int_{\mathbb{T}_0} x(t, \cdot)\Delta t = \left(\int_{\mathbb{T}_0} X(t)\Delta t \right)(\cdot).$$

(b) ([13, *Lemma III.11.16*]). *Let* $1 \leq p < \infty$ *and let* $x(\cdot,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ *be a* $\mathcal{L} \times \mathcal{A}$*-measurable function such that* $x(t,\cdot) \in \mathcal{L}^p(\Omega)$ *for* λ*-a.e.* $t \in \mathbb{T}_0$. *Then the function* $X : \mathbb{T}_0 \to L^p(\Omega)$, *defined by* $X(t) = \widehat{x}(t,\cdot)$, *is strongly measurable on* $\mathbb{T}_0$.

A $\mathcal{L} \times \mathcal{A}$-measurable function $x(\cdot,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ will be called a *measurable random process.*

Remark 2.2. Let $x(\cdot,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ be a measurable random process such that, for each fixed $t \in \mathbb{T}_0$, $x(t,\cdot) \in \mathcal{L}^p(\Omega)$. If we denote $\widehat{x}(t,\cdot)$ by $X(t)$, then $X(t) : \Omega \to \mathbb{R}$ is a random variable such that $X(t) \in L^p(\Omega)$ and $x(t,\omega) = X(t)(\omega)$ for P-a.e. $\omega \in \Omega$. In the following, using a common abuse of notation in measure theory, we will denote $x(t,\cdot)$ by $X(t)$ for each fixed $t \in \mathbb{T}_0$. In this way, a measurable random process $x(\cdot,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ such that $x(t,\cdot) \in \mathcal{L}^p(\Omega)$ for all $t \in \mathbb{T}_0$ can be identified with a strongly measurable function $X : \mathbb{T}_0 \to L^p(\Omega)$.

Let us denote by $C_c = C(\mathbb{T}_0, L^p(\Omega))$ the space of continuous functions $X : \mathbb{T}_0 \to L^p(\Omega)$ with the compact open topology. We recall that if K is a compact subset of $\mathbb{T}_0$ and U is an open subset of $L^p(\Omega)$ and we put

$$S(K,U) = \{X : K \to L^p(\Omega) \,|\, X(K) \subset U\},$$

then the sets

$$S(K_1,\ldots,K_n; U_1,\ldots,U_n) = \bigcap_{i=1}^{n} S(K_i,U_i),$$

where $n \in \mathbb{N}$, form a basis for the compact open topology. In fact, this topology coincides with the topology of uniform convergence on any compact subset of $\mathbb{T}_0$. The space C_c is a locally convex space [28] whose topology is defined by means of the following family of seminorms:

$$\|X\|_n = \sup_{t \in K_n} \|X(t)\|_{L^p(\Omega)},$$

where $K_n = [0,t_n] \subset \mathbb{T}_0$, $n \in \mathbb{N}$ and $(t_n)_n$ is a strictly increasing sequence of elements of $\mathbb{T}_0$ such that $t_n \to \infty$ as $n \to \infty$.

A distance function can be defined on C_c by

$$d_c(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|X-Y\|_{L^p(\Omega)}}{1 + \|X-Y\|_{L^p(\Omega)}}.$$

The topology induced by this distance function is the same topology of uniform convergence on any compact subset of $\mathbb{T}_0$.

Further, consider a continuous function $g : \mathbb{T}_0 \to (0,\infty)$. By $C_g = C_g(\mathbb{T}_0, L^p(\Omega))$ we denote the space of all continuous functions from $\mathbb{T}_0$ into $L^p(\Omega)$ such that

$$\sup_{t \in \mathbb{T}_0} \left\{ \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} : t \in \mathbb{T}_0 \right\} < \infty.$$

Then

$$\|X\|_{C_g} := \sup_{t\in\mathbb{T}_0} \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} \tag{2.1}$$

is a norm of C_g.

