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# THE LQ-CONTROLLER SYNTHESIS PROBLEM FOR INFINITE–DIMENSIONAL SYSTEMS IN FACTOR FORM

Piotr Grabowski

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**Abstract.** The general lq-problem with infinite time horizon for well-posed infinite-dimensional systems has been investigated by George Weiss and Martin Weiss and by Olof Staffans with a complement by Kalle Mikkola and Olof Staffans.

Our aim in this paper is to present a solution of a general lq-optimal controller synthesis problem for infinite-dimensional systems in factor form. The systems in factor form are an alternative to additive models, used in the theory of well-posed systems, which rely on leading the analysis exclusively within the basic state space. As a result of applying the simplified analysis in terms of the factor systems and an another derivation technique, we obtain an equivalent, however, astonishingly not the same formulae expressing the optimal controller in the time-domain and the method of spectral factorization.

The results are illustrated by two examples of the construction of both the optimal control and optimal controller for some standard lq-problems met in literature: a control problem for a class of boundary controlled hyperbolic equations initiated by Chapelon and Xu, to which we give full solution and an example of the synthesis of the optimal control/controller for the standard lq-problem with infinite-time horizon met in the problem of improving a river water quality by artificial aeration, proposed by Żołopa and the author.

**Keywords:** control of infinite-dimensional systems, semigroups, infinite-time lq-control problem.

**Mathematics Subject Classification:** 49N10, 93B05, 93C25.

## 1. INTRODUCTION

Consider a control system governed by the model in factor form

$$\begin{cases} \dot{x}(t) = \mathcal{A}\left[x(t) + \mathcal{D}u(t)\right], \\ y(t) = \mathcal{C}x(t), \end{cases} \tag{1.1}$$

 

where the *state operator* $\mathcal{A}$ generates an exponentially stable (EXS) semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$, , i.e., there exit $M \geq 1$ and $\alpha > 0$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \leq Me^{-\alpha t}\|x_0\|_{\mathrm{H}}, \qquad \forall t \geq 0, \quad \forall x_0 \in \mathrm{H}. \tag{1.2}$$

Since $s \mapsto (sI - \mathcal{A})^{-1}x_0$ is the Laplace transform of $t \mapsto S(t)x_0$ then, by (1.2), the half-plane $\{s \in \mathbb{C} : \ \operatorname{Re} s > -\alpha\}$ is contained in the resolvent set of $\mathcal{A}$ which, in particular, implies that $\mathcal{A}$ is invertible with bounded and everywhere defined inverse, $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$. Next, $\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$, $\mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$, $\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{H})$ with $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$. Here Y and U are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, respectively.

The lq-optimal control problem with infinite time horizon is to minimize the quadratic integral performance index

$$J(x_0,u) = \int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \tag{1.3}$$

where $Q = Q^* \in \mathbf{L}(\mathrm{Y})$, $N \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $R = R^* \in \mathbf{L}(\mathrm{U})$, on trajectories of (1.1).

To solve this problem we shall assume that:

(**A1**) $\mathcal{C}$ is an admissible *observation operator*, i.e., $\mathcal{R}(\mathcal{Z}) \subset D(\mathcal{L}_{\mathrm{Y}})$, where

$$\mathcal{Z} \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y})), \ (\mathcal{Z}x_0)(t) := \mathcal{C}\mathcal{A}^{-1}S(t)x_0;$$
$$\mathcal{L}_{\mathrm{Y}}f = f', \ D(\mathcal{L}_{\mathrm{Y}}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{Y}).$$

Since $\mathcal{L}_{\mathrm{Y}}$ generates the *semigroup of left-shifts* on $\mathrm{L}^2(0,\infty;\mathrm{Y})$ then, by the closed-graph theorem, the admissibility of $\mathcal{C}$ holds iff

$$\Psi = \mathcal{L}_{\mathrm{Y}}\mathcal{Z} \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y})),$$

and $\Psi$ is called the system *observability map.*

(**A2**) $\mathcal{D}$ is an admissible *factor control operator*, i.e., $\mathcal{R}(\mathcal{W}) \subset D(\mathcal{A})$, where

$$\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}), \qquad \mathcal{W}f := \int_0^\infty S(t)\mathcal{D}f(t)\mathrm{d}t.$$

By the closed-graph theorem, the admissibility of $\mathcal{D}$ holds iff

$$\Phi = \mathcal{A}\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}),$$

and $\Phi$ is the system *reachability map.*

(**A3**) The system *transfer function* $\hat{G}(s) := s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D} = s^2\left(\mathcal{C}\mathcal{A}^{-1}\right)(sI - \mathcal{A})^{-1}\mathcal{D} - s\left(\mathcal{C}\mathcal{A}^{-1}\right)\mathcal{D} - \mathcal{C}\mathcal{D}$ (thus $\hat{G}$ is well-defined for $\operatorname{Re} s > -\alpha$) satisfies

$$\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))$$

(recall that $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathrm{Z})$, for some Banach space Z, if $\hat{G} : \mathbb{C}^+ \ni s \longmapsto \hat{G}(s) \in \mathrm{Z}$ is holomorphic and $\left\|\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Z})} = \sup_{s\in\mathbb{C}^+} \left\|\hat{G}(s)\right\|_{\mathrm{Z}} < \infty$; this definition applies as $\mathrm{Z} = \mathbf{L}(\mathrm{U}, \mathrm{Y})$ is a Banach space). If the latter is met then the *input-output operator*, given by

$$(\mathbb{F}u)(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (\Psi[\mathcal{D}u(\tau)])(t-\tau)\mathrm{d}\tau - (\mathcal{C}\mathcal{D})u(t),$$

satisfies $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y}))$. This follows from the Paley-Wiener theorem [1, Theorem 1.8.3, p. 48; this version of the Paley-Wiener theorem does not require *separability* of a Hilbert space. It is attached in Appendix A] upon taking the Laplace transforms: $(\widehat{\mathbb{F}u})(s) = \hat{G}(s)\hat{u}(s)$, $s \in \mathbb{C}^+$.

Let us remark that $\hat{G}$ is analytic on a set containing $\overline{\mathbb{C}^+}$, which jointly with (**A3**) yields

$$\left\|\hat{G}(j\omega)\right\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \le \left\|\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}, \qquad \forall\omega \in \mathbb{R}, \quad j^2 = -1.$$

**Remark 1.1.** If $\mathcal{C}$ is not admissible, the operator $\Psi = \mathcal{L}_\mathrm{Y}\mathcal{Z}$ with natural domain $D(\Psi) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L}_\mathrm{Y})\}$ is *closed* and *densely defined*, with $\Psi|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$ (for $x_0 \in D(\mathcal{A})$, $\Psi x_0$ is *homogeneous part* of the system output), and therefore it has *closed* and *densely defined* adjoint operator $\Psi^* = \mathcal{A}^*\mathcal{Z}^*$ with natural domain $D(\Psi^*) = \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathcal{Z}^*y \in D(\mathcal{A}^*)\}$, with $\Psi^*|_{D(\mathcal{R}_\mathrm{Y})} = \mathcal{Z}^*\mathcal{R}_\mathrm{Y}$, $\mathcal{R}_\mathrm{Y} = \mathcal{L}_\mathrm{Y}^*$.

Similarly, if $\mathcal{D}$ is not admissible, the operator $\Phi = \mathcal{A}\mathcal{W}$ with natural domain $D(\Phi) = \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathcal{W}u \in D(\mathcal{A})\}$ is *closed* and *densely defined*, with $\Phi|_{D(\mathcal{R}_\mathrm{U})} = \mathcal{W}\mathcal{R}_\mathrm{U}$, $\mathcal{R}_\mathrm{U} = \mathcal{L}_\mathrm{U}^*$, and therefore it has *closed* and *densely defined* adjoint operator $\Phi^* = \mathcal{L}_\mathrm{U}\mathcal{W}^*$ with natural domain $D(\Phi^*) = \{x \in \mathrm{H} : \mathcal{W}^*x \in D(\mathcal{L}_\mathrm{U})\}$, with $\Phi^*|_{D(\mathcal{A}^*)} = \mathcal{W}^*\mathcal{A}^*$.

