Opuscula Math. 33, no. 2 (2013), 293–306
http://dx.doi.org/10.7494/OpMath.2013.33.2.293 Opuscula Mathematica

MULTIPLE SOLUTIONS FOR SYSTEMS OF MULTI-POINT BOUNDARY VALUE PROBLEMS

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Communicated by Vicentiu D. Radulescu

Abstract. In this paper, we establish the existence of at least three solutions of the multi-point boundary value system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(x, u_1, \ldots, u_n),\ t \in (0,1), \\ u_i(0) = \sum_{j=1}^m a_j u_i(x_j),\ u_i(1) = \sum_{j=1}^m b_j u_i(x_j), \end{cases} \quad i = 1, \ldots, n.$$

The approaches used are based on variational methods and critical point theory.

Keywords: multiple solutions, multi-point boundary value problem, critical point theory.

Mathematics Subject Classification: 34B10, 34B15.

1. INTRODUCTION

In this paper, we study the existence of multiple solutions to the multi-point boundary value system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(x, u_1, \ldots, u_n),\ t \in (0,1), \\ u_i(0) = \sum_{j=1}^m a_j u_i(x_j),\ u_i(1) = \sum_{j=1}^m b_j u_i(x_j), \end{cases} \quad i = 1, \ldots, n, \tag{1.1}$$

where $p_i > 1$ and $\phi_{p_i}(t) = |t|^{p_i-2}t$ for $i = 1, \ldots, n$, λ is a positive parameter, $m, n \geq 1$ are integers, a_j, $b_j \in \mathbb{R}$ for $j = 1, \ldots, m$, and $0 < x_1 < x_2 < x_3 < \ldots < x_m < 1$. Here, $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2, \ldots, t_n) \to F(x, t_1, t_2, \ldots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in [0,1]$, F_{t_i} is continuous in $[0,1] \times \mathbb{R}^n$ for $i = 1, \ldots, n$, where F_{t_i} denotes the partial derivative of F with respect to t_i, and $F(x, 0, \ldots, 0) = 0$ for all $x \in [0,1]$.

Throughout this paper, we let X be the Cartesian product of n spaces

$$X_i = \left\{ \xi \in W^{1,p_i}([0,1]) \ : \ \xi(0) = \sum_{j=1}^{m} a_j \xi(x_j), \quad \xi(1) = \sum_{j=1}^{m} b_j \xi(x_j) \right\}, \ i = 1, \ldots, n,$$

i.e., $X = X_1 \times \ldots \times X_n$, endowed with the norm

$$\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{p_i} \quad \text{for } u = (u_1, \ldots, u_n) \in X,$$

where

$$\|u_i\|_{p_i} = \left(\int_0^1 |u_i'(x)|^{p_i} dx \right)^{1/p_i}, \quad i = 1, \ldots, n.$$

Clearly, X is a reflexive Banach space. Here, X^* denoted the dual space of X.

By a *classical solution* of the system (1.1), we mean a function $u = (u_1, \ldots, u_n)$ such that, for $i = 1, \ldots, n$, $u_i \in C^1[0,1]$, $\phi_{p_i}(u_i') \in C^1[0,1]$, and $u_i(x)$ satisfies (1.1). We say that a function $u = (u_1, \ldots, u_n) \in X$ is a *weak solution* of (1.1) if

$$\int_0^1 \sum_{i=1}^{n} \phi_{p_i}(u_i'(x)) v_i'(x) dx - \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx = 0$$

for any $v = (v_1, \ldots, v_n) \in W_0^{1,p_1}([0,1]) \times W_0^{1,p_2}([0,1]) \times \ldots \times W_0^{1,p_n}([0,1])$. We will show that a weak solution of (1.1) is indeed a classical solution (see Lemma 1.3 below).

Multi-point boundary value problems appear in a number of applications and have been studied by many researchers in recent years; see, for example, [4, 7–21] for some recent results on this topic. Our goal in this paper is to obtain some sufficient conditions for system (1.1) to have at least three classical solutions. Our analysis is mainly based on two recent critical points theorems; see Lemmas 1.1 and 1.2 below. Lemmas 1.1 and 1.2 are essential to the proofs of our main results, and while they appeared in [2] and [1], respectively, we recall them as they are given in [6]. Other contributions related to the method and results here can be found in [3, 5, 22, 23].