Lemma 2.3. $(C_g, \|\cdot\|_{C_g})$ *is a Banach space.*

Proof. Let (X_n) be a Cauchy sequence in C_g. Then for each $\varepsilon > 0$ there exists a $N = N(\varepsilon) > 0$ such that $\|X_n - X\|_{C_g} < \varepsilon$ for all $n, m \geq N$. Hence, by (2.1), it follows that

$$\|X_n(t) - X_m(t)\|_{L^p(\Omega)} < \varepsilon g(t), \tag{2.2}$$

for all $t \in \mathbb{T}_0$ and $n, m \geq N$. Since $L^p(\Omega)$ is a complete metric space, it follows that, for any fixed $t \in \mathbb{T}_0$, $(X_n(t))$ is a convergent sequence in $L^p(\Omega)$. Therefore, for any fixed $t \in \mathbb{T}_0$, there exists $X(t) \in L^p(\Omega)$ such that $X(t) = \lim_{n\to\infty} X_n(t)$ in $L^p(\Omega)$. Moreover, it follows from (2.2) that $X(t) = \lim_{n\to\infty} X_n(t)$ in $L^p(\Omega)$, uniformly on any compact subset of $\mathbb{T}_0$. Hence, X is a continuous function from $\mathbb{T}_0$ into $L^p(\Omega)$. Further, we show that $X \in C_g$. Let us keep n fixed and take $m \to \infty$ in (2.2). Then we obtain that $X_n - X \in C_g$ for all $n \geq N$. Since $X = (X - X_n) + X_n$ and $X - X_n$, $X_n \in C_g$, it follows that $X \in C_g$. □

Remark 2.4. The topology of C_g is stronger than the topology of C_c. Indeed, if $X_n \to X$ in C_g as $n \to \infty$, then for each $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $\|X_n(t) - X(t)\|_{L^p(\Omega)} < \varepsilon g(t)$, for all $t \in \mathbb{T}_0$ and $n \geq N(\varepsilon)$. Since g is bounded on any compact subset of $\mathbb{T}_0$, it allows that $X_n(t) \to X(t)$ as $n \to \infty$, uniformly on any compact subset of $\mathbb{T}_0$. In other words, convergence in C_g implies convergence in C_c. If $g(t) = 1$ on $\mathbb{T}_0$, then C_g becomes the space $C = C(\mathbb{T}_0, L^p(\Omega))$ of all continuous and bounded functions from $\mathbb{T}_0$ into $L^p(\Omega)$. The norm on C is given by

$$\|X\|_c = \sup_{t\in\mathbb{T}_0} \|X(t)\|_{L^p(\Omega)}.$$

Note that the following inclusions hold $C \subset C_g \subset C_c$.

Let (B, D) be a pair of Banach spaces such that $B, D \subset C_c$ and let $\mathcal{T}$ be a linear operator from C_c to itself. The pair of Banach spaces (B, D) is called *admissible* with respect to the operator $\mathcal{T} : C_c \to C_c$ if $\mathcal{T}(B) \subset D$ ([13]).

Remark 2.5. If the pair (B, D) is admissible with respect to the linear operator $\mathcal{T} : C_c \to C_c$ then, by Lemma 2.1.1 from [21], it follows that $\mathcal{T}$ is a continuous operator from B to D. Therefore, there exists a $M > 0$ such that

$$\|\mathcal{T}X\|_D \leq M \|X\|_B, \quad X \in B.$$

3. RANDOM INTEGRAL EQUATION OF VOLTERRA TYPE

In this section we study the existence and uniqueness of a random solution of a random integral equation of Volterra type.

$$x(t,\omega) = h(t,\omega) + \lambda \int_{t_0}^{t} k(t,s,\omega) f(s, x(s,\omega), \omega)\Delta s, \quad t \in \mathbb{T}_0, \tag{3.1}$$

where P-a.e. $\omega \in \Omega$, $x(\cdot,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ is the unknown random process, $h : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ is a measurable random process, $f : \mathbb{T}_0 \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a random function, $k : \Gamma \times \Omega \to \mathbb{R}$ is the random kernel, $\lambda \in \mathbb{R}^*$, and $\Gamma := \{(t,s) \in \mathbb{T}_0 \times \mathbb{T}_0 \colon t_0 \le s \le t < \infty\}$.

In what follows, we will use the notations $X(t) = x(t,\cdot)$, $H(t) = h(t,\cdot)$, $K(t,s) = k(t,s,\cdot)$, $F(t,X(t)) = f(t,x(t,\cdot),\cdot)$.

Let us consider the following assumptions:

(h1) $K(t,s) \in L^\infty(\Omega)$ for all $(t,s) \in \Gamma$, $K(\cdot,\cdot) : \Gamma \to L^\infty(\Omega)$ continuous in its first variable and rd-continuous in its second variable, there exists $k_0 > 0$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ such that

$$\|K(t,s)\|_{L^\infty(\Omega)} \le k_0 e_{-\alpha}(t,\sigma(s))$$

for $(t,s) \in \Gamma$.

