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A VERTEX OPERATOR REPRESENTATION OF SOLUTIONS TO THE GUREVICH-ZYBIN HYDRODYNAMICAL EQUATION

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Abstract. An approach based on the spectral and Lie – algebraic techniques for constructing vertex operator representation for solutions to a Riemann type hydrodynamical hierarchy is devised. A functional representation generating an infinite hierarchy of dispersive Lax type integrable flows is obtained.

Keywords: Lax type integrability, vertex operator representation, Lax integrability, Lie-algebraic approach.

Mathematics Subject Classification: 58A30, 56B05, 34B15.

1. INTRODUCTION

Nonlinear hydrodynamic equations are of constant interest from the classical works by B. Riemann, who had extensively studied them in the general three-dimensional case, having paid special attention to their one-dimensional spatial reduction, for which he devised the generalized method of characteristics and Riemann invariants. These methods appeared to be very effective [1] in investigating many types of nonlinear spatially one-dimensional systems of hydrodynamical type and, in particular, the characteristics method in the form of a "reciprocal" transformation of variables has been used recently in studying a so called Gurevich-Zybin system [2, 3] in [9] and a Whitham type system in [5, 6]. Moreover, this method was further effectively applied to studying solutions to a generalized [5] (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical system

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \tag{1.1}$$

where $N \in \mathbb{Z}_+$, $u \in M^1 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ is a smooth function on a periodic functional manifold M^1 and $t \in \mathbb{R}$ is the evolution parameter. Making use of novel methods, devised in [8, 18, 25] and based both on the spectral theory [10, 17, 19, 20] and differential algebra techniques, the Lax type representations for the cases $N = \overline{1,4}$ were constructed in explicit form.

In this work we are interested in constructing a so called vertex operator representation [14,15,22,24] for solutions to the Gurevich-Zybin hydrodynamical hierarchy naturally related to the Riemann type hydrodynamical flow (1.1) on a periodic functional manifold $M^2 \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$ at $N = 2$:

$$\begin{cases} D_t u = u_t + u u_x = v, \\ D_t v = v_t + u v_x = 0, \end{cases} \tag{1.2}$$

making use of an approach recently devised in [23, 24] for the case of the classical AKNS hierarchy of integrable flows, and which can be easily generalized for treating the problem for arbitrary integers $N \in \mathbb{Z}_+$. Namely, the following proposition, effectively describing the whole infinite hierarchy of the Gurevich-Zybin type commuting to each other Hamiltonian flows, holds.

Proposition 1.1. *The Gurevich-Zybin hydrodynamical system* (1.2) *possesses an infinite hierarchy*

$$d(u,v)^\intercal/dt_j = K_j[u,v],$$

$j \in \mathbb{N}$, *of commuting to each other Hamiltonian flows* $K_j : M^2 \to T(M^2)$, *representable in the following vertex operator form:*

$$\begin{aligned} \alpha^-(x,\tau;\lambda) &= u_x + [u_x^2 - 2v_x + \lambda^{-1}\alpha^-(x,\tau;\lambda)]^{1/2}, \\ \beta^+(x,\tau;\lambda) &= u_x - [u_x^2 - 2v_x + \lambda^{-1}\beta^+(x,\tau;\lambda)]^{1/2}, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \alpha^-(x,\tau;\lambda) &= \exp(-D_\lambda)\alpha(x,\tau), & \beta^+(x,\tau;\lambda) &:= \exp(D_\lambda)\beta(x,\tau), \\ D_\lambda &:= \sum_{j\in\mathbb{N}} \lambda^{-j}\frac{d}{j dt_j}, & \alpha(x,\tau) &= u_x(x,\tau) + \varphi(x,\tau), \\ \beta(x,\tau) &= u_x(x,\tau) - \varphi(x,\tau), & \varphi(x,\tau) &:= \sqrt{u_x^2(x,\tau) - 2v_x(x,\tau)}, \end{aligned} \tag{1.4}$$

$\tau := (t_1, t_2 = t, t_3, \ldots) \in \mathbb{R}^{\mathbb{N}}$ *and* $|\lambda| \to \infty$.