Lemma 1.1 ([6, Theorem 3.2]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, and*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there is a positive constant r and $\overline{v} \in X$, with $2r < \Phi(\overline{v})$, such that:

(C1) $\frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)}{r} < \frac{2}{3} \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(C2) *for all* $\lambda \in \left(\frac{3}{2} \frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \ \frac{r}{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)} \right)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Lemma 1.2 ([6, Theorem 3.3])**.** *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a convex, coercive, and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact,*

(D1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$,
(D2) *for each* $\lambda > 0$ *and for every* u_1, u_2 *that are local minima for the functional* $\Phi - \lambda\Psi$ *and are such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *we have*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume further that there exist $\overline{v} \in X$ *and positive constants* r_1 *and* r_2, *with* $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, *such that:*

(D3) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,
(D4) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}, \frac{r_2}{2\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}\right\}\right),$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points that lie in* $\Phi^{-1}(-\infty, r_2)$.

Note that the coercivity of the functional $\Phi - \lambda\Psi$ is required in Lemma 1.1 and a suitable sign hypothesis on Ψ is assumed in Lemma 1.2.

We also need the following lemma in this paper.

Lemma 1.3 ([11, Lemma 2.5])**.** *A weak solution of* (1.1) *coincides with a classical solution of* (1.1).

In this paper, we assume throughout, and without further mention, that the following condition holds:

(H1) Either $\underline{p} \geq 2$ or $\overline{p} < 2$, where $\underline{p} = \min\{p_1, \ldots, p_n\}$ and $\overline{p} = \max\{p_1, \ldots, p_n\}$.
(H2) $\sum_{j=1}^m a_j \neq 1$ and $\sum_{j=1}^m b_j \neq 1$.

In Section 2, we present our main results and their proofs.

2. MAIN RESULTS

Let

$$c = \max\left\{\sup_{u_i\in X_i\setminus\{0\}} \frac{\max_{x\in[0,1]}|u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} \ : \ i = 1, \ldots, n\right\}. \tag{2.1}$$

Since $p_i > 1$ for $i = 1, \dots, n$, the embedding $X = X_1 \times \ldots \times X_n \hookrightarrow (C^0([0,1]))^n$ is compact, and so $c < +\infty$. In addition, if (H2) holds, then from [7, Lemma 3.1],

$$\sup_{v \in X_i \setminus \{0\}} \frac{\max_{x\in[0,1]} |v(x)|}{\|v\|_{p_i}} \leq \frac{1}{2}\left(1 + \frac{\sum_{j=1}^m |a_j|}{|1 - \sum_{j=1}^m a_j|} + \frac{\sum_{j=1}^m |b_j|}{|1 - \sum_{j=1}^m b_j|}\right) \text{ for } i = 1, \dots, n.$$

For any $\gamma > 0$, we define the set $K(\gamma)$ by

$$K(\gamma) = \left\{(t_1, \dots, t_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \gamma\right\}.$$

We will use this set in some of our hypotheses with appropriate choices of γ.

Here is our first existence result.

Theorem 2.1. *Assume that there exist a function $w = (w_1, \dots, w_n) \in X$ and a positive constant r such that:*

(A1) $\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i} > 2r$,

(A2) $\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)} F(x,t_1,\dots,t_n)dx}{r} < \frac{2}{3}\frac{\int_0^1 F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}$,

(A3) $\limsup_{|t_1|\to\infty,\dots,|t_n|\to\infty} \frac{F(x,t_1,\dots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} < \frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)} F(x,t_1,\dots,t_n)dx}{cr}$.

