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A UNIFIED REPRESENTATION OF SOME STARLIKE AND CONVEX HARMONIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper we introduce a unified representation of starlike and convex harmonic functions with negative coefficients, related to uniformly starlike and uniformly convex analytic functions. We obtain extreme points, distortion bounds, convolution conditions and convex combinations for this family.

Keywords: harmonic, analytic, univalent, sense-preserving, extreme points.

Mathematics Subject Classification: 30C45.

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$, $z \in D$ (see [2]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z^{'}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.2}$$

In 1984 Clunie and Sheil-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.

Also let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n, \quad g(z) = -\sum_{n=1}^{\infty} |b_n|\, z^n, \quad |b_1| < 1. \tag{1.3}$$

Recently Kanas and Wisniowska [6] (see also Kanas and Srivastava [5]), studied the class of $k-$uniformly convex analytic functions, denoted by $k - UCV$, $k \geq 0$, so that $\phi \in k - UCV$ if and only if

$$\operatorname{Re}\left\{1 + \frac{(z-\zeta)\phi^{''}(z)}{\phi^{'}(z)}\right\} \geq 0, \quad |\zeta| \leq k,\ z \in U. \tag{1.4}$$

For $\theta \in \mathbb{R}$, if we let $\zeta = -kze^{i\theta}$, then condition (1.4) can be written as

$$\operatorname{Re}\left\{1 + (1 + ke^{i\theta})\frac{z\phi^{''}(z)}{\phi^{'}(z)}\right\} \geq 0. \tag{1.5}$$

In 2002 Kim *et al.* [7] introduced and studied the class $HCV(k, \alpha)$, consisting of functions $f = h + \overline{g}$ such that h and g are given by (1.3), and f satisfies the condition

$$\operatorname{Re}\left\{1 + (1 + ke^{i\theta})\frac{zf^{''}(z)}{f^{'}(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1,\ \theta \in \mathbb{R},\ k \geq 0. \tag{1.6}$$

Replacing $h + \overline{g}$ for f in (1.6), we have

$$\operatorname{Re}\left\{1 + (1 + ke^{i\theta})\frac{z^2h^{''}(z) + \overline{2zg^{'}(z) + z^2g^{''}(z)}}{zh^{'}(z) - \overline{zg^{'}(z)}}\right\} \geq \alpha, \quad 0 \leq \alpha < 1,\ \theta \in \mathbb{R},\ k \geq 0. \tag{1.7}$$

Also the class of $k - UST$ uniformly starlike functions is defined by using (1.5) as the class of all functions $\psi(z) = z\phi^{'}(z)$ such that $\phi \in k - UCV$, then $\psi(z) \in k - UST$ if and only if

$$\operatorname{Re}\left\{(1 + ke^{i\theta})\frac{z\psi^{'}(z)}{\psi(z)} - ke^{i\theta}\right\} \geq 0.$$

Generalizing the class $k - UST$ to include harmonic functions, we let $HST(k, \alpha)$ denote the class of functions $f = h + \overline{g}$ such that h and g are given by (1.3), which satisfy the condition

$$\operatorname{Re}\left\{(1 + ke^{i\theta})\frac{zf^{'}(z)}{f(z)} - ke^{i\theta}\right\} \geq \alpha, \quad 0 \leq \alpha < 1,\ \theta \in \mathbb{R},\ k \geq 0. \tag{1.8}$$

Observe that the class $HST(1, \alpha)$ was introduced by Rosy *et al.* [8] and also by Aghalary [1, with $m = 1$ and $n = 0$].

Now we shall need the following lemmas.

Lemma 1.1 ([7]). *A function $f = h + \overline{g}$ such that h and g are given by* (1.3) *is in the class $HCV(k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} n \left[\frac{(1+k)n - k - \alpha}{1-\alpha} |a_n| + \frac{(1+k)n + k + \alpha}{1-\alpha} |b_n| \right] \leq 2. \tag{1.9}$$

The condition (1.9) *is sharp.*

Lemma 1.2 ([1, with $m = 1$ and $n = 0$]). *A function $f = h + \overline{g}$ such that h and g are given by* (1.3) *is in the class $HST(k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left[\frac{(1+k)n - k - \alpha}{1-\alpha} |a_n| + \frac{(1+k)n + k + \alpha}{1-\alpha} |b_n| \right] \leq 2. \tag{1.10}$$

The condition (1.10) *is sharp.*

In view of Lemma 1.1 and Lemma 1.2, we introduce and study an interesting unification of the classes $HCV(k,\alpha)$ and $HST(k,\alpha)$.