2. A VERTEX OPERATOR ANALYSIS

We begin with a Lax type linear spectral problem [4, 5, 9] for the equation (1.1) at $N = 2$:

$$\begin{cases} D_t u = u_t + u u_x = v, \\ D_t v = v_t + u v_x = 0, \end{cases} \tag{2.1}$$

defined on the space of smooth real-valued 2π-periodic functions $(u,v)^\intercal \in M \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$:

$$df/dx = \ell[u,v;\lambda]f, \quad \ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x/2 & -v_x \\ \lambda^2/2 & \lambda u_x/2 \end{pmatrix}, \tag{2.2}$$

where, by definition, $v := D_t u$, $f \in L_\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is a spectral parameter. Assume that a vector function $(u,v)^\top \in M$ depends parametrically on the infinite set $\tau := \{t_1, t_2, t_3, \ldots\} \in \mathbb{R}^{\mathbb{Z}_+}$ in such a way that the generalized Floquet spectrum [10, 16, 19] $\sigma(\ell) := \{\lambda \in \mathbb{C} : \sup_{x\in\mathbb{R}} \|f(x;\lambda)\|_\infty < \infty\}$ of the linear problem (2.2) persists in being parametrically iso-spectral, that is $d\sigma(\ell)/dt_j = 0$ for all $t_j \in \mathbb{R}, j \in \mathbb{N}$. The iso-spectrality condition imbedded in problem (2.2) gives rise [14, 16–18, 21] naturally to a hierarchy of commuting to each other nonlinear bi-Hamiltonian dynamical systems on the functional manifold M in the general form

$$\frac{d}{dt_j}(u(t), v(t))^\top = -\vartheta grad H_j[u,v] := K_j[u(t), v(t)], \tag{2.3}$$

where $K_j : M^2 \to T(M^2)$ and $H_j \in \mathcal{D}(M^2), j \in \mathbb{N}$, are, respectively, vector fields and conservation laws on the manifold M^2, which were before described in [4, 5, 8],

$$\vartheta := \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{2.4}$$

is a Poisson structure on the manifold M^2 and, by definition,

$$\begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix} := \begin{pmatrix} u(x, t_1, t_2, t_3, \ldots) \\ v(x, t_1, t_2, t_3, \ldots) \end{pmatrix} \tag{2.5}$$

for $\tau \in \mathbb{R}^{\mathbb{N}}$.

It is well known [10, 16, 17, 19] that the Casimir invariants, determining conservation laws for dynamical systems (2.3), are generated by the suitably normalized monodromy matrix $\tilde{S}(x;\lambda) \in End\ \mathbb{C}^2$ of the linear problem (2.2)

$$\tilde{S}(x;\lambda) = k(\lambda)S(x;\lambda) - \frac{k(\lambda)}{2}\mathrm{tr}S(x;\lambda), \tag{2.6}$$

where $F(y,x;\lambda) \in End\ \mathbb{C}^2$ is the matrix solution to the Cauchy problems

$$\frac{d}{dy}F(y,x;\lambda) = \ell(y;\lambda)F(y,x;\lambda), \quad F(y,x;\lambda)|_{y=x} = \mathbf{I}, \tag{2.7}$$

for all $\lambda \in \mathbb{C}$ and $x, y \in \mathbb{R}$, where $\mathbf{I} \in End\ \mathbb{C}^2$ is the identity matrix, $S(x;\lambda) := F(x+2\pi, x;\lambda)$ is the usual monodromy matrix for equation (2.7). Here the parameter $k(\lambda) \in \mathbb{C}$ is invariant with respect to flows (2.3) and is chosen in such a way that the asymptotic condition