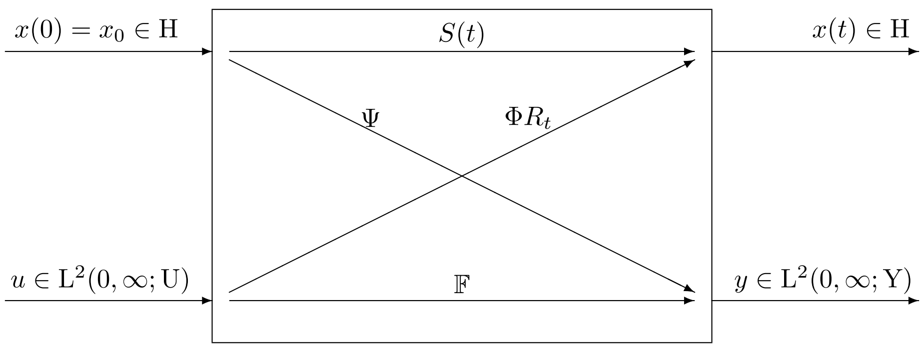


**Fig. 1.** Basic control-theoretic operators and their action

The structure of the remaining part of this paper is as follows.

In Section 2 the time-domain theory of the lq-problem is developed. We start from existence of weak and classical solutions and the well-posedness of the output. The main result of this section is Theorem 2.3, which says that if the operators $\mathcal{R}$ and $R_-$, defined therein, are coercive then a solution $\mathcal{H}$ to the Riccati operator equation (2.7) determines an implicit form of the optimal feedback controller solving the lq-problem, provided that it generates a $\mathrm{L}^2(0,\infty,\mathrm{U})$-function of time $t$ or if the operator-valued function $s \mapsto R_- + s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}$ is in $\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U}))$ jointly with its $\mathbf{L}(\mathrm{U})$-inverse, where $\mathcal{G}$ is dictated by $\mathcal{H}$ via (2.8). If, in addition, $\mathcal{G}$ has an extension $\mathcal{G}_\Lambda$ such that $R(\mathcal{D}) \subset \mathcal{G}_\Lambda$ and $(R_- + \mathcal{G}_\Lambda\mathcal{D})$ is a Banach isomorphism then Remark 2.4 enables us to represent the optimal controller in its explicit form (2.11). Furthermore, the optimal controller gives rise to a closed-loop state operator generating an EXS semigroup.

The frequency-domain theory of the lq-problem is presented in Section 3. In particular, we show that if the Popov spectral function $\Pi$ given by (3.1) is coercive then there exists a special spectral factorization (3.2) with spectral factor $\Xi \in \mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U}))$ jointly with its $\mathbf{L}(\mathrm{U})$-inverse. Moreover, since $\Xi^*(0)\Xi(s) = R_- + s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}$ this implies the existence of the implicitly given optimal feedback controller, while the regularity of the spectral factor implies that the explicit form of the optimal feedback controller is valid.

In Section 4 the realization formula (3.4) is being reversed as a procedure of finding the optimal controller formula via the knowledge of a spectral factor. This procedure requires, however, that the pair $(\mathcal{A},\mathcal{D})$ is approximately controllable, which is a restrictive assumption. The method of spectral factorization was not discussed in [19, 20, 24] and [13], however it was used in [6], a paper which was a motivation for developing the Riccati equation theory of [24] and [19,20]. Other aspects of comparison of our results with those existing in literature are listed in Section 5.

Sections 6, 7 and 8 bring some physically meaningful examples illustrating all results of the previous sections. In Section 6 a full solution to the lq-problem formulated in [3] is presented. The optimal controller is built using both: the method of spectral factorization and the time-domain approach employing the operator Riccati equation. In examples of Sections 7 and 8, the pair $(\mathcal{A},\mathcal{D})$ is not approximately controllable, so the method of spectral factorization is not applicable. However, the optimal cost operator can be found using direct methods, whence the easy-realizable optimal feedback controller is constructed via the Riccati operator equation. It is also examined that the method of spectral factorization partially characterizes the optimal controller.

Some conclusions and a short discussion of the results are presented in Section 9. In particular, we show therein how the lq-theory can be extended to the unstable case where the state operator does not generate an EXS $\mathrm{C}_0$-semigroup.

## 2. TIME-DOMAIN CONSIDERATIONS

We start from two lemmas characterizing weak and classical solutions of (1.1), respectively. Proofs of all results of this section are given in Appendix B.

**Lemma 2.1.** *By* (**A2**)*, for every* $x_0 \in \mathrm{H}$ *and* $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$

$$x(t) = S(t)x_0 + \underbrace{\Phi}_{=\mathcal{AW}} R_t u, \qquad (R_t u)(\tau) := \begin{cases} u(t-\tau) & \text{if} \quad \tau \le t, \\ 0 & \text{if} \quad \tau < t, \end{cases} \tag{2.1}$$

*is a weak solution of* (1.1)*, and* $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$ *is called the operator of reflection at* $t$*. If in addition, the semigroup* $\{S(t)\}_{t\ge 0}$ *is EXS, then the weak solution* (2.1) *is for every* $x_0 \in \mathrm{H}$ *and* $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ *in* $\mathrm{BUC}_0([0,\infty),\mathrm{H})$*, and* $t \longmapsto \langle z, x(t)\rangle_{\mathrm{H}}$ *is in* $\mathrm{L}^2(0,\infty)$ *for every* $z \in \mathrm{H}$*,* $x_0 \in \mathrm{H}$ *and* $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$*.*

**Lemma 2.2.** *If* (**A2**) *holds then for every* $u \in \mathrm{W}^{1,2}([0,\infty);\mathrm{U})$ *and* $x_0 \in \mathrm{H}$ *such that* $x_0 + \mathcal{D}u(0) \in D(\mathcal{A})$*,* (2.1) *is a* classical *solution of* (1.1)*.*

*The output equation*

$$y(t) = \mathcal{C}x(t) = \mathcal{C}[x(t) + \mathcal{D}u(t)] - \mathcal{CD}u(t) \tag{2.2}$$

*is well-posed and is a continuous function of* $t$*. If, in addition* (**A1**) *holds, then*

$$y(t) = (\Psi x_0)(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi[\mathcal{D}u(\tau)])(t-\tau)\mathrm{d}\tau - \mathcal{CD}u(t). \tag{2.3}$$

*Finally, if all assumptions* (**A1**)*,* (**A2**) *and* (**A3**) *are met then for every* $x_0 \in \mathrm{H}$ *and* $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$*:*

$$y = \Psi x_0 + \mathbb{F}u. \tag{2.4}$$

Now we are in position to present the main result of this section.

**Theorem 2.3.** *Let* $\mathcal{A}$ *generates an EXS semigroup on* H *and the assumptions* (**A1**)*,* (**A2**) *and* (**A3**) *hold. If the operator*

$$\mathcal{R} := R + N^*\mathbb{F} + \mathbb{F}^*Q\mathbb{F} + \mathbb{F}^*N = \mathcal{R}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$$

*is coercive then there exists a unique optimal control, given by*

$$u_{\mathrm{opt}} = \mathfrak{M}x_0, \quad \mathfrak{M} := -\mathcal{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{U})), \tag{2.5}$$

*on which the performance index* $J$ *achieves its minimum. The minimal value is*

$$J(x_0) = \langle x_0, \mathcal{H}_{\mathrm{opt}}x_0\rangle_{\mathrm{H}},$$

*where*

$$\mathcal{H}_{\mathrm{opt}} := \Psi^*Q\Psi - \Psi^*(Q\mathbb{F} + N)\mathcal{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi = \mathcal{H}^*_{\mathrm{opt}} \in \mathbf{L}(\mathrm{U}). \tag{2.6}$$

*Next, define*

$$N_- := N - Q(\mathcal{CD}), \qquad R_- := R - (\mathcal{CD})^*N - N^*(\mathcal{CD}) + (\mathcal{CD})^*Q(\mathcal{CD}) = R_-^*$$