Let the class $HGN(k,\alpha,\gamma)$, $0 \leq \alpha < 1$, $0 \leq \gamma \leq 1$, $k \geq 0$ be the class of functions $f = h + \overline{g}$ such that h and g are given by (1.3), which satisfy the condition

$$\sum_{n=1}^{\infty} n^{\gamma} \left[\frac{(1+k)n - k - \alpha}{1-\alpha} |a_n| + \frac{(1+k)n + k + \alpha}{1-\alpha} |b_n| \right] \leq 2. \tag{1.11}$$

Specializing the parameters α, γ and k, we obtain the following subclasses studied by various authors:

(i) $HGN(k,\alpha,1) = HCV(k,\alpha)$ (see Kim *et al.* [7]),
(ii) $HGN(k,\alpha,0) = HST(k,\alpha)$ (see Aghalary [1, with $m = 1$ and $n = 0$]),
(iii) $HGN(1,\alpha,0) = G_H(\alpha)$ (see Rosy *et al.* [8]),
(iv) $HGN(0,\alpha,1) = HK(\alpha)$ (see Jahangiri [3]),
(v) $HGN(0,\alpha,0) = HS(\alpha)$ (see Silverman [9], Jahangiri [4], and Silverman and Silvia [10]).

Also we note that:

(i) $HGN(k,0,\gamma) = HG(k,\gamma)$,
(ii) $HGN(0,0,\gamma) = HG(\gamma)$.

In the following section we obtain the distortion bounds and extreme points for functions in the class $HGN(k,\alpha,\gamma)$.

2. DISTORTION BOUNDS AND EXTREME POINTS

Unless otherwise mentioned, we assume in the reminder of this paper that, $0 \leq \alpha < 1$, $0 \leq \gamma \leq 1$, $k \geq 0$ and $z \in U$.

Now we obtain the distortion bounds for functions in the class $HGN(k,\alpha,\gamma)$.

Theorem 2.1. *Let $f = h+\overline{g}$, where h and g are given by (1.3) and $f \in HGN(k,\alpha,\gamma)$. Then for $|z| = r < 1$, we have*

$$|f(z)| \leq (1+|b_1|)\, r + \frac{1}{2^\gamma}\left[\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}\,|b_1|\right] r^2 \tag{2.1}$$

and

$$|f(z)| \geq (1-|b_1|)\, r - \frac{1}{2^\gamma}\left[\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}\,|b_1|\right] r^2, \tag{2.2}$$

where

$$|b_1| \leq \frac{1-\alpha}{2+k-\alpha}.$$

The result is sharp.

Proof. We shall prove the first inequality. Let $f \in HGN(k,\alpha,\gamma)$. Then we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)\, r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^n \leq \\
&\leq (1+|b_1|)\, r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^2 = \\
&= (1+|b_1|)\, r + \tfrac{1-\alpha}{2^\gamma[2(1+k)-k-\alpha]} \sum_{n=2}^{\infty} \tfrac{2^\gamma[2(1+k)-k-\alpha]}{1-\alpha} (|a_n| + |b_n|)\, r^2,
\end{aligned}$$

and so

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)\, r + \tfrac{1-\alpha}{2^\gamma[2+k-\alpha]}\left[\sum_{n=2}^{\infty} n^\gamma \left[\tfrac{(1+k)n-k-\alpha}{1-\alpha}\,|a_n| + \tfrac{(1+k)n+k+\alpha}{1-\alpha}\,|b_n|\right]\right] r^2 \leq \\
&\leq (1+|b_1|)\, r + \frac{1-\alpha}{2^\gamma[2+k-\alpha]}\left[1 - \frac{1+2k+\alpha}{1-\alpha}\,|b_1|\right] r^2 = \\
&= (1+|b_1|)\, r + \frac{1}{2^\gamma}\left[\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}\,|b_1|\right] r^2.
\end{aligned}$$

The proof of the inequality (2.2) is similar, thus we omit it.