$$\tilde{S}(x;\lambda) \in \tilde{\mathcal{G}}_- \tag{2.8}$$

as $\lambda \to \infty$ holds for all $x \in \mathbb{R}$. The latter allows us to define the corresponding momentum mapping $\ell : M^2 \to \tilde{\mathcal{G}}_-$ and to imbed the whole hierarchy of the Hamiltonian flows (2.3), related with the linear problem (1.1), into the standard [14, 16–18, 21] Lie-algebraic scheme. Here $\tilde{\mathcal{G}}_- \subset \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ is the natural splitting into two affine subalgebras of positive and negative λ-expansions of the centrally extended [16, 26] affine current $\mathfrak{sl}(2)$-algebra $\hat{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$:

$$\tilde{\mathcal{G}} := \Big\{ a = \sum_{j \in \mathbb{Z},\, j \ll \infty} a^{(j)} \otimes \lambda^j : a^{(j)} \in C^{\infty}\left(\mathbb{R}/2\pi\mathbb{Z}; \mathfrak{sl}(2;\mathbb{C})\right) \Big\}. \tag{2.9}$$

The latter is endowed with the Lie commutator

$$[(a_1, c_1), (a_2, c_2)] := ([a_1, a_2], \langle a_1, da_2/dx\rangle), \tag{2.10}$$

where the scalar product is defined as

$$\langle a_1, a_2\rangle := \mathrm{res}_{\lambda=\infty} \int_0^{2\pi} \mathrm{tr}(a_1 a_2) dx \tag{2.11}$$

for any two elements $a_1, a_2 \in \tilde{\mathcal{G}}$ with "res" and "tr" being the usual residue and trace maps, respectively. As the spectrum $\sigma(\ell) \subset \mathbb{C}$ of the problem (2.2) is supposed to be parametrically independent, flows (2.3) are naturally associated with evolution equations

$$d\tilde{S}/dt_j = [(\lambda^{j+1}\tilde{S})_+, \tilde{S}] \tag{2.12}$$

for all $j \in \mathbb{N}$, which are generated by the set $I(\hat{\mathcal{G}}^*)$ of Casimir invariants of the coadjoint action of the current algebra $\hat{\mathcal{G}}$ on a given element $\ell(x;\lambda) \in \tilde{\mathcal{G}}_-^* \cong \tilde{\mathcal{G}}_+$ contained in the space of smooth functionals $\mathcal{D}(\hat{\mathcal{G}})$. In particular, a functional $\gamma(\lambda) \in I(\hat{\mathcal{G}})$ if and only if

$$[\tilde{S}(x;\lambda), \ell(x;\lambda)] + \frac{d}{dx}\tilde{S}(x;\lambda) = 0, \tag{2.13}$$

where the gradient $\tilde{S}(x;\lambda) := \mathrm{grad}\gamma(\lambda)(\ell) \in \tilde{\mathcal{G}}_-$ is defined with respect to the scalar product (2.11) by means of the variation

$$\delta\gamma(\lambda) := \langle \mathrm{grad}\gamma(\lambda)(\ell), \delta\ell\rangle\,. \tag{2.14}$$

To construct the solution to matrix equation (2.13), we find a preliminary partial solution $\tilde{F}(y,x;\lambda) \in End\ \mathbb{C}^2$, $x, y \in \mathbb{R}$, to equation (2.7) satisfying the asymptotic Cauchy data

$$\tilde{F}(y,x;\lambda)|_{y=x} = \mathbf{I} + O(1/\lambda) \tag{2.15}$$

as $\lambda \to \infty$. It is easy to check that

$$\tilde{F}(y,x;\lambda) = \begin{pmatrix} \tilde{e}_1(y,x;\lambda) & -\frac{\tilde{\beta}(y;\lambda)}{\lambda}\tilde{e}_2(y,x;\lambda) \\ -\frac{\lambda}{\tilde{\alpha}(y;\lambda)}\tilde{e}_1(y,x;\lambda) & \tilde{e}_2(y,x;\lambda) \end{pmatrix}, \tag{2.16}$$