(h2) $f(\cdot,x,\cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ is a $\mathcal{L} \times \mathcal{A}$-measurable function for each $x \in \mathbb{R}$, and there exist an $a > 0$ and a positive random variable $L : \Omega \to \mathbb{R}$ such that $P(\{\omega \in \Omega \colon L(\omega) > a\}) = 0$ and

$$|f(t,x,\omega) - f(t,y,\omega)| \le L(\omega)\,|x-y|$$

for all $t \in \mathbb{T}_0$ and $x, y \in \mathbb{R}$.

(h3) $F(t,0) \in L^p(\Omega)$ for all $t \in \mathbb{T}_0$ and there exists $\beta \in (0,\alpha)$ with $-\beta \in \mathcal{R}^+$ such that

$$r := \sup_{t \in \mathbb{T}_0} \frac{\|F(t,0)\|_{L^p(\Omega)}}{e_{-\beta}(t,0)} < \infty.$$

In what follows, consider $g(t) := e_{-\beta}(t,0)$, $t \in \mathbb{T}_0$, where $0 < \beta < \alpha$. Also, we will use the notation C_β instead of C_g.

Lemma 3.1. *If* (h2) *and* (h3) *hold, then*

$$\sup_{t \in \mathbb{T}_0} \frac{\|F(t,X(t))\|_{L^p(\Omega)}}{e_{-\beta}(t,0)} \le a\,\|X\|_{C_\beta} + r < \infty \tag{3.2}$$

for every $X \in C_\beta$, *and*

$$\|F(t,X(t)) - F(t,Y(t))\|_{L^p(\Omega)} \le a\,\|X(t) - Y(t)\|_{L^p(\Omega)} \tag{3.3}$$

for all $t \in \mathbb{T}_0$ *and* $X, Y \in C_\beta$.

Proof. If we denote $\{\omega \in \Omega\colon L(\omega) \le a\}$ by Ω_a, then from (h2) we have that $P(\Omega_a) = 1$. If $X, Y \in C_\beta$, using the Minkowski's inequality, (h2) and (h3), we have

$$\begin{aligned}
&\|F(t, X(t))\|_{L^p(\Omega)} = \|f(t, x(t,\cdot),\cdot)\|_{L^p(\Omega)} \le \\
&\le \left(\int_\Omega |f(t, x(t,\omega),\omega) - f(t,0,\omega)|^p \, dP(\omega)\right)^{1/p} + \left(\int_\Omega |f(t,0,\omega)|^p \, dP(\omega)\right)^{1/p} \le \\
&\le \left(\int_{\Omega_a} |L(\omega)|^p \, |x(t,\omega)|^p \, dP(\omega)\right)^{1/p} + \|F(t,0)\|_{L^p(\Omega)} \le \\
&\le a \, \|X(t)\|_{L^p(\Omega)} + \|F(t,0)\|_{L^p(\Omega)} .
\end{aligned}$$

Dividing both sides of the last inequality by $e_{-\beta}(t, 0) > 0$ and taking the supremum with respect to $t \in \mathbb{T}_0$, we obtain (3.2). Also,

$$\begin{aligned}
&\|F(t, X(t)) - F(t, X(t))\|_{L^p(\Omega)} = \|f(t, x(t,\cdot),\cdot) - f(t, y(t,\cdot),\cdot)\|_{L^p(\Omega)} = \\
&= \left(\int_\Omega |f(t, x(t,\omega),\omega) - f(t, y(t,\omega),\omega)|^p \, dP(\omega)\right)^{1/p} \le \\
&\le \left(\int_{\Omega_a} |L(\omega)|^p \, |x(s,\omega) - y(s,\omega)|^p \, dP(\omega)\right)^{1/p} \le a \, \|X(t) - Y(t)\|_{L^p(\Omega)} . \qquad \square
\end{aligned}$$

Remark 3.2. It follows from Lemma 3.1 that $F(t, X(t)) \in L^p(\Omega)$ for all $t \in \mathbb{T}_0$ and $X \in C_\beta$. Moreover, (3.2) implies that the function $t \mapsto F(t, X(t))$ belong to C_β for all $X \in C_\beta$.