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\int_0^1 F(x, w_1(x), \dots, w_n(x))dx}, \ \frac{r}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)} F(x, t_1, \dots, t_n)dx}\right),$$

the system (1.1) *has at least three classical solutions.*

Proof. We wish to apply Lemma 1.1 to our problem. To this end, for each $u = (u_1, \dots, u_n) \in X$ we introduce the functionals Φ, $\Psi : X \to \mathbb{R}$ as follows:

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{2.2}$$

and

$$\Psi(u) = \int_0^1 F(x, u_1(x), \dots, u_n(x))dx. \tag{2.3}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals and their derivatives at the point $u = (u_1, \dots, u_n) \in X$ are the functionals $\Phi'(u)$, $\Psi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^1 \sum_{i=1}^n |u_i'(x)|^{p_i-2} u_i'(x) v_i'(x)dx$$

and

$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^n F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx$$

for every $v = (v_1, \ldots, v_n) \in X$. Moreover, Φ is coercive, Φ' admits a continuous inverse on X^* (see [11, Lemma 2.6]), and since Φ' is monotone, Φ is sequentially weakly lower semicontinuous (see [24, Proposition 25.20]). Furthermore, $\Psi' : X \to X^*$ is a compact operator and

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

From (A1), we see that $\Phi(w) > 2r$. For each $(u_1, \ldots, u_n) \in X$, note from (2.1) that

$$\sup_{x\in[0,1]} |u_i(x)|^{p_i} \leq c\|u_i\|_{p_i}^{p_i} \quad \text{for } i = 1, \ldots, n.$$

Then, we have

$$\sup_{x\in[0,1]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq c \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \tag{2.4}$$

and so

$$\begin{aligned}
\Phi^{-1}(-\infty, r) &= \{u = (u_1, u_2, \ldots, u_n) \in X \ : \ \Phi(u) < r\} = \\
&= \left\{ u = (u_1, u_2, \ldots, u_n) \in X \ : \ \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r \right\} \subseteq \\
&\subseteq \left\{ u = (u_1, u_2, \ldots, u_n) \in X \ : \ \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq cr \ \text{ for each } x \in [0,1] \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r)} \Psi(u) &= \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r)} \int_0^1 F(x, u_1(x), \ldots, u_n(x)) dx \leq \\
&\leq \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(cr)} F(x, t_1, \ldots, t_n) dx.
\end{aligned}$$

Therefore, in view of (A2), it follows that

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}=\frac{\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r)}\int_0^1 F(x,u_1(x),\dots,u_n(x))dx}{r}\le$$

$$\le\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)}F(x,t_1,\dots,t_n)dx}{r}<$$

$$<\frac{2}{3}\frac{\int_0^1 F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^n\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}=\frac{2}{3}\frac{\Psi(w)}{\Phi(w)},$$

i.e., (C1) of Lemma 1.1 holds with $\overline{v}=w$.

From (A3), there exist two constants $\eta,\ \vartheta\in\mathbb{R}$ with

$$\eta<\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)}F(x,t_1,\dots,t_n)dx}{r}$$

such that

$$cF(x,t_1,\dots,t_n)\le\eta\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}+\vartheta\quad\text{for all } x\in[0,1] \text{ and } (t_1,\dots,t_n)\in\mathbb{R}^n.$$

Let $(u_1,\dots,u_n)\in X$ be fixed. Then

$$F(x,u_1(x),\dots,u_n(x))\le\frac{1}{c}\left(\eta\sum_{i=1}^n\frac{|u_i(x)|^{p_i}}{p_i}+\vartheta\right)\quad\text{for all } x\in[0,1]. \tag{2.5}$$

Now, in order to prove the coercivity of the functional $\Phi-\lambda\Psi$, first we assume that $\eta>0$. Then, for any fixed

$$\lambda\in\left(\frac{3}{2}\frac{\sum_{i=1}^n\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\int_0^1 F(x,w_1(x),\dots,w_n(x))dx},\ \frac{r}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)}F(x,t_1,\dots,t_n)dx}\right),$$

from (2.4) and (2.5), we have

$$
\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda \int_0^1 F(x, u_1(x), \dots, u_n(x))dx \geq \\
&\geq \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\eta}{c}\left(\sum_{i=1}^{n} \frac{1}{p_i}\int_0^1 |u_i(x)|^{p_i} dx\right) - \frac{\lambda\vartheta}{c} \geq \\
&\geq \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\eta}{c}\left(c\sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}\right) - \frac{\lambda\vartheta}{c} = \\
&= \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \lambda\eta\sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\vartheta}{c} \geq \\
&\geq \left(1 - \eta\frac{r}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr)} F(x, t_1, \dots, t_n)dx}\right)\sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} - \frac{\lambda\vartheta}{c}.
\end{aligned}
$$