is an exact functional solution to (2.7) satisfying condition (2.15), where we have defined

$$\begin{aligned} \tilde{e}_1(y,x;\lambda) &:= \exp\Big\{\frac{\lambda}{2}[u(x)-u(y)] + \lambda \int\limits_x^y \tilde{\alpha} dv(s)\Big\}, \\ \tilde{e}_2(y,x;\lambda) &:= \exp\Big\{\frac{\lambda}{2}[u(y)-u(x)] - \frac{\lambda}{2} \int\limits_x^y \tilde{\beta} ds\Big\}, \end{aligned} \tag{2.17}$$

with the vector-functions $\alpha^{\pm} \in C^{\infty}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ satisfying the following determining functional relationships:

$$\begin{aligned} \tilde{\alpha} &= u_x + (u_x^2 - 2v_x + \xi\tilde{\alpha})^{1/2}, \\ \tilde{\beta} &= u_x - (u_x^2 - 2v_x + \xi\tilde{\beta})^{1/2}, \end{aligned} \tag{2.18}$$

as $\xi := 1/\lambda \to 0$ and existing when the condition $\varphi(x,\tau) := \sqrt{u_x^2 - 2v_x} \neq 0$ on the manifold M^2 at $\tau = \emptyset \in \mathbb{R}^{\mathbb{N}}$.

Remark 2.1. It is easy to observe that the condition $\varphi(x,\tau) := \sqrt{u_x^2 - 2v_x} \neq 0$ holds for all $\tau \in \mathbb{R}^{\mathbb{N}}$, if $\varphi(x,\tau)|_{\tau=\emptyset} \neq 0$.

The fundamental matrix $F(y,x;\lambda) \in End\ \mathbb{C}^2$ can be represented for all $x, y \in \mathbb{R}$ in the form

$$F(y,x;\lambda) = \tilde{F}(y,x;\lambda)\tilde{F}^{-1}(x,x;\lambda). \tag{2.19}$$

Consequently, if one sets $y = x + 2\pi$ in this formula and defines the expression

$$k(\lambda) := \lambda^{-1}[\tilde{e}_1(x+2\pi,x;\lambda) - \tilde{e}_2(x+2\pi,x;\lambda)]^{-1}, \tag{2.20}$$

it follows from (2.6), (2.16) and (2.19) that the exact functional matrix representation

$$\tilde{S}(x;\lambda) = \begin{pmatrix} \frac{[\tilde{\alpha}(x;\lambda)+\tilde{\beta}(x;\lambda)]}{2\lambda[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} & \frac{\tilde{\alpha}\tilde{\beta}}{\lambda^2[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} \\ -\frac{1}{[\tilde{\alpha}(x;\lambda)-\tilde{\beta}(x;\lambda)]} & \frac{[\tilde{\beta}(x;\lambda)+\tilde{\alpha}(x;\lambda)]}{2\lambda[\tilde{\beta}(x;\lambda)-\tilde{\alpha}(x;\lambda)]} \end{pmatrix}, \tag{2.21}$$

satisfies the necessary condition (2.8) as $\lambda \to \infty$.

Remark 2.2. The invariance of the expression (2.20) with respect to the generating vector field (2.3) on the manifold M derives from the representation (2.19), equations (2.13) and

$$\frac{d}{d\tau}\tilde{F}(y,x_0;\mu) = \frac{\lambda^3}{\mu-\lambda}\tilde{S}(x;\lambda)\tilde{F}(y,x_0;\mu), \tag{2.22}$$

which follows naturally from the determining matrix flows (2.12) upon applying the translation $y \to y + 2\pi$.