*and assume, in addition, that $R_-$ is coercive. Assume that $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$ solves the Riccati operator equation*

$$\begin{aligned} &\langle \mathcal{A}z, \mathcal{H}z\rangle_{\mathrm{H}} + \langle z, \mathcal{H}\mathcal{A}z\rangle_{\mathrm{H}} + \langle Q\mathcal{C}z, \mathcal{C}z\rangle_{\mathrm{Y}} = \\ &= \left\langle -\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z, R_-^{-1}\left(-\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z\right)\right\rangle_{\mathrm{U}}, \quad z \in D(\mathcal{A}). \end{aligned} \tag{2.7}$$

*Define*

$$\mathcal{G}z := -\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z, \qquad z \in D(\mathcal{A}) \tag{2.8}$$

*and consider the feedback control law*

$$u(t) = -R_-^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}\mathcal{A}^{-1}x(t)\right], \tag{2.9}$$

*resulting in the closed-loop system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{A}^{-1}x\right] = x - \mathcal{D}R_-^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}\mathcal{A}^{-1}x(t)\right] \iff \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{A}^{-1}x + \mathcal{D}R_-^{-1}\mathcal{G}\mathcal{A}^{-1}x\right] = x. \tag{2.10}$$

(I) *If $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ then $u = u_{\mathrm{opt}}$, $\mathcal{H} = \mathcal{H}_{\mathrm{opt}}$ (in particular, this means that $\mathcal{H}_{\mathrm{opt}}$ solves* (2.7)*), $\mathcal{G} = \mathcal{G}_{\mathrm{opt}}$, $s \longmapsto R_- + s\mathcal{G}_{\mathrm{opt}}(sI - \mathcal{A})^{-1}\mathcal{D}$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ and the solution $x_{\mathrm{opt}}$ of* (2.10) *with initial condition $x_0$, corresponding to $u_{\mathrm{opt}}$ reads as $x_{\mathrm{opt}}(t) = S_{\mathrm{opt}}(t)x_0 = [S(t) + \Phi R_t\mathfrak{M}]x_0$, and $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$ is an* EXS *semigroup on* H.

(II) *If a solution $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$ to the Riccati operator equation* (2.7) *is such that for the corresponding $\mathcal{G}$, defined by* (2.8)*, the operator-valued function $s \longmapsto [R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}]$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its $\mathbf{L}(\mathrm{U})$-inverse $s \longmapsto [R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}]^{-1}$, then the implicitly defined feedback control* (2.9) *is in $\mathrm{L}^2(0,\infty;\mathrm{U})$ and therefore it is optimal, i.e., $u = u_{\mathrm{opt}}$, $\mathcal{H} = \mathcal{H}_{\mathrm{opt}}$ and $\mathcal{G} = \mathcal{G}_{\mathrm{opt}}$.*

**Remark 2.4.** If $\mathcal{G}_{\mathrm{opt}}$, originally defined on $D(\mathcal{A})$, extends to an operator $\mathcal{G}_\Lambda$ with domain $D(\mathcal{G}_\Lambda)$ such that: (i) $R(\mathcal{D}) \subset D(\mathcal{G}_\Lambda)$ and (ii) $(R_- + \mathcal{G}_\Lambda\mathcal{D})$, $(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1} \in \mathbf{L}(\mathrm{U})$ then the equation $z + \mathcal{D}R_-^{-1}\mathcal{G}_{\mathrm{opt}}z = x$, in definition of $D(\mathcal{A}_{\mathrm{opt}})$, can be explicitly solved:

$$\begin{aligned} z + \mathcal{D}R_-^{-1}\mathcal{G}_{\mathrm{opt}}z = x \Longrightarrow &\mathcal{G}_\Lambda z + \mathcal{G}_\Lambda\mathcal{D}R_-^{-1}\mathcal{G}_\Lambda z = (R_- + \mathcal{G}_\Lambda\mathcal{D})\,R_-^{-1}\mathcal{G}_\Lambda z = \mathcal{G}_\Lambda x \Longrightarrow \\ \Longrightarrow &R_-^{-1}\mathcal{G}_\Lambda z = (R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda x \Longrightarrow \\ \Longrightarrow &z = x - \mathcal{D}\,(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda x. \end{aligned}$$

Consequently, the *closed-loop state operator* can be rewritten as

$$\begin{aligned} \mathcal{A}_{\mathrm{opt}}x &= \mathcal{A}\left[x - \mathcal{D}\,(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda x\right], \\ D(\mathcal{A}_{\mathrm{opt}}) &= \left\{x \in D(\mathcal{G}_\Lambda) : \; x - \mathcal{D}\,(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda x \in D(\mathcal{A})\right\} \subset D(\mathcal{C}). \end{aligned}$$

This form of $\mathcal{A}_{\mathrm{opt}}x$ suggests that the optimal feedback reads as

$$u = -\,(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\,\mathcal{G}_\Lambda x, \qquad x \in D(\mathcal{G}_\Lambda), \tag{2.11}$$

what can easily be confirmed by the Laplace transformation.

A part of a proof of the Hille-Phillips-Yosida generation theorem is to show that the operator $\mathcal{A}_s \in \mathbf{L}(\mathrm{H})$, $\mathcal{A}_s f := s\mathcal{A}(sI-\mathcal{A})^{-1}f$ satisfies $\lim_{s\to\infty, s\in\mathbb{R}} \mathcal{A}_s f = \mathcal{A}f$ for every $f \in D(\mathcal{A})$ [15, Lemma 3.3, p. 10]. Therefore $\mathcal{A}_s$ has been called the *Yosida approximation* of $\mathcal{A}$. Since $\mathcal{G}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{U})$ the limit $\lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}z$ exists for $z \in D(\mathcal{A})$ and it is well-known that it may exist on some domain larger than $D(\mathcal{A})$. Thus the *Yosida approximation* of $\mathcal{G}_{\mathrm{opt}}$,

$$\mathcal{G}_\Lambda z := \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}_{\mathrm{opt}}(sI-\mathcal{A})^{-1}z,$$
$$D(\mathcal{G}_\Lambda) = \left\{ z \in \mathrm{H} :\ \text{there exists} \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}_{\mathrm{opt}}(sI-\mathcal{A})^{-1}z \right\},$$

or even its *restriction* to $R(\mathcal{D})$, may serve as the needed extension of $\mathcal{G}_{\mathrm{opt}}$, provided that the limit

$$(R_- + \mathcal{G}_\Lambda\mathcal{D})u = \lim_{s\to\infty, s\in\mathbb{R}} (R_- u + s\mathcal{G}_{\mathrm{opt}}(sI-\mathcal{A})^{-1}\mathcal{D}u), \quad u \in \mathrm{U}$$

defines a Banach isomorphism on U, and if the latter holds then it follows from the proof of Theorem 2.3 (see Appendix B) that the optimal cost operator $\mathcal{H}$ satisfies also the *closed-loop Lyapunov/Riccati operator equation*

$$\langle \mathcal{A}_{\mathrm{opt}}x, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}\mathcal{A}_{\mathrm{opt}}x\rangle_{\mathrm{H}} = \\ = -\begin{bmatrix} \mathcal{C}x \\ -(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} \mathcal{C}x \\ -(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x \end{bmatrix}, \quad x \in D(\mathcal{A}_{\mathrm{opt}}). \tag{2.12}$$

## 3. THE FREQUENCY-DOMAIN APPROACH

By the Paley-Wiener theorem [1, Theorem 1.8.3, p. 48],

$$J(u,x_0) = J(\hat{u},x_0) = \langle \hat{u}, \Pi\hat{u}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{U})} + \langle \hat{u}, \left[\hat{G}^*Q + N^*\right]\widehat{\Psi x_0}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{U})} + \\ + \langle \widehat{\Psi x_0}, \left[Q\hat{G} + N\right]\hat{u}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{Y})} + \\ + \langle \widehat{\Psi x_0}, Q\widehat{\Psi x_0}\rangle_{\mathrm{L}^2(j\mathbb{R},\mathrm{Y})}, \qquad \hat{u} \in \mathrm{L}^2(j\mathbb{R},\mathrm{U}),\ x_0 \in \mathrm{H},$$

where $\Pi$ stands for the *Popov spectral function*,

$$\Pi(j\omega) := R + 2\,\mathrm{Re}[N^*\hat{G}(j\omega)] + \hat{G}^*(j\omega)Q\hat{G}(j\omega) = \Pi^*(j\omega), \tag{3.1}$$

which, thanks to the continuity and boundedness of $\hat{G}$ on $j\mathbb{R}$, is $\mathbf{L}(\mathrm{U})$-valued bounded and continuous on $j\mathbb{R}$. Here we use the notation $2\,\mathrm{Re}\,Z := Z + Z^*$, $Z \in \mathbf{L}(\mathrm{U})$.