Thus,

$$\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = \infty.$$

On the other hand, if $\eta \leq 0$, then it is clear that $\lim_{\|u\|\to\infty}(\Phi(u)-\lambda\Psi(u)) = \infty$. Then, both cases lead to the coercivity of functional $\Phi - \lambda\Psi$, i.e., (C2) of Lemma 1.1 holds with $\overline{v} = w$. Hence, by Lemma 1.1, $\Phi(u) - \lambda\Psi(u)$ has at least three distinct critical points. Then, taking into account the fact that the weak solutions of the system (1.1) are exactly critical points of $\Phi(u) - \lambda\Psi(u)$ and applying Lemma 1.3, we obtain the conclusion of the theorem. □

Our next result considers the existence of three nonnegative solutions of the system (1.1).

Theorem 2.2. *Assume that:*

(B1) $a_j, b_j \in [0, 1)$ *for* $j = 1, \dots, m$ *with* $\sum_{j=1}^{m} a_j \in [0, 1)$ *and* $\sum_{j=1}^{m} b_j \in [0, 1)$,
(B2) $F_{t_i}(x, t_1, \dots, t_n) \geq 0$ *for all* $(x, t_1, \dots, t_n) \in [0, 1] \times [0, \infty)^n$ *and* $i = 1, \dots, n$,

and there exist a function $w = (w_1, \dots, w_n) \in X$ *and two positive constants* r_1 *and* r_2 *with* $2r_1 < \sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i} < \frac{r_2}{2}$ *such that:*

(B3) $\dfrac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr_1)} F(x,t_1,\dots,t_n)dx}{r_1} < \dfrac{2}{3}\dfrac{\int_0^1 F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}$,

(B4) $\dfrac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr_2)} F(x,t_1,\dots,t_n)dx}{r_2} < \dfrac{1}{3}\dfrac{\int_0^1 F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}$.

Then, for each

$$\lambda \in \left(\frac{3}{2} \frac{\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\int_0^1 F(x, w_1(x), \dots, w_n(x))dx},\ \Theta_1 \right),$$

where

$$\Theta_1 = \min \left\{ \frac{r_1}{\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr_1)} F(x,t_1,\dots,t_n)dx}, \frac{r_2}{2\int_0^1 \sup_{(t_1,\dots,t_n)\in K(cr_2)} F(x,t_1,\dots,t_n)dx} \right\},$$

the system (1.1) *has at least three nonnegative classical solutions* $v^j = (v_1^j, \dots, v_n^j)$, $j = 1, 2, 3$ *such that*

$$\sum_{i=1}^{n} \frac{|v_i^j(x)|^{p_i}}{p_i} \le cr_2 \quad \textit{for each } x \in [0,1] \textit{ and } j = 1, 2, 3.$$

We need the following comparison principle in the proof of Theorem 2.2.

Lemma 2.3 ([20, Lemma 2.1])**.** *Let* (B1) *hold. Assume that* $y \in C^1[0,1]$ *satisfies* $\phi_p(y') \in AC[0,1]$ *with* $p > 1$ *and*

$$\begin{cases} -(\phi_p(y'))' \ge 0, \quad t \in (0,1), \\ y(0) = \sum_{j=1}^{m} a_j y(x_j),\ y(1) = \sum_{j=1}^{m} b_j y(x_j). \end{cases}$$

Then, $y(t) \ge 0$ *for* $t \in [0,1]$.