The matrix expression (2.21) gives rise to the following important functional relationships:

$$\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}} = \tilde{\alpha}, \ \frac{-2\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})} = \tilde{\beta}, \tag{2.23}$$

which allow us to introduce in a natural way the vertex operator vector fields

$$X_\lambda^\pm = \exp(\pm D_\lambda), \quad D_\lambda := \sum_{j\in\mathbb{N}} \lambda^{-j}\frac{d}{jdt_j}, \tag{2.24}$$

acting on an arbitrary smooth function $\eta \in C^\infty(\mathbb{R}^\mathbb{N};\mathbb{R})$ by means of the shifting mappings:

$$\begin{aligned} &X_\lambda^\pm\ \eta(x,t_1,t_2,\ldots,t_j,\ldots) := \eta^\pm(x,\tau;\lambda) = \\ &= \eta(x,t_1 \pm 1/\lambda, t_2 \pm /(2\lambda^2), t_3 \pm 1/(3\lambda^3),\ldots,t_j \pm 1/(j\lambda^j),\ldots) \end{aligned} \tag{2.25}$$

as $\lambda \to \infty$. Namely, the following proposition holds.

Proposition 2.3. *The functional vertex operator expressions*

$$\begin{aligned} \tilde{\alpha}(x,\tau;\lambda) &= X_\lambda^- \alpha(x,\tau) = \alpha^-(x,\tau;\lambda), \\ \tilde{\beta}(x,\tau;\lambda) &= X_\lambda^+ \beta(x,\tau)) = \beta^+(x,\tau;\lambda) \end{aligned} \tag{2.26}$$

as $\xi = 1/\lambda \to 0$ *solve the functional equations* (2.18)*, that is*

$$\begin{aligned} \alpha^- &= u_x + (u_x^2 - 2v_x + \xi\alpha^-)^{1/2}, \\ \beta^+ &= u_x - (u_x^2 - 2v_x + \xi\beta^+)^{1/2}. \end{aligned} \tag{2.27}$$

Proof. To state this proposition it is enough to show that the following relationships hold:

$$\begin{aligned} \frac{d}{d\xi}\left[\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}}\right]_{\lambda=1/\xi} &= \frac{d}{d\tau}\left[\frac{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}{2\tilde{s}_{21}}\right]_{\lambda=1/\xi}, \\ \frac{d}{d\xi}\left[\frac{-8\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}\right]_{\lambda=1/\xi} &= \frac{d}{d\tau}\left[\frac{-8\lambda^2\tilde{s}_{12}}{1-\lambda(\tilde{s}_{11}-\tilde{s}_{22})}\right]_{\lambda=1/\xi} \end{aligned} \tag{2.28}$$

as $\xi \to 0$, where by definition

$$\frac{d}{d\tau} := \frac{d}{d\xi}D_\lambda\bigg|_{\lambda=1/\xi} = \sum_{j\in\mathbb{N}}\xi^{j-1}\frac{d}{dt_j} \tag{2.29}$$

is a generating evolution vector field. Before doing this we find the evolution equation

$$\frac{d}{d\tau}\tilde{S}(x;\mu) = \left[\lambda^3\frac{d}{d\lambda}\tilde{S}(x;\mu), \tilde{S}(x;\lambda)\right] \tag{2.30}$$

on the matrix $\tilde{S}(x;\mu)$ as $\mu,\lambda \to \infty$, which entails the following differential relationships:

$$\begin{aligned} d\tilde{s}_{11}/d\tau &= \lambda^3(\tilde{s}_{21}d\tilde{s}_{12}/d\lambda - \tilde{s}_{12}d\tilde{s}_{21}/d\lambda), \\ d\tilde{s}_{22}/d\tau &= \lambda^3(\tilde{s}_{12}d\tilde{s}_{21}/d\lambda - \tilde{s}_{21}d\tilde{s}_{12}/d\lambda), \\ d\tilde{s}_{22}/d\tau &= \lambda^3\big[\tilde{s}_{12}\frac{d}{d\lambda}(\tilde{s}_{11}-\tilde{s}_{22}) - (\tilde{s}_{11}-\tilde{s}_{22})\frac{d\tilde{s}_{12}}{d\lambda}\big], \\ d\tilde{s}_{11}/d\tau &= \lambda^3\big[\tilde{s}_{21}\frac{d}{d\lambda}(\tilde{s}_{22}-\tilde{s}_{11}) - (\tilde{s}_{22}-\tilde{s}_{11})\frac{d\tilde{s}_{21}}{d\lambda}\big]. \end{aligned} \tag{2.31}$$