**Theorem 3.1.** *Assume that the assumptions* (**A1**), (**A2**) *and* (**A3**) *hold, and* $\mathcal{A}$ *generates an EXS semigroup. Let* $\Pi$ *be coercive. Then the following facts hold.*

(I) $\mathcal{R}$ *is coercive and, by Theorem 2.3, the lq-problem has a unique* $\mathrm{L}^2(0,\infty;\mathrm{U})$*-minimizer, whence, by the Paley-Wiener theorem, a unique* $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$*-minimizer.*
*There exists a spectral factorization*

$$\Pi(j\omega) = \Xi^*(j\omega)\Xi(j\omega), \tag{3.2}$$

*where* $\Xi \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ *jointly with* $\mathbb{C}^+ \ni s \longmapsto \Xi^{-1}(s) \in \mathbf{L}(\mathrm{U})$*. This spectral factorization is uniquely determined up to a constant, i.e., independent of* $s$*, unitary operator multiplier which belongs to* $\mathbf{L}(\mathrm{U})$*.*
*Let* $P_+$ *stand for the projection from* $\mathrm{L}^2(j\mathbb{R};\mathrm{U})$ *onto its closed subspace* $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$*. Then the* $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$*-minimizer is given by*

$$\hat{u}(s) = -\Xi^{-1}(s)P_+\left\{\Xi^{-*}(j\omega)\left[\hat{G}^*(j\omega)Q + N^*\right]\widehat{(\Psi x_0)}(j\omega)\right\}. \tag{3.3}$$

(II) $R_- = \Pi(0) = \Xi^*(0)\Xi(0)$ *is coercive, so we can discuss the operator Riccati equation* (2.7)*. To each its solution* $\mathcal{H}$*, or to each* $\mathcal{G}$ *given by* (2.8)*, there corresponds a spectral factorization* (3.2)*, where*

$$\Xi(s) := V + V^{-*}\mathcal{G}s(sI-\mathcal{A})^{-1}\mathcal{D} \in \mathbf{L}(\mathrm{U}) \tag{3.4}$$

*and* $s \longmapsto \Xi(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$*. Furthermore,* $V^{-*}\mathcal{G}$ *is admissible.*
*If* $\mathbf{L}(\mathrm{U})$*-inverse of* $\Xi$ *is in* $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ *then the implicit formula* (2.9) *defines optimal feedback controller.*
*Finally,*

$$\text{there exists} \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}u := \mathcal{G}_\Lambda\mathcal{D}u \iff$$
$$\iff \text{there exists} \lim_{s\to\infty, s\in\mathbb{R}} \Xi(s)u := Du,$$

*and then* $V^{-*}(R_- + \mathcal{G}_\Lambda\mathcal{D}) = D$*. Thus* $R_- + \mathcal{G}_\Lambda\mathcal{D}$ *is invertible iff so is* $D$*, a fact important for verification whether the explicit formula for the optimal feedback controller* (2.11) *holds true.*

Theorem 3.1 is proved in Appendix C.

## 4. THE METHOD OF SPECTRAL FACTORIZATION

Let us treat (3.4) not as a definition of a spectral factor but an equation determining $\mathcal{G}$. Such the equation is said to be the *realization identity* or *equation.* Then, by (2.8) and (3.4) a unique spectral factor corresponds to the optimal cost, thus this spectral factor is necessarily in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its inverse and is determined up to a unitary operator which is hidden in $V$. Thus if the LHS of (3.4) is a spectral factor in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its inverse then the realization identity must be satisfied, out of uniqueness, by $\mathcal{G}$, corresponding to the optimal control/controller.

It should be emphasized that the realization equation is generally not uniquely solvable. Nevertheless, if the system is *approximately controllable*, i.e., if $\overline{\mathcal{R}(\Phi)} = \mathrm{H}$ (iff $\ker \Phi^* = \{0\}$), then the realization identity cannot have more then one solution, so it determines uniquely the optimal controller (in its implicit form), provided that the LHS of the realization identity is a spectral factor belonging to $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its inverse.

Thus if, in addition, the system is *approximately controllable*, then $\mathcal{G}_\Lambda$ or $D^{-1}V^{-*}\mathcal{G}_\Lambda$ are *uniquely* determined by the following equivalent *realization equations*

$$\begin{aligned} \Xi^*(0)\Xi(s) = R_- + \mathcal{G}_\Lambda \mathcal{D} + \hat{G}_{\mathcal{G}}(s) \iff \Xi^*(0)\left[\Xi(s) - D\right] = \hat{G}_{\mathcal{G}}(s) \iff \\ \iff \Xi(s) = D\left[I + D^{-1}V^{-*}\mathcal{G}_\Lambda \mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}\right], \end{aligned} \tag{4.1}$$

where $\hat{G}_{\mathcal{G}}(s) := \mathcal{G}_\Lambda \mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}$, and the second line arises by acting with the operator $D^{-1}V^{-*}$ on both sides of the last identity in the first line.

**Remark 4.1.** If $\tau$ is the *operator of boundary control* (see [11] for a definition) then, since $D(\mathcal{A}) \subset \ker \tau$, $\tau\mathcal{D} = -I$, one has

$$\tau\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D} = s\tau(sI - \mathcal{A})^{-1}\mathcal{D} - \tau\mathcal{D} = I$$

and (4.1) can also be written as

$$\Xi(s) = (D\tau + V^{-*}\mathcal{G}_\Lambda)\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}.$$

## 5. COMPARISON WITH EARLIER WORKS

Consider the *tower* (or *scale*) of Hilbert spaces

$$\mathrm{H}_1 \hookrightarrow \mathrm{H}(= \mathrm{H}^*) \hookrightarrow \mathrm{H}_{-1},$$

with continuous dense embeddings, where $\mathrm{H}_1 = (D(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$, $\|x\|_{\mathcal{A}} := \|\mathcal{A}x\|_{\mathrm{H}}$ whilst $\mathrm{H}_{-1}$ stands for the completion of H under the norm $\|x\|_{\mathrm{H}_{-1}} := \|\mathcal{A}^{-1}x\|_{\mathrm{H}}$; the latter arises by taking the limits of all sequences of H, which are Cauchy sequences with respect to $\|x\|_{\mathrm{H}_{-1}}$.

Parallely, consider also the *tower* of Hilbert spaces

$$\mathrm{Z}_{-1} \hookleftarrow \mathrm{H}(= \mathrm{H}^*) \hookleftarrow \mathrm{Z}_1,$$

with continuous dense embeddings, where $\mathrm{Z}_1 = (D(\mathcal{A}^*), \|\cdot\|_{\mathcal{A}^*})$, $\|x\|_{\mathcal{A}^*} := \|\mathcal{A}^*x\|_{\mathrm{H}}$ whilst $\mathrm{Z}_{-1}$ stands for the completion of H under the norm $\|x\|_{\mathrm{Z}_{-1}} := \|\mathcal{A}^{-*}x\|_{\mathrm{H}}$; the latter arises by taking the limits of all sequences of H, which are Cauchy sequences with respect to $\|x\|_{\mathrm{Z}_{-1}}$.

The bilinear form

$$\langle x, z\rangle_{\mathrm{H}_{-1}\times \mathrm{Z}_1} := \left\langle \mathcal{A}_{\mathrm{e}}x, \mathcal{A}^{-*}z\right\rangle_{\mathrm{H}\times\mathrm{H}},$$

where $\mathcal{A}_{\mathrm{e}} \in \mathbf{L}(\mathrm{H}, \mathrm{H}_{-1})$ denotes the extension of $\mathcal{A} \in \mathbf{L}(\mathrm{H}_1, \mathrm{H})$, an isometry from $\mathrm{H}_1$, onto H, defines duality pairing between $\mathrm{H}_{-1}$ and $\mathrm{Z}_1$. Here $\mathrm{H}_{-1}$ is isomorphic with $[D(\mathcal{A}^*)]^*$ whilst $\mathrm{Z}_{-1}$ is isomorphic with $[D(\mathcal{A})]^*$.

It is proved in [24] that if $\Pi$ has the spectral factorization $\Pi(j\omega) = [\Xi(j\omega)]^*\Xi(j\omega)$, where $\Xi$, $\Xi^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ and $\Xi(s) \longrightarrow D$ as $s \to \infty$, $s \in \mathbb{R}$ with $D$ and $D^{-1} \in \mathbf{L}(\mathrm{U})$ (*regular spectral function*), then the optimal cost operator $X$ solves the operator Riccati [24, Theorem 12.8, p. 322, especially formula (12.7)] and [19, Corollary 45, p. 3712]; see also [13, Theorem 3, especially formula (6)]

$$\mathcal{A}^*X + X\mathcal{A} + C^*QC = (B^*_{\Lambda_w}X + NC)^*(D^*D)^{-1}(B^*_{\Lambda_w}X + NC), \tag{5.1}$$

where all terms are in $\mathbf{L}(\mathrm{H}_1, \mathrm{Z}_{-1})$ and, actually, $X$ maps $D(\mathcal{A})$ into $D(B^*_{\Lambda_w})$. Here $B \in \mathbf{L}(\mathrm{U}, \mathrm{H}_{-1})$ iff $B^* \in \mathbf{L}(\mathrm{Z}_1, \mathrm{U})$, $C \in \mathbf{L}(\mathrm{H}_1, \mathrm{Y})$ iff $C^* \in \mathbf{L}(\mathrm{Y}, \mathrm{Z}_{-1})$, $B^*_{\Lambda_w}$ $(B^*_\Lambda)$ denotes weak (strong) extension of $B^*$, defined as the weak (strong) limit of $sB^*(sI - \mathcal{A})^{-1}x$ as $s \to \infty$, $s \in \mathbb{R}$ and $D(B^*_{\Lambda_w})$ consists of those $x \in \mathrm{H}$ for which the weak limit exists ($D(B_{\Lambda^*})$ consists of those $x \in \mathrm{H}$ for which the strong limit exists). The optimal controller is given on $D(\mathcal{A})$ as

$$Fx = -(D^*D)^{-1}(B^*_{\Lambda_w}X + NC)x, \qquad x \in D(\mathcal{A}).$$

The spectral factor $\Xi$ can be realized as a transfer function of the system with the state operator $\mathcal{A}$, control operator $B$, observation operator $-DF_\Lambda$ and the feedtrough operator $D$ [24, p. 329, formula (12.5)], i.e.,

$$\Xi(s) = D - DF_\Lambda(sI - \mathcal{A})^{-1}B = D\left[I - F_\Lambda(sI - \mathcal{A})^{-1}B\right]. \tag{5.2}$$

Finally, the state operator of the optimal closed-loop system reads as

$$\mathcal{A}_{\mathrm{opt}} = \mathcal{A} + BF_\Lambda, \qquad D(\mathcal{A}_{\mathrm{opt}}) = \{x_0 \in D(F_\Lambda) : \ (\mathcal{A} + BF_\Lambda)x_0 \in \mathrm{H}\},$$

so the optimal controller is $u = F_{\mathrm{opt}}x_0$, where $F_{\mathrm{opt}}x_0 = F_\Lambda x_0$ for $x_0 \in D(\mathcal{A}_{\mathrm{opt}})$.

In the case of (1.1), the results of [24] follows from Theorems 3.1 and 2.3, and our Riccati operator equation (2.7) slightly differs from (5.1) as:

(a) it does not employ the feedthrough operator $D$,
(b) it is stated in a weak sense within the state space H,
(c) even if we identify $X$ with $\mathcal{H}$ (both operators express the minimal cost), $C$ with $\mathcal{C}$ and notify that $B^*_{\Lambda_w}$ is an extension of $\mathcal{D}^*\mathcal{A}^*$ then the ordering of operators defining $\mathcal{G}$ and $F$ is not the same and in (2.7) the operator $N_-$ appears instead of $N$ in (5.1). Thus our Riccati equation (2.7) is astonishingly not the same as (5.1).

Next, EXS of $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$ is not shown in [24], though we still do not know whether it decays with the same rate or faster than $\{S(t)\}_{t\geq 0}$. Here our Theorem 2.3 jointly with Remark 2.4 confirm the implication (ii) $\Rightarrow$ (i) of [13, Theorem 3].

On the other side our results and those of [24] are very close in the frequency-domain aspects as:

(d) the idea of Remark 2.4 coincides with the concept of a *regular spectral factor*,
(e) comparing the second line of (4.1) with (5.2) we get a relationship between $\mathcal{G}_\Lambda$ and $F_\Lambda$,

$$F_\Lambda = -D^{-1}V^{-*}\mathcal{G}_\Lambda. \tag{5.3}$$

In next sections we shall solve certain exemplary *standard lq-problems* for which $Q = R = I$ and $N = 0$. In this case $N_- = -\mathcal{CD}$ whilst $\mathcal{R}$, $R_-$ and the Popov spectral function $\Pi$ are coercive:

$$\mathcal{R} = I+\mathbb{F}^*\mathbb{F} \geq I,\ R_- := I+(\mathcal{CD})^*(\mathcal{CD}) \geq I;\ \ \Pi(j\omega) = I+\hat{G}^*(j\omega)\hat{G}(j\omega) \geq I,\ \ \forall\omega\in\mathbb{R}.$$

## 6. FULL SOLUTION OF THE EXAMPLE BY CHAPELON AND XU

In this section we revise the example of [3] where the standard lq-problem has been formulated for a system with the state operator $\mathcal{A}$ acting in $\mathrm{H} = \mathrm{L}^2(0,1) \oplus \mathrm{L}^2(0,1)$,

$$\mathcal{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -m_1 x_1' \\ m_2 x_2' \end{bmatrix}, \quad m_1 > 0,\ m_2 > 0,$$
$$D(\mathcal{A}) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) :\ x_1(0) = \alpha x_2(0), x_2(1) = \beta x_1(1) \right\},$$

which generates an EXS $\mathrm{C}_0$-semigroup, provided that $\alpha^2\beta^2 < 1$. This fact is not explicitly proved in [3], where the authors recall an older result due to D. Russell [3, Proposition 3.1, p. 592], however we are able to give a separate Lyapunov-type proof. For that, define the following matrix operators of multiplication $\mathcal{E}_1 = \mathcal{E}_1^*$, $\mathcal{E}_2 = \mathcal{E}_2^*$ and $\mathcal{E}_3 = \mathcal{E}_3^* \in \mathbf{L}(\mathrm{H})$, $\mathcal{E}_3^* \geq 0$:

$$(\mathcal{E}_1 x)(\theta) := \frac{1}{1-\alpha^2\beta^2}\,\mathrm{diag}\left\{\frac{1}{m_1}, 0\right\} x(\theta), \quad (\mathcal{E}_2 x)(\theta) := \frac{1}{1-\alpha^2\beta^2}\,\mathrm{diag}\left\{0, \frac{1}{m_2}\right\} x(\theta)$$

and

$$(\mathcal{E}_3 x)(\theta) = \mathrm{diag}\left\{\frac{1-\theta}{m_1}, \frac{\theta}{m_2}\right\} x(\theta), \qquad x \in \mathrm{H}.$$

Notice that its linear combination $k_1\mathcal{E}_1 + k_2\mathcal{E}_2 + \mathcal{E}_3$ satisfies

$$\begin{aligned} \langle \mathcal{A}x, (k_1\mathcal{E}_1 + k_2\mathcal{E}_2 + \mathcal{E}_3)x\rangle_{\mathrm{H}} + \langle x, (k_1\mathcal{E}_1 + k_2\mathcal{E}_2 + \mathcal{E}_3)\mathcal{A}x\rangle_{\mathrm{H}} = \\ = -\|x\|_{\mathrm{H}}^2 + \left\{\beta^2 - \frac{k_1}{1-\alpha^2\beta^2} + \frac{k_2\beta^2}{1-\alpha^2\beta^2}\right\} x_1^2(1) + \\ + \left\{\alpha^2 + \frac{k_1\alpha^2}{1-\alpha^2\beta^2} - \frac{k_2}{1-\alpha^2\beta^2}\right\} x_2^2(0), \qquad x \in D(\mathcal{A}). \end{aligned} \tag{6.1}$$

Solving an appropriate linear system of equations determining $k_1$, $k_2$ we establish that $\mathcal{E} := (\alpha^2+1)\beta^2\mathcal{E}_1 + (\beta^2+1)\alpha^2\mathcal{E}_2 + \mathcal{E}_3$ satisfies the Lyapunov operator equation

$$\langle \mathcal{A}x, \mathcal{E}x\rangle_{\mathrm{H}} + \langle x, \mathcal{E}\mathcal{A}x\rangle_{\mathrm{H}} = -\|x\|_{\mathrm{H}}^2, \qquad x \in D(\mathcal{A}).$$

Now EXS for $\alpha^2\beta^2 < 1 \Leftrightarrow \mathcal{E} \geq 0$ easily follows from either Datko's theorem. In this example $\mathrm{Y} = \mathrm{H}$ and $\mathcal{C} = I$, whence admissibility of $\mathcal{C}$ is equivalent to EXS, and $\mathcal{E}$ is the system *observability gramian.* We proved that the semigroup generated by $\mathcal{A}$ is EXS semigroup and (**A1**) is met.

The authors of [3] have used the framework of well-posed systems rather than our model (1.1), so it is worth to note that, here, the *operator of boundary control* (see [11] for more details) reads as

$$\tau \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1(0) - \alpha x_2(0) \\ x_2(1) - \beta x_1(1) \end{bmatrix}, \quad D(\tau) \subset \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1). \tag{6.2}$$

A control takes its values in $\mathrm{U} = \mathbb{R}^2$, and the factor control operator $\mathcal{D}$ is given by

$$\mathcal{D}u = \mathbf{D}u, \qquad \mathbf{D} = \frac{1}{\alpha\beta - 1} \begin{bmatrix} \mathbf{1} & \alpha\mathbf{1} \\ \beta\mathbf{1} & \mathbf{1} \end{bmatrix}.$$

Standard computations yield

$$\mathcal{A}^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} m_1 v_1' \\ -m_2 v_2' \end{bmatrix},$$
$$D(\mathcal{A}^*) = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) : \begin{Bmatrix} m_1 v_1(1) = \beta m_2 v_2(1) \\ \alpha m_1 v_1(0) = \ m_2 v_2(0) \end{Bmatrix} \right\}.$$

Thus we have

$$\mathcal{D}^* \mathcal{A}^* v = \mathbf{D}^T \int_0^1 (\mathcal{A}^* v)(\theta) \mathrm{d}\theta = \begin{bmatrix} m_1 v_1(0) \\ m_2 v_2(1) \end{bmatrix}$$

and this observation operator is admissible. Indeed, the operator

$$(\mathcal{H}_\Phi v)(\theta) = \frac{1}{1 - \alpha^2 \beta^2} \operatorname{diag}\left\{(\alpha^2 + 1)m_1, (\beta^2 + 1)m_2\right\} v(\theta), \qquad v \in \mathrm{H}$$

is the system *controllability gramian*, because it solves the Lyapunov operator equation

$$\langle \mathcal{A}^* v, \mathcal{H}_\Phi v \rangle_{\mathrm{H}} + \langle v, \mathcal{H}_\Phi \mathcal{A}^* v \rangle_{\mathrm{H}} = -m_1^2 v_1^2(0) - m_2 v_2^2(1) = -\left\| \mathcal{D}^* \mathcal{A}^* v \right\|_{\mathrm{U}}^2, \quad v \in D(\mathcal{A}^*).$$

By duality, the factor control operator $\mathcal{D}$ is admissible, whence (**A2**) is met. Moreover, the system is infinite-time exactly controllable as $\mathcal{H}_\Phi$ is a coercive operator.

Next,

$$(\mathcal{A}(sI - \mathcal{A})^{-1} z)(\theta) = \begin{bmatrix} s e^{-\frac{s\theta}{m_1}} \mathfrak{c} - z_1(\theta) + \frac{s}{m_1} \int_0^\theta e^{-\frac{s(\theta - \tau)}{m_1}} z_1(\tau) \mathrm{d}\tau \\ \frac{1}{\alpha} s e^{\frac{s\theta}{m_2}} \mathfrak{c} - z_2(\theta) - \frac{s}{m_2} \int_0^\theta e^{\frac{s(\theta - \tau)}{m_2}} z_2(\tau) \mathrm{d}\tau \end{bmatrix},$$

where

$$\mathfrak{c} = \frac{1}{\frac{1}{\alpha} e^{\frac{s}{m_2}} - \beta e^{-\frac{s}{m_1}}} \left[ \frac{\beta}{m_1} \int_0^1 e^{-\frac{s(1-\tau)}{m_1}} z_1(\tau) \mathrm{d}\tau + \frac{1}{m_2} \int_0^1 e^{\frac{s(1-\tau)}{m_2}} z_2(\tau) \mathrm{d}\tau \right].$$

On constant functions $z_i(\theta) \equiv z_i$, $i = 1, 2$:

$$\mathfrak{c} = \frac{z_1\beta(1 - e^{-\frac{s}{m_1}}) + z_2(e^{\frac{s}{m_2}} - 1)}{\frac{1}{\alpha}e^{\frac{s}{m_2}} - \beta e^{-\frac{s}{m_1}}},$$

$$\mathcal{A}(sI - \mathcal{A})^{-1}z = \begin{bmatrix} \dfrac{z_1(\alpha\beta e^{-\frac{s}{m_2}} - 1) + z_2\alpha(1 - e^{-\frac{s}{m_2}})}{1 - \alpha\beta e^{-\left(\frac{s}{m_1} + \frac{s}{m_2}\right)}}\, e^{-\frac{s\theta}{m_1}} \\ \dfrac{z_1\beta e^{-\frac{s}{m_2}}(1 - e^{-\frac{s}{m_1}}) + z_2 e^{-\frac{s}{m_2}}(\alpha\beta e^{-\frac{s}{m_1}} - 1)}{1 - \alpha\beta e^{-\left(\frac{s}{m_1} + \frac{s}{m_2}\right)}}\, e^{\frac{s\theta}{m_2}} \end{bmatrix},$$

whence, taking $z = \mathbf{D}u$, we get

$$\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D} = \hat{G}(s) = \frac{1}{1 - \alpha\beta e^{-s\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}} \begin{bmatrix} e^{-\frac{s\theta}{m_1}} & \alpha e^{-\frac{s}{m_2}} e^{-\frac{s\theta}{m_1}} \\ \beta e^{-\frac{s}{m_1}} e^{-\frac{s(1-\theta)}{m_2}} & e^{-\frac{s(1-\theta)}{m_2}} \end{bmatrix} \tag{6.3}$$

with $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$, so (**A3**) is satisfied.

The transfer function can be represented in the *right coprime* form $\hat{G}(s) = \mathbf{U}(s)\mathbf{M}^{-1}(s)$ with

$$\mathbf{U}(s) = \begin{bmatrix} e^{-\frac{s\theta}{m_1}} & 0 \\ 0 & e^{-\frac{s(1-\theta)}{m_2}} \end{bmatrix}, \quad \mathbf{M}(s) = \begin{bmatrix} 1 & -\alpha e^{-\frac{s}{m_2}} \\ -\beta e^{-\frac{s}{m_1}} & 1 \end{bmatrix}.$$

Denoting by $\mathbf{Z}_*(s) := \mathbf{Z}^T(-s)$ the *para-Hermitian adjoint* of $\mathbf{Z}(s)$, we see that $\mathbf{U}_*(s) = \mathbf{U}^T(-s) = \mathbf{U}(-s) = \mathbf{U}^{-1}(s)$, so $\mathbf{U}(s)$ is *para-unitary*. Now

$$\mathbf{\Pi}(s) = I + \hat{G}_*(s)\hat{G}(s) = I + \mathbf{M}^{-T}(-s)\mathbf{U}^T(-s)\mathbf{U}(s)\mathbf{M}^{-1}(s) = I + \mathbf{M}^{-T}(-s)\mathbf{M}^{-1}(s)$$

which facilitates finding a spectral factor of $\Pi(j\omega) \geq I$ by reducing the problem to finding a spectral factor of an entire matrix-valued function

$$\mathbf{M}^T(-s)\mathbf{\Pi}(s)\mathbf{M}(s) = I + \mathbf{M}_*(s)\mathbf{M}(s) = \begin{bmatrix} 2 + \beta^2 & -\alpha e^{-\frac{s}{m_2}} - \beta e^{\frac{s}{m_1}} \\ -\alpha e^{\frac{s}{m_2}} - \beta e^{-\frac{s}{m_1}} & 2 + \alpha^2 \end{bmatrix}.$$

We shall seek for a factorization $I + \mathbf{M}_*(s)\mathbf{M}(s) = \mathbf{X}_*(s)\mathbf{X}(s)$ with

$$\mathbf{X}(s) = \begin{bmatrix} \mathfrak{m} & -\mathfrak{n}e^{-\frac{s}{m_2}} \\ -\mathfrak{p}e^{-\frac{s}{m_1}} & \mathfrak{q} \end{bmatrix}.$$

This leads to the system of equations:

$$\mathfrak{m}^2 + \mathfrak{p}^2 = 2 + \beta^2, \tag{6.4}$$

$$\mathfrak{n}\mathfrak{m} = \alpha, \tag{6.5}$$

$$\mathfrak{p}\mathfrak{q} = \beta, \tag{6.6}$$

$$\mathfrak{n}^2 + \mathfrak{q}^2 = 2 + \alpha^2. \tag{6.7}$$

Eliminating $\mathfrak{n}$, $\mathfrak{p}$ from (6.7) and (6.4) with the aid of (6.5) and (6.6), respectively, we get

$$\mathfrak{n} = \frac{\alpha}{\mathfrak{m}}, \quad \mathfrak{p} = \frac{\beta}{\mathfrak{q}}, \quad \mathfrak{q}^2 = 2 + \alpha^2 - \frac{\alpha^2}{\mathfrak{m}^2} \Longrightarrow \mathfrak{p}^2 = \frac{\beta^2\mathfrak{m}^2}{(2+\alpha^2)\mathfrak{m}^2 - \alpha^2},$$

and a biquadratic equation determining $\mathfrak{m}$:

$$(2+\alpha^2)\mathfrak{m}^4 - \left[\alpha^2 - \beta^2 + (2+\alpha^2)(2+\beta^2)\right]\mathfrak{m}^2 + \alpha^2(2+\beta^2) = 0. \tag{6.8}$$

Observe that the LHS of (6.8) at $\mathfrak{m}^2 = 0$ equals: $\alpha^2(2+\beta^2) \geq 0$. Let

$$\mu := \alpha^2 - \beta^2 + (2+\alpha^2)(2+\beta^2) = (2+\beta^2) + (2+\alpha^2) + \alpha^2(2+\beta^2) \geq 4 + 2\alpha^2 > 0$$

and observe that the determinant of (6.8) satisfies

$$\begin{aligned} \mu^2 \geq \Delta = \mu^2 - 4\alpha^2(2+\alpha^2)(2+\beta^2) > \\ > \left[(2+\alpha^2) + \alpha^2(2+\beta^2)\right]^2 - 4\alpha^2(2+\alpha^2)(2+\beta^2) = \\ = \left[(2+\alpha^2) - \alpha^2(2+\beta^2)\right]^2 \geq 0. \end{aligned}$$

Hence (6.8) has four real roots $\mathfrak{m}_B > \mathfrak{m}_A \geq 0 \geq -\mathfrak{m}_A > -\mathfrak{m}_B$ with equality signs iff $\alpha = 0$. Furthermore, the LHS of (6.8) at $\mathfrak{m}^2 = 2$ equals: $-\beta^2(2+\alpha^2) \leq 0$, so $\mathfrak{m}_B \geq \sqrt{2}$ ($= \iff \beta = 0$) and $\mathfrak{q}_B \geq \sqrt{2}$ ($= \iff \alpha = 0$). Take the solution

$$\begin{aligned} \mathfrak{m} = \mathfrak{m}_B := \sqrt{\frac{\mu + \sqrt{\Delta}}{2(2+\alpha^2)}}, \quad \mathfrak{n} = \frac{\alpha}{\mathfrak{m}_B}, \\ \mathfrak{p} = \frac{\beta\mathfrak{m}_B}{\sqrt{(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2}}, \quad \mathfrak{q} = \frac{\sqrt{(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2}}{\mathfrak{m}_B}. \end{aligned}$$

Since

$$\mathbf{X}^{-1}(s) = \frac{1}{\mathfrak{m}\mathfrak{q}\left[1 - \frac{\mathfrak{n}\mathfrak{p}}{\mathfrak{m}\mathfrak{q}}e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)}\right]} \begin{bmatrix} \mathfrak{q} & \mathfrak{n}e^{-\frac{s}{m_2}} \\ \mathfrak{p}e^{-\frac{s}{m_1}} & \mathfrak{m} \end{bmatrix},$$

then $s \longmapsto \mathbf{X}(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathbb{C}^2))$ jointly with $s \longmapsto \mathbf{X}^{-1}(s)$ iff

$$1 > \frac{\mathfrak{n}^2\mathfrak{p}^2}{\mathfrak{m}^2\mathfrak{q}^2} = \frac{\alpha^2\beta^2}{\left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2} \iff \left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2 > \alpha^2\beta^2, \tag{6.9}$$

but the last inequality holds as $\alpha^2\beta^2 < 1$ and

$$(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2 = \frac{\mu + \sqrt{\Delta} - 2\alpha^2}{2} \geq \frac{\mu - 2\alpha^2}{2} > 2 \Longrightarrow \left[(2+\alpha^2)\mathfrak{m}_B^2 - \alpha^2\right]^2 \geq 4.$$

Consequently the spectral factor $\Xi(s)$ of $\Pi$ reads as

$$\begin{aligned} \Xi(s) &= \mathbf{X}(s)\mathbf{M}^{-1}(s) = \\ &= \frac{1}{1 - \alpha\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)}} \begin{bmatrix} \mathfrak{m} & -\mathfrak{n}e^{-\frac{s}{m_2}} \\ -\mathfrak{p}e^{-\frac{s}{m_1}} & \mathfrak{q} \end{bmatrix} \begin{bmatrix} 1 & \alpha e^{-\frac{s}{m_2}} \\ \beta e^{-\frac{s}{m_1}} & 1 \end{bmatrix} = \\ &= \frac{1}{1 - \alpha\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)}} \begin{bmatrix} \mathfrak{m} - \mathfrak{n}\beta e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} & (\mathfrak{m}\alpha - \mathfrak{n})e^{-\frac{s}{m_2}} \\ (\mathfrak{q}\beta - \mathfrak{p})e^{-\frac{s}{m_1}} & \mathfrak{q} - \mathfrak{p}\alpha e^{-\left(\frac{s}{m_1}+\frac{s}{m_2}\right)} \end{bmatrix} \end{aligned}$$

and belongs to $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathbb{C}^2))$ jointly with $\Xi^{-1}(s)$,

$$\Xi^{-1}(s) = \frac{1}{\mathfrak{m}\mathfrak{q} - \mathfrak{n}\mathfrak{p}e^{-(\frac{s}{m_1}+\frac{s}{m_2})}} \begin{bmatrix} \mathfrak{q} - \mathfrak{p}\alpha e^{-(\frac{s}{m_1}+\frac{s}{m_2})} & (\mathfrak{n} - \mathfrak{m}\alpha)e^{-\frac{s}{m_2}} \\ (\mathfrak{p} - \mathfrak{q}\beta)e^{-\frac{s}{m_1}} & \mathfrak{m} - \mathfrak{n}\beta e^{-(\frac{s}{m_1}+\frac{s}{m_2})} \end{bmatrix}.$$

For obtaining the optimal controller we get

$$D = \lim_{s\to\infty, s\in\mathbb{R}} \Xi(s) = \begin{bmatrix} \mathfrak{m} & 0 \\ 0 & \mathfrak{q} \end{bmatrix}, \quad \Xi(0) = \frac{1}{1-\alpha\beta} \begin{bmatrix} \mathfrak{m} - \mathfrak{n}\beta & \mathfrak{m}\alpha - \mathfrak{n} \\ \mathfrak{q}\beta - \mathfrak{p} & \mathfrak{q} - \mathfrak{p}\alpha \end{bmatrix}$$

and, since $\mathfrak{m}\mathfrak{q} = \sqrt{(2+\alpha)^2\mathfrak{m}_B^2 - \alpha^2} \geq \sqrt{2}$,

$$D^{-1} = \begin{bmatrix} \frac{1}{\mathfrak{m}} & 0 \\ 0 & \frac{1}{\mathfrak{q}} \end{bmatrix} \left( = \lim_{s\to\infty, s\in\mathbb{R}} \Xi^{-1}(s) \right), \quad \Xi^{-1}(0) = \frac{1}{\mathfrak{m}\mathfrak{q} - \mathfrak{n}\mathfrak{p}} \begin{bmatrix} \mathfrak{q} - \mathfrak{p}\alpha & \mathfrak{n} - \mathfrak{m}\alpha \\ \mathfrak{p} - \mathfrak{q}\beta & \mathfrak{m} - \mathfrak{n}\beta \end{bmatrix}.$$

From the realization identity (4.1), which here takes the form:

$$I - D^{-1} \underbrace{\Xi(s)}_{=\mathbf{X}(s)\mathbf{M}^{-1}(s)} = \overbrace{-D^{-1} \underbrace{V^{-*}}_{=\Xi^{-*}(0)} \mathcal{G}_\Lambda}^{=F_\Lambda} \overbrace{\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}}^{=\hat{G}(s)} = F_\Lambda \underbrace{\hat{G}(s)}_{=\mathbf{U}(s)\mathbf{M}^{-1}(s)} \iff$$

$$\iff \mathbf{M}(s) - D^{-1}\mathbf{X}(s) = F_\Lambda \mathbf{U}(s) \iff$$

$$\begin{bmatrix} 1 & -\alpha e^{-\frac{s}{m_2}} \\ -\beta e^{-\frac{s}{m_1}} & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\mathfrak{m}} & 0 \\ 0 & \frac{1}{\mathfrak{q}} \end{bmatrix} \begin{bmatrix} \mathfrak{m} & -\mathfrak{n}e^{-\frac{s}{m_2}} \\ -\mathfrak{p}e^{-\frac{s}{m_1}} & \mathfrak{q} \end{bmatrix} = F_\Lambda \begin{bmatrix} e^{-\frac{s\theta}{m_1}} & 0 \\ 0 & e^{-\frac{s(1-\theta)}{m_2}} \end{bmatrix},$$

and (5.3), we determine (uniquely as infinite-time exact controllability implies approximate controllability) the optimal controller

$$u = F_\Lambda x = \begin{bmatrix} (\frac{\mathfrak{n}}{\mathfrak{m}} - \alpha)x_2(0) \\ (\frac{\mathfrak{p}}{\mathfrak{q}} - \beta)x_1(1) \end{bmatrix} \Longrightarrow V^{-*}\mathcal{G}_\Lambda x = -DF_\Lambda x = \begin{bmatrix} (\alpha\mathfrak{m} - \mathfrak{n})x_2(0) \\ (\beta\mathfrak{q} - \mathfrak{p})x_1(1) \end{bmatrix} \Longrightarrow$$

$$\Longrightarrow \mathcal{G}_\Lambda x = \frac{1}{1-\alpha\beta} \begin{bmatrix} (\mathfrak{m} - \mathfrak{n}\beta)(\alpha\mathfrak{m} - \mathfrak{n})x_2(0) + (\mathfrak{q}\beta - \mathfrak{p})^2 x_1(1) \\ (\alpha\mathfrak{m} - \mathfrak{n})^2 x_2(0) + (\mathfrak{q} - \mathfrak{p}\alpha)(\mathfrak{q}\beta - \mathfrak{p})x_1(1) \end{bmatrix},$$

$$D(F_\Lambda) = D(\mathcal{G}_\Lambda) \supset \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) \supset R(\mathcal{D}).$$

A unique (by EXS) solution to the Lyapunov operator equation

$$\langle \mathcal{A}x, \mathcal{H}x \rangle_{\mathrm{H}} + \langle x, \mathcal{H}\mathcal{A}x \rangle_{\mathrm{H}} = -\|\mathcal{C}x\|_{\mathrm{H}}^2 + \|V^{-*}\mathcal{G}_\Lambda x\|_{\mathrm{U}}^2 = -\|\mathcal{C}x\|_{\mathrm{H}}^2 + \|V^{-*}\mathcal{G}x\|_{\mathrm{U}}^2 =$$
$$= -\|x\|_{\mathrm{H}}^2 + (\alpha\mathfrak{m} - \mathfrak{n})^2 x_2^2(0) + (\beta\mathfrak{q} - \mathfrak{p})^2 x_1^2(1), \qquad x \in D(\mathcal{A}) \tag{6.10}$$

has the form of a linear combination $k_1\mathcal{E}_1 + k_2\mathcal{E}_2 + \mathcal{E}_3$. Indeed, comparing the right side of (6.1) with the right side of the second line in (6.10) we get the linear equations determining $k_1$, $k_2$

$$\begin{bmatrix} -1 & \beta^2 \\ \alpha^2 & -1 \end{bmatrix} \begin{bmatrix} k_1(1-\alpha^2\beta^2)^{-1} \\ k_2(1-\alpha^2\beta^2)^{-1} \end{bmatrix} = \begin{bmatrix} (\beta\mathfrak{q} - \mathfrak{p})^2 - \beta^2 \\ (\alpha\mathfrak{m} - \mathfrak{n})^2 - \alpha^2 \end{bmatrix}$$

with a unique solution:

$$k_1 = \beta^2 - (\beta\mathfrak{q} - \mathfrak{p})^2 + \alpha^2\beta^2 - \beta^2(\alpha\mathfrak{m} - \mathfrak{n})^2, \quad k_2 = \alpha^2 - (\alpha\mathfrak{m} - \mathfrak{n})^2 + \alpha^2\beta^2 - \alpha^2(\beta\mathfrak{q} - \mathfrak{p})^2,$$

which can be simplified, using consecutively (6.4), (6.5), (6.6) and (6.7) (elimination of $\mathfrak{p}^2$, $\mathfrak{mn}$, $\mathfrak{qp}$ and $\beta^2(\mathfrak{q}^2 + \mathfrak{n}^2)$ for $k_1$ and elimination of $\alpha^2(\mathfrak{m}^2 + \mathfrak{p}^2)$, $\mathfrak{mn}$, $\mathfrak{qp}$ and $\mathfrak{n}^2$ and for $k_2$, respectively) to

$$k_1 = (\mathfrak{m}^2 - 2)(1 - \alpha^2\beta^2), \qquad k_2 = (\mathfrak{q}^2 - 2)(1 - \alpha^2\beta^2).$$

Thus $k_1 \geq 0$ ($= \iff \beta = 0$) and $k_2 \geq 0$ ($= \iff \alpha = 0$) and consequently $\mathcal{H} \geq 0$ or even coercive if $\alpha\beta \neq 0$. Hence

$$\mathcal{H} = (\mathfrak{m}^2 - 2)(1 - \alpha^2\beta^2)\mathcal{E}_1 + (\mathfrak{q}^2 - 2)(1 - \alpha^2\beta^2)\mathcal{E}_2 + \mathcal{E}_3, \qquad \mathbf{L}(\mathrm{H}) \ni \mathcal{H} = \mathcal{H}^* \geq 0$$

and we claim that $\mathcal{