Proof of Theorem 2.2. Our aim is to apply Lemma 1.2 to our problem. To this end, let Φ and Ψ be defined by (2.2) and (2.3), respectively. Clearly, Φ and Ψ satisfy (D1) of Lemma 1.2. To show that (D2) of Lemma 1.2 holds, let $u^* = (u_1^\star, \dots, u_n^\star)$ and $u^{\star\star} = (u_1^{\star\star}, \dots, u_n^{\star\star})$ be two local minima for $\Phi - \lambda\Psi$. Then, u^* and $u^{\star\star}$ are critical points of $\Phi - \lambda\Psi$, and so they are weak solutions for the system (1.1). Thus, by Lemma 1.3, u^* and $u^{\star\star}$ are classical solutions of (1.1). Note that the fact $F(x, 0, \dots, 0) = 0$ and (B2) imply that $F(x, t_1, \dots, t_n) \ge 0$ for all $(x, t_1, \dots, t_n) \in [0,1] \times [0,\infty)^n$. Then, for $i = 1, \dots, n$, from (B1) and Lemma 2.3, we see that $u_i^\star(x) \ge 0$ and $u_i^{\star\star}(x) \ge 0$ on $[0,1]$, which imply that $su_i^\star + (1-s)u_i^{\star\star} \ge 0$ on $[0,1]$. Thus, $F(x, su^\star + (1-s)u^{\star\star}) \ge 0$, and consequently, $\Psi(su^\star + (1-s)u^{\star\star}) \ge 0$ for all $s \in [0,1]$, i.e., (D2) holds.

Now, from the condition $2r_1 < \sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i} < \frac{r_2}{2}$, we observe that $2r_1 < \Phi(w) < \frac{r_2}{2}$. Next, note that (2.4) holds, so

$$\begin{aligned} \Phi^{-1}(-\infty, r_1) &= \{u = (u_1, u_2, \dots, u_n) \in X \colon \Phi(u) < r_1\} = \\ &= \left\{ u = (u_1, u_2, \dots, u_n) \in X \colon \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_1 \right\} \subseteq \\ &\subseteq \left\{ u = (u_1, u_2, \dots, u_n) \in X \colon \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le cr_1 \text{ for each } x \in [0,1] \right\}. \end{aligned}$$

Thus,

$$\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_1)} \Psi(u) = \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_1)} \int_0^1 F(x,u_1(x),\ldots,u_n(x))dx \le$$
$$\le \int_0^1 \sup_{(t_1,\ldots,t_n)\in K(cr_1)} F(x,t_1,\ldots,t_n)dx.$$

Therefore, from (B3), it follows that

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} &= \frac{\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_1)}\int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r_1} \le \\ &\le \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(cr_1)} F(x,t_1,\ldots,t_n)dx}{r_1} < \\ &< \frac{2}{3}\frac{\int_0^1 F(x,w_1(x),\ldots,w_n(x))dx}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} = \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

i.e., (D3) of Lemma 1.2 holds with $\overline{v} = w$.

Using (B4) and arguing as above, we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} &= \frac{\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(-\infty,r_2)}\int_0^1 F(x,u_1(x),\ldots,u_n(x))dx}{r_2} \le \\ &\le \frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(cr_2)} F(x,t_1,\ldots,t_n)dx}{r_2} < \\ &< \frac{1}{3}\frac{\int_0^1 F(x,w_1(x),\ldots,w_n(x))dx}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} = \frac{1}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

i.e., (D4) of Lemma 1.2 holds with $\overline{v} = w$.

Therefore, by Lemma 1.2, $\Phi(u) - \lambda\Psi(u)$ has at least three distinct critical points, which are all nonnegative by Lemma 2.3. Then, taking into account the fact that the weak solutions of (1.1) are exactly critical points of $\Phi(u) - \lambda\Psi(u)$ and applying Lemma 1.3 and (2.4), we finish the proof of the theorem. □

Now, we present some fairly easily verifiable consequences of the main results where the test function w is specified. Let

$$\sigma_n = \left[2^{p_n-1}\left(x_1^{1-p_n}\left|1-\sum_{j=1}^m a_j\right|^{p_n} + (1-x_m)^{1-p_n}\left|1-\sum_{j=1}^m b_j\right|^{p_n}\right)\right]^{1/p_n}.$$

Define

$$B_{1,n}(x) = \begin{cases} [x\sum_{j=1}^m a_j, x]^n, & \text{if } \sum_{j=1}^m a_j < 1, \\ [x, x\sum_{j=1}^m a_j]^n, & \text{if } \sum_{j=1}^m a_j > 1, \end{cases}$$

and

$$B_{2,n}(x)=\begin{cases}[x\sum_{j=1}^m b_j, x]^n, & \text{if } \sum_{j=1}^m b_j<1,\\ [x, x\sum_{j=1}^m b_j]^n, & \text{if } \sum_{j=1}^m b_j>1,\end{cases}$$

where $[\cdot,\cdot]^n=[\cdot,\cdot]\times\ldots\times[\cdot,\cdot]$.