Lemma 3.3. *Let us consider the integral operator $\mathcal{T} : C_c \to C_c$ defined by*

$$(\mathcal{T}X)(t) = \int_0^t K(t,s) X(s) \Delta s, \quad t \in \mathbb{T}_0. \tag{3.4}$$

If (h1) *holds, then $\mathcal{T}(C_\beta) \subset C_\beta$.*

Proof. Let $X \in C_\beta$. We have that

$$\begin{aligned}
\|(\mathcal{T}X)(t)\|_{L^p(\Omega)} &\le \int_0^t \|K(t,s)X(s)\|_{L^p(\Omega)} \Delta s \le \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \|X(s)\|_{L^p(\Omega)} \Delta s = \\
&= \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \frac{\|X(s)\|_{L^p(\Omega)}}{e_{-\beta}(s,0)} e_{-\beta}(s,0) \Delta s \le \\
&\le \|X\|_{C_\beta} \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e_{-\beta}(s,0) \Delta s.
\end{aligned}$$

Take into account (h1), we infer that

$$\int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e_{-\beta}(s,0)\Delta s \le k_0 \int_0^t e_{-\alpha}(t,\sigma(s))e_{-\beta}(s,0)\Delta s =$$

$$= \frac{k_0}{\alpha-\beta}[e_{-\beta}(t,0) - e_{-\alpha}(t,0)].$$

Since $-\alpha, -\beta \in \mathcal{R}^+$ and $-\alpha < -\beta$, then (see [6, Corollary 2.10]) we have that $e_{-\beta}(t,0) > e_{-\alpha}(t,0)$, $t \in \mathbb{T}_0$, and it follows that

$$\int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e_{-\beta}(s,0)\Delta s \le \frac{k_0}{\alpha-\beta} e_{-\beta}(t,0), \quad t \in \mathbb{T}_0. \tag{3.5}$$

Consequently,

$$\|(\mathcal{T}X)(t)\|_{L^p(\Omega)} \le \frac{k_0}{\alpha-\beta} \|X\|_{C_\beta} e_{-\beta}(t,0), \quad t \in \mathbb{T}_0,$$

and thus $\mathcal{T}X \in C_\beta$ for every $X \in C_\beta$, that is, $\mathcal{T}(C_\beta) \subset C_\beta$. □

Remark 3.4. Since, by Lemma 3.3, the pair (C_β, C_β) is admissible with respect to the linear operator $\mathcal{T} : C_c \to C_c$ then, by Remark 2.5, it follows that $\mathcal{T}$ is a continuous operator from C_β to C_β. Therefore, there exists a $M > 0$ such that

$$\|\mathcal{T}X\|_{C_\beta} \le M \|X\|_{C_\beta}, \quad X \in C_\beta.$$

In fact, it easy to see that $M = \frac{k_0}{\alpha-\beta}$ is the norm of $\mathcal{T}$ as a linear operator from C_β into C_β.

A solution $X \in C_\beta$ of the integral equation (3.1) is called *asymptotically exponentially stable* if there exists a $\rho > 0$ and a $\beta > 0$ such that $-\beta \in \mathcal{R}^+$ and

$$\|X(t)\|_{L^p(\Omega)} \le \rho e_{-\beta}(t,0), \quad t \in \mathbb{T}_0.$$

Remark 3.5. The admissibility concept is related to stability in various senses (see [17]). Let $\mathcal{T} : C_c \to C_c$ be a linear operator. Roughly speaking we say that the pair of function spaces $B, D \subset C_c$ is admissible with respect to the equation

$$X = H + \mathcal{T}X, \tag{3.6}$$

if this equation has its solution in the space D, for each $H \in D$. Therefore, if we choose $D = C_\beta$ and if $X \in C_\beta$ is a solution of the equation (3.6), then there exists a $\rho > 0$ such that $\|X\|_{C_\beta} \le \rho$. Using (2.1) we infer that

$$\|X(t)\|_{L^p(\Omega)} \le \rho e_{-\beta}(t,0)$$

for all $t \in \mathbb{T}_0$, that is, the solution of the equation (3.6) is asymptotically exponentially stable. For several results concerning the admissibility theory for Volterra integral equations see [11].

These preliminaries being completed, we shall state the following result.