The upper bound given for $f \in HGN(k,\alpha,\gamma)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1|\,\overline{z} + \tfrac{1}{2^\gamma}\left[\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha}\,|b_1|\right] \overline{z}^2, \tag{2.3}$$

$$|b_1| \leq \frac{1-\alpha}{2+k-\alpha}. \tag{2.4}$$

This completes the proof of Theorem 2.1. □

Now we determine the extreme points of the closed convex hull of the class $HGN(k,\alpha,\gamma)$ denoted by $clcoHGN(k,\alpha,\gamma)$.

Theorem 2.2. *Let* $f = h + \overline{g}$, *where* h *and* g *are given by* (1.3), *then* $f(z) \in clcoHGN(k, \alpha, \gamma)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty} X_n h_n + Y_n g_n, \tag{2.5}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_n(z) &= z - \frac{1-\alpha}{n^{\gamma}\left[(1+k)n - k - \alpha\right]} z^n \quad (n = 2, 3, \ldots), \\ g_n &= z - \frac{1-\alpha}{n^{\gamma}\left[(1+k)n + k + \alpha\right]} z^n \quad (n = 1, 2, \ldots), \end{aligned}$$

$\sum_{n=1}^{\infty} X_n + Y_n = 1$, $X_n \geq 0$ *and* $Y_n \geq 0$. *In particular, the extreme points of the class* $HGN(k, \alpha, \gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$, *respectively.*

Proof. For a function f of the form (2.5), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} X_n h_n + Y_n g_n = \\ &= \sum_{n=1}^{\infty} (X_n + Y_n)\, z - \sum_{n=2}^{\infty} \tfrac{1-\alpha}{n^{\gamma}[(1+k)n-k-\alpha]} X_n z^n - \sum_{n=1}^{\infty} \tfrac{1-\alpha}{n^{\gamma}[(1+k)n+k+\alpha]} Y_n z^n = \\ &= z - \sum_{n=2}^{\infty} \tfrac{1-\alpha}{n^{\gamma}[(1+k)n-k-\alpha]} X_n z^n - \sum_{n=1}^{\infty} \frac{1-\alpha}{n^{\gamma}\left[(1+k)n + k + \alpha\right]} Y_n z^n. \end{aligned}$$

But

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n^{\gamma}\left[(1+k)n - k - \alpha\right]}{1-\alpha} \left(\frac{1-\alpha}{n^{\gamma}\left[(1+k)n - k - \alpha\right]} X_n \right) + \\ &\quad + \sum_{n=1}^{\infty} \frac{n^{\gamma}\left[(1+k)n + k + \alpha\right]}{1-\alpha} \left(\frac{1-\alpha}{n^{\gamma}\left[(1+k)n + k + \alpha\right]} Y_n \right) = \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1. \end{aligned}$$

Thus $f(z) \in clcoHGN(k, \alpha, \gamma)$.

Conversely, suppose that $f(z) \in clcoHGN(k, \alpha, \gamma)$. Set

$$X_n = \frac{n^{\gamma}\left[(1+k)n - k - \alpha\right]}{1-\alpha} |a_n| \quad (n = 2, 3, \ldots) \tag{2.6}$$

and

$$Y_n = \frac{n^{\gamma}\left[(1+k)n + k + \alpha\right]}{1-\alpha} |b_n| \quad (n = 1, 2, \ldots). \tag{2.7}$$

Then by the inequality (1.11), we have $0 \leq X_n \leq 1$ $(n = 2, 3, \ldots)$ and $0 \leq Y_n \leq 1$ $(n = 1, 2, \ldots)$. Define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that $X_1 \geq 0$. Thus we obtain $f(z) = \sum_{n=1}^{\infty} X_n h_n + Y_n g_n$. This completes the proof of Theorem 2.2. □

3. CONVOLUTION AND CONVEX COMBINATIONS

For two harmonic functions

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n - \sum_{n=1}^{\infty} |b_n|\, \overline{z}^n$$

and

$$F(z) = z - \sum_{n=2}^{\infty} |A_n|\, z^n - \sum_{n=1}^{\infty} |B_n|\, \overline{z}^n,$$

we define the modified convolution of f and F as

$$(f * F)(z) = z - \sum_{n=2}^{\infty} |a_n A_n|\, z^n - \sum_{n=1}^{\infty} |b_n B_n|\, \overline{z}^n = (F * f)(z). \tag{3.1}$$