Using these relationships (2.31), one can easily obtain by means of simple, but rather cumbersome calculations, the necessary relationships (2.28). As a direct consequence the *vertex operator representations* (2.26) for the vector functions $\tilde{\alpha}, \tilde{\beta} \in M^2 \subset C(\mathbb{R}^{\mathbb{Z}_+}; \mathbb{R})$ hold. $\square$

Now we take into account that, owing to the determining functional representations (2.18), that the limits

$$\begin{aligned} &\lim_{\lambda \to \infty} \alpha^-(x, \tau; \lambda) = u_x(x, \tau) + \varphi(x, \tau), \\ &\lim_{\lambda \to \infty} \beta^+(x, \tau; \lambda) = u_x(x, \tau) - \varphi(x, \tau), \quad \varphi(x, \tau) := \sqrt{u_x^2(x, \tau) - 2v_x(x, \tau)}, \end{aligned} \tag{2.32}$$

exist on the manifold M^2. Moreover, having iterated the functional relationships (2.18), one can find that

$$\begin{aligned} X_\lambda^- \alpha = \alpha^- &= u_x + \varphi + \xi\Big(\frac{u_{xx}}{\varphi} + \frac{\varphi_x}{\varphi}\Big) + \\ &+ \frac{\xi^2}{2}\Big(\frac{u_{xx}^2 + 2u_{xx}\varphi_x - u_{3x}\varphi}{\varphi^3} + \frac{\varphi_{xx}\varphi + 5\varphi_x^2}{\varphi^3}\Big) + O(\xi^3), \\ X_\lambda^+ \beta = \beta^+ &= u_x - \varphi - \xi\Big(\frac{u_{xx}}{\varphi} - \frac{\varphi_x}{\varphi}\Big) - \\ &- \frac{\xi^2}{2}\Big(\frac{u_{xx}^2 - 2u_{xx}\varphi_x + u_{3x}\varphi}{\varphi^3} + \frac{\varphi_{xx}\varphi + 5\varphi_x^2}{\varphi^3}\Big) + O(\xi^3), \end{aligned} \tag{2.33}$$

which immediately yield the higher Riemann type commuting nonlinear Lax integrable dispersive dynamical systems on the functional manifold M^2.

Now, based on Proposition 2.3, one can formulate our main result, effectively describing the whole infinite hierarchy of the Gurevich-Zybin type commuting to each other Hamiltonian flows, in the following form.

Proposition 2.4. *The Gurevich-Zybin hydrodynamical system* (1.2) *possesses an infinite hierarchy*

$$d(u, v)^\intercal / dt_j = K_j[u, v],$$

$j \in \mathbb{N}$, *of commuting to each other Hamiltonian flows* $K_j : M^2 \to T(M^2)$, *representable in the following vertex operator form:*

$$\begin{aligned} \alpha^-(x, \tau; \lambda) &= u_x + [u_x^2 - 2v_x + \lambda^{-1}\alpha^-(x, \tau; \lambda)]^{1/2}, \\ \beta^+(x, \tau; \lambda) &= u_x - [u_x^2 - 2v_x + \lambda^{-1}\beta^+(x, \tau; \lambda)]^{1/2}, \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} &\alpha^-(x, \tau; \lambda) = \exp(-D_\lambda)\alpha(x, \tau), \quad \beta^+(x, \tau; \lambda) := \exp(D_\lambda)\beta(x, \tau), \\ &D_\lambda := \sum_{j \in \mathbb{N}} \lambda^{-j} \frac{d}{j dt_j}, \qquad \alpha(x, \tau) = u_x(x, \tau) + \varphi(x, \tau), \\ &\beta(x, \tau) = u_x(x, \tau) - \varphi(x, \tau), \qquad \varphi(x, \tau) := \sqrt{u_x^2(x, \tau) - 2v_x(x, \tau)}, \end{aligned} \tag{2.35}$$

$\tau := (t_1, t_2, t_3, \ldots) \in \mathbb{R}^{\mathbb{N}}$ *and* $|\lambda| \to \infty$.

As a simple corollary of relationship (2.34) one finds that

$$\lim_{\lambda\to\infty}[\alpha^{-}(x,\tau;\lambda)\pm\beta^{+}(x,\tau;\lambda)]/2=\begin{cases}u_x(x,\tau),\\ \varphi(x,\tau),\end{cases} \tag{2.36}$$

giving rise to the following infinite hierarchy of Hamiltonian vector fields on the functional manifold M^2:

$$\frac{d}{dt_1}\begin{pmatrix}u_x\\ \varphi\end{pmatrix}=\begin{pmatrix}-u_{xx}/\varphi\\ -\varphi_x/\varphi\end{pmatrix},\ \frac{d}{dt_2}\begin{pmatrix}u_x\\ \varphi\end{pmatrix}=\begin{pmatrix}(u_{xx}^2+7\varphi_x^2)/\varphi^3\\ (2u_{3x}\varphi-4u_x\varphi_x)/\varphi^3\end{pmatrix},\ldots, \tag{2.37}$$

and so on, where $\varphi=\sqrt{u_x^2-2v_x}$ and we took into account that owing to the following the asymptotic expansions hold

$$\begin{aligned}X_\lambda^{-}\alpha(x,\tau;\lambda)&=u_x+\varphi-\xi(u_{x,t_1}+\varphi_{t_1})+\\&\quad+\frac{\xi^2}{2}(u_{x,t_1,t_1}+\varphi_{t_1,t_1}-u_{x,t_2}-\varphi_{t_2})+O(\xi^3),\\ X_\lambda^{+}\beta(x,\tau;\lambda)&=u_x-\varphi+\xi(u_{x,t_1}-\varphi_{t_1})+\\&\quad+\frac{\xi^2}{2}(u_{x,t_1,t_1}-\varphi_{t_1,t_1}+u_{x,t_2}-\varphi_{t_2})+O(\xi^3)\end{aligned} \tag{2.38}$$

as $\xi=1/\lambda\to 0$.

It is worth also mentioning that the scheme devised above for finding the corresponding vertex operator representations for the Gurevich-Zybin (2.1) can be similarly generalized for treating other equations of the infinite hierarchy (1.1) when $N\geq 3$, having taking into account the existence of their suitable Lax type representations found before in recent works [4, 5, 25].

3. CONCLUDING REMARKS

The vertex operator functional representations of the solution to the Riemann type hydrodynamical equation (2.1) in the form (2.27) is crucially based on the representations (2.23) and evolution equations (2.28), which provide a very straightforward and transparent explanation of many of “miraculous” vertex operator calculations presented before both in [14, 15] and in [22]. It should be noted that the effectiveness of our approach to studying the vertex operator representation of the Riemann type hierarchy owes much to the important exact representation (2.21) for the corresponding monodromy matrix, whose properties are described by means of applying the standard [16, 17, 19, 21] Lie-algebraic techniques. As an indication of possible future research, it should also be mentioned that it would be interesting to generalize the vertex operator approach devised in this work to other linear spectral problems such as those related to dynamical systems with a parametrical spectral [12, 18, 19] dependence, spatially two-dimensional [11], Pavlov’s and heavenly [13] dynamical systems.

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