Theorem 3.6. *If the assumptions* (h1)–(h3) *hold and* $H \in C_\beta$*, then the integral equation* (3.1) *has a unique asymptotically exponentially stable solution, provided that* $|\lambda|\, aM < 1$*, where* $M > 0$ *is the norm of the operator* $\mathcal{T}$*.*

Proof. Let us consider the operator $\mathcal{V} : C_\beta \to C_c$ defined by

$$(\mathcal{V}X)(t) = H(t) + \lambda \int_0^t K(t,s)F(s,X(s))\Delta s, \quad t \in \mathbb{T}_0. \tag{3.7}$$

Then we can rewrite the operator $\mathcal{V}$ as

$$(\mathcal{V}X)(t) = H(t) + \lambda(\mathcal{T}G)(t), \quad t \in \mathbb{T}_0, \tag{3.8}$$

where $G(t) := F(t, X(t))$, $t \in \mathbb{T}_0$ and $\mathcal{T}$ is the operator given by (3.4). Since by Remark 3.2 and Lemma 3.1 we have that $\|G\|_{C_\beta} \le a\, \|X\|_{C_\beta} + r$, then

$$\|(\mathcal{T}G)(t)\|_{L^p(\Omega)} \le bMe_{-\beta}(t,0), \quad t \in \mathbb{T}_0, \tag{3.9}$$

where $b := a\, \|X\|_{C_\beta} + r$. From (3.8) and (3.9) we obtain that

$$\|(\mathcal{V}X)(t)\|_{L^p(\Omega)} \le \|H(t)\|_{L^p(\Omega)} + b\, |\lambda|\, Me_{-\beta}(t,0),$$

for all $t \in \mathbb{T}_0$. Dividing both sides of the last inequality by $e_{-\beta}(t,0) > 0$ and taking the supremum with respect to $t \in \mathbb{T}_0$, it follows that

$$\|\mathcal{V}X\|_{C_\beta} \le \|H\|_{C_\beta} + b\, |\lambda|\, M, \tag{3.10}$$

and so $\mathcal{V}X \in C_\beta$ for all $X \in C_\beta$. Further, we show that the operator $\mathcal{V}$ is a contraction on C_β. Indeed, using (3.3) and (3.5), we have

$$\begin{aligned}
\|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} &\le |\lambda| \int_0^t \|K(t,s)[F(s,X(s)) - F(s,Y(s))]\|_{L^p(\Omega)}\Delta s \le \\
&\le |\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \|F(s,X(s)) - F(s,Y(s))\|_{L^p(\Omega)}\Delta s \le \\
&\le a\, |\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \frac{\|X(s) - Y(s)\|_{L^p(\Omega)}}{e_{-\beta}(s,0)} e_{-\beta}(s,0)\Delta s \le \\
&\le a\, |\lambda|\, \|X - Y\|_{C_\beta} \int_0^t \|K(t,s)\|_{L^\infty(\Omega)}\, e_{-\beta}(s,0)\Delta s \le \\
&\le \frac{a\, |\lambda|\, k_0}{\alpha - \beta}\, \|X - Y\|_{C_\beta}\, e_{-\beta}(t,0) = \\
&= a\, |\lambda|\, M\, \|X - Y\|_{C_\beta}\, e_{-\beta}(t,0).
\end{aligned}$$

Thus

$$\|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} \le a\,|\lambda|\,M\,\|X - Y\|_{C_\beta}$$

for all $t \in \mathbb{T}_0$, and so

$$\|\mathcal{V}X - \mathcal{V}Y\|_{C_\beta} \le a\,|\lambda|\,M\,\|X - Y\|_{C_\beta},$$

with $a\,|\lambda|\,M < 1$, that is, $\mathcal{V}$ is a contraction on C_β. From Banach's Fixed Point Theorem, it follows that there exist a unique solution $X \in C_\beta$ of the integral equation (3.1). From Remark 3.5, we infer that the solution is asymptotically exponentially stable. □

Corollary 3.7. *If all the hypotheses of Theorem 3.6 hold for $\beta = 0$, then the integral equation* (3.1) *has a unique solution $X \in C$.*

Corollary 3.8. *If all the hypotheses of Theorem 3.6, then the solution of the integral equation* (3.1) *is asymptotically stable in mean, that is, $E[|X(t)|] \to 0$ as $t \to \infty$.*

Proof. Since $-\beta < 0$, then $e_{-\beta}(t, 0)$ decreases monotonically towards zero as $t \to \infty$, and therefore $\|X(t)\|_{L^p(\Omega)} \to 0$ as $t \to \infty$. Since $E[|X(t)|^p] = \|X(t)\|^p_{L^p(\Omega)}$ then, using the Jensen's inequality, we infer that $E[|X(t)|] \to 0$ as $t \to \infty$. □

Remark 3.9. Let $\mathbb{T}_0 = [0, \infty)$. Then, for $g(t) = q(t) = e^{-\beta t}$, $t \ge 0$, we obtain Theorem 2.2 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 3.1 from [26]. Let $\mathbb{T}_0 = \mathbb{N}$. Then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.3.1 from [27].