Theorem 3.1. *For $0 \leq \beta \leq \alpha < 1$, let $f \in HGN(k, \alpha, \gamma)$ and $F \in HGN(k, \beta, \gamma)$. Then*

$$f * F \in HGN(k, \alpha, \gamma) \subset HGN(k, \beta, \gamma). \tag{3.2}$$

Proof. Let $f(z) = z - \sum\limits_{n=2}^{\infty} |a_n|\, z^n - \sum_{n=1}^{\infty} |b_n|\, \overline{z}^n$ be in the class $HGN(k, \alpha, \gamma)$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n|\, z^n - \sum_{n=1}^{\infty} |B_n|\, \overline{z}^n$ be in the class $HGN(k, \beta, \gamma)$. But f and F satisfy conditions similar to (1.11).

For the coefficients of the function $f * F$, noting that $|A_n| \leq 1$ and $|B_n| \leq 1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\gamma} \left\{ [(1+k)n - k - \alpha]\, |a_n A_n| + [(1+k)n + k + \alpha]\, |b_n B_n| \right\} \leq \\
&\leq \sum_{n=1}^{\infty} n^{\gamma} \left\{ [(1+k)n - k - \alpha]\, |a_n| + [(1+k)n + k + \alpha]\, |b_n| \right\} \leq \\
&\leq 2(1-\alpha) \leq 2(1-\beta).
\end{aligned}$$

Then we conclude that $f * F \in HGN(k, \alpha, \gamma) \subset HGN(k, \beta, \gamma)$. This completes the proof of Theorem 3.1. □

Finally, we show that the class $HGN(k, \alpha, \gamma)$ is closed under convex combination of its members.

Theorem 3.2. *The family $HGN(k, \alpha, \gamma)$ is closed under convex combination.*

Proof. For $i = 2, 3, \ldots$, let $f_i \in HGN(k, \alpha, \gamma)$, where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n_i}|\, z^n - \sum_{n=1}^{\infty} |b_{n_i}|\, \overline{z}^n, \quad |b_{n_i}| < 1. \tag{3.3}$$

Then, by (1.11), we have

$$\sum_{n=1}^{\infty} n^{\gamma} \left[\tfrac{(1+k)n-k-\alpha}{1-\alpha} |a_{n_i}| + \tfrac{(1+k)n+k+\alpha}{1-\alpha} |b_{n_i}| \right] \leq 2. \tag{3.4}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of t_i is written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i\, |a_{n_i}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i\, |b_{n_i}| \right) \overline{z}^n. \tag{3.5}$$

Then, by using (3.4), we have

$$\sum_{n=1}^{\infty} n^{\gamma} \left[\tfrac{(1+k)n-k-\alpha}{1-\alpha} \left| \sum_{i=1}^{\infty} t_i\, |a_{n_i}| \right| + \tfrac{(1+k)n+k+\alpha}{1-\alpha} \left| \sum_{i=1}^{\infty} t_i\, |b_{n_i}| \right| \right] =$$
$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=1}^{\infty} n^{\gamma} \left[\tfrac{(1+k)n-k-\alpha}{1-\alpha} |a_{n_i}| + \tfrac{(1+k)n+k+\alpha}{1-\alpha} |b_{n_i}| \right] \right\} \leq$$
$$\leq 2 \sum_{i=1}^{\infty} t_i = 2.$$

Thus the proof of Theorem 3.2 is completed. □

Remark 3.3. (i) Putting $\gamma = 1$ in all the above results, we obtain the corresponding results obtained by Kim *et al.* [7].

(ii) Putting $\gamma = 0$ in all the above results, we obtain the corresponding results obtained by Aghalary [1, with $m = 1$ and $n = 0$].

(iii) Putting $k = 1$ and $\gamma = 0$ in all the above results, we obtain the corresponding results obtained by Rosy *et al.* [8].

(iv) Putting $k = 0$ and $\gamma = 1$ in all the above results, we obtain the corresponding results obtained by Jahangiri [3].

(v) Putting $k = 0$ and $\gamma = 0$ in all the above results, we obtain the corresponding results obtained by Silverman [9], Jahangiri [4], and Silverman and Silvia [10].

(vi) Putting $\alpha = 0$ in all the above results, we obtain the corresponding results for the class $HG(k, \gamma)$.

(vii) Putting $\alpha = 0$ and $k = 0$ in all the above results, we obtain the corresponding results for the class $HG(\gamma)$.

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