Corollary 2.4. *Assume that there exist two positive constants θ and τ such that*

(E1) $(\sigma_n\tau)^{p_n}>\frac{2\theta^{p_n}}{c\prod_{i=1}^{n-1}p_i}$,

(E2) $F(x,t_1,\ldots,t_n)\geq 0$ *for each* $x\in[0,x_1/2]\cup[(1+x_m)/2,1]$ *and* $(t_1,\ldots,t_n)\in B_{1,n}(\tau)\cup B_{2,n}(\tau)$,

(E3) $\frac{\int_0^1 \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx}{\theta^{p_n}}<\frac{2}{3c(\sigma_n\tau)^{p_n}\prod_{i=1}^{n-1}p_i}\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,0,\ldots,0,\tau)dx,$

(E4) $\limsup_{|t_1|\to\infty,\ldots,|t_n|\to\infty}\frac{F(x,t_1,\ldots,t_n)}{\sum_{i=1}^n\frac{|t_i|^{p_i}}{p_i}}<\frac{\prod_{i=1}^n p_i}{\theta^{p_n}}\int_0^1 \sup_{(t_1,\ldots,t_n)\in K(\frac{\theta^{p_n}}{\prod_{i=1}^n p_i})} F(x,t_1,\ldots,t_n)dx.$

Then, for each

$$\lambda\in\left(\frac{3}{2}\frac{(\sigma_n\tau)^{p_n}}{p_n\int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}}F(x,0,\ldots,0,\tau)dx},\ \frac{\theta^{p_n}}{c\prod_{i=1}^n p_i\int_0^1\sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta^{p_n}}{\prod_{i=1}^n p_i}\right)}F(x,t_1,\ldots,t_n)dx}\right),$$

the system (1.1) *has at least three classical solutions.*

Proof. Under the conditions (E1)–(E4), the assumptions (A1)–(A3) of Theorem 2.1 are satisfied by choosing $w=(0,\ldots,0,w_n(x))$ with

$$w_n(x)=\begin{cases}\tau\left(\sum_{j=1}^m a_j+\frac{2(1-\sum_{j=1}^m a_j)}{x_1}x\right), & \text{if } x\in[0,\frac{x_1}{2}),\\ \tau, & \text{if } x\in[\frac{x_1}{2},\frac{1+x_m}{2}],\\ \tau\left(\frac{2-\sum_{j=1}^m b_j-x_m\sum_{j=1}^m b_j}{1-x_m}-\frac{2(1-\sum_{j=1}^m b_j)}{1-x_m}x\right), & \text{if } x\in(\frac{1+x_m}{2},1],\end{cases}\tag{2.6}$$

and $r=\frac{\theta^{p_n}}{c\prod_{i=1}^n p_i}$. It is easy to see that $w=(0,\ldots,0,w_n)\in X$ and, in particular, that

$$\|w_n\|_{p_n}^{p_n}=(\sigma_n\tau)^{p_n}.$$

Thus,

$$\Phi(w)=\sum_{i=1}^n\frac{\|w_i\|_{p_i}^{p_i}}{p_i}=\frac{(\sigma_n\tau)^{p_n}}{p_n}.$$

Then, (E1) implies (A1). On the other hand, since

$$\tau\sum_{j=1}^{m} a_j \le w_n(x) \le \tau \quad \text{for each} \quad x\in[0,x_1/2] \quad \text{if} \quad \sum_{j=1}^{m} a_j < 1,$$

$$\tau \le w_n(x) \le \tau\sum_{j=1}^{m} a_j \quad \text{for each} \quad x\in[0,x_1/2] \quad \text{if} \quad \sum_{j=1}^{m} a_j > 1,$$

$$\tau\sum_{j=1}^{m} b_j \le w_n(x) \le \tau \quad \text{for each} \quad x\in[(1+x_m)/2,1] \quad \text{if} \quad \sum_{j=1}^{m} b_j < 1,$$

and

$$\tau \le w_n(x) \le \tau\sum_{j=1}^{m} b_j \quad \text{for each} \quad x\in[(1+x_m)/2,1] \quad \text{if} \quad \sum_{j=1}^{m} b_j > 1,$$

condition (E2) ensures that

$$\int_0^{\frac{x_1}{2}} F(x,w_1(x),\dots,w_n(x))dx + \int_{\frac{1+x_m}{2}}^{1} F(x,w_1(x),\dots,w_n(x))dx \ge 0,$$

and so,

$$\int_0^1 F(x,w_1(x),\dots,w_n(x))dx \ge \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,0,\dots,0,\tau)dx.$$

Now, from this inequality and (E3), it is easy to see that (A2) holds. Finally, note that (E4) implies (A3). The conclusion then follows from Theorem 2.1. □

Corollary 2.5. *Assume that* (B1) *and* (B2) *hold and there exist three positive constants* θ_1*,* θ_2*, and* τ *with*

$$2\theta_1^{p_n} < (\sigma_n\tau)^{p_n} c \prod_{i=1}^{n-1} p_i < \frac{\theta_2^{p_n}}{2}$$

such that:

(F1) $$\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_1^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x,t_1,\dots,t_n)dx}{\theta_1^{p_n}} < \frac{2}{3c(\sigma_n\tau)^{p_n}\prod_{i=1}^{n-1} p_i} \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,0,\dots,0,\tau)dx,$$

(F2) $$\frac{\int_0^1 \sup_{(t_1,\dots,t_n)\in K\left(\frac{\theta_2^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x,t_1,\dots,t_n)dx}{\theta_2^{p_n}} < \frac{1}{3c(\sigma_n\tau)^{p_n}\prod_{i=1}^{n-1} p_i} \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,0,\dots,0,\tau)dx.$$

Then, for each

$$\lambda \in \left(\frac{3}{2} \frac{(\sigma_n \tau)^{p_n}}{p_n \int_{\frac{x_1}{2}}^{\frac{1+x_m}{2}} F(x,0,\ldots,0,\tau)dx}, \ \Theta_2 \right),$$

where

$$\begin{aligned}\Theta_2 \quad = \quad \min \Bigg\{ & \frac{\theta_1^{p_n}}{c \prod_{i=1}^n p_i \int_0^1 \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta_1^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx}, \\ & \frac{\theta_2^{p_n}}{c \prod_{i=1}^n p_i \int_0^1 \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\ldots,t_n)dx} \Bigg\},\end{aligned}$$

the system (1.1) *has at least three nonnegative classical solutions* $v^j = (v_1^j, \ldots, v_n^j)$, $j = 1,2,3$ *such that*

$$\sum_{i=1}^n \frac{|v_i^j(x)|^{p_i}}{p_i} \le \frac{\theta_2^{p_n}}{\prod_{i=1}^n p_i} \quad \text{for each } x \in [0,1] \text{ and } j = 1,2,3.$$

Proof. Let $w = (0,\ldots,0,w_n(x))$ with $w_n(x)$ defined by (2.6), $r_1 = \frac{\theta_1^{p_n}}{c\prod_{i=1}^n p_i}$, and $r_2 = \frac{\theta_2^{p_n}}{c\prod_{i=1}^n p_i}$. Then, under the conditions (F1) and (F2), it is easy to verify that (B3) and (B4) of Theorem 2.2 hold. The conclusion then follows from Theorem 2.2. The details are omitted here. $\square$

Acknowledgments
Research of Shapour Heidarkhani was supported in part by a grant from IPM (*No. 91470046*).

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Received: August 29, 2012.
Revised: October 23, 2012.
Accepted: October 25, 2012.