In what follows, using the concept of admissibility, we prove a general result of the existence and uniqueness for the integral equation (3.1). From this result it is possible to derive many existence results, by particularizing the spaces B and D.

Let us consider the integral equation (3.1) under the following conditions:

($\tilde{\text{h}}$1) $K(t, s) \in L^\infty(\Omega)$ for all $(t, s) \in \Gamma$, $K(\cdot, \cdot) : \Gamma \to L^\infty(\Omega)$ continuous in its first variable and rd-continuous in its second variable.

($\tilde{\text{h}}$2) $B, D \subset C_c$ are Banach spaces stronger than C_c such that the pair (B, D) is admissible with respect to the linear operator $\mathcal{T} : C_c \to C_c$ defined by (3.4).

($\tilde{\text{h}}$3) For each $X \in D$, the function $t \mapsto F(t, X(t))$ belong to B, and the operator $\mathcal{G} : D \to B$, defined by $(\mathcal{G}X)(t) = F(t, X(t))$ for all $t \in \mathbb{T}_0$, satisfies the Lipschitz condition

$$\|\mathcal{G}X - \mathcal{G}Y\|_B \le a\,\|X - Y\|_D$$

for all $X, Y \in D$ and some $a > 0$.

Theorem 3.10. *If the assumptions* ($\tilde{\text{h}}$1)–($\tilde{\text{h}}$3) *hold and $H \in D$, then the integral equation* (3.1) *has a unique solution $X \in D$, provided that $|\lambda|\,aM < 1$, where $M > 0$ is the norm of the operator $\mathcal{T}$.*

Proof. Let us consider the operator $\mathcal{V} : D \to C_c$ defined by $\mathcal{V}X = H + \lambda\mathcal{T}\mathcal{G}X$. Since the pair (B, D) is admissible with respect to the linear operator $\mathcal{T}$, it follows from Remark 2.5 that there exists a $M > 0$ such that $\|\mathcal{T}X\|_D \leq M\|X\|_B$ for all $X \in B$. Using ($\tilde{\text{h}}$3) and the fact that $H \in D$ it follows from Minkowski's inequality that

$$\begin{aligned}\|\mathcal{V}X\|_D &\leq \|H\|_D + |\lambda|\, M\, \|\mathcal{G}X\|_B \leq \|H\|_D + |\lambda|\, M\, \|\mathcal{G}X - \mathcal{G}0\|_B + |\lambda|\, M\, \|\mathcal{G}0\|_B \leq \\ &\leq \|H\|_D + a\,|\lambda|\, M\, \|X\|_D + |\lambda|\, M\, \|\mathcal{G}0\|_B < \infty,\end{aligned}$$

that is, $\mathcal{V}X \in D$ for all $X \in D$. Next, all $X, Y \in D$ we have that $\mathcal{V}X - \mathcal{V}Y = \lambda\mathcal{T}(\mathcal{G}X - \mathcal{G}Y)$. Obviously, $\mathcal{G}X - \mathcal{G}Y \in B$ and $\mathcal{V}X - \mathcal{V}Y \in D$. It follows that

$$\|\mathcal{V}X - \mathcal{V}Y\|_D \leq |\lambda|\, M\, \|\mathcal{G}X - \mathcal{G}Y\|_B \leq |\lambda|\, aM\, \|X - Y\|_D\,,$$

with $|\lambda|\, aM < 1$, that is, $\mathcal{V}$ is a contraction on D. From Banach's Fixed Point Theorem, it follows that there exist a unique solution $X \in D$ of the integral equation (3.1). □

Remark 3.11. If $\mathbb{T}_0 = [0, \infty)$, we obtain Theorem 2.4 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 2.1.2 from [27]. If $\mathbb{T}_0 = \mathbb{N}$, then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.1.2 from [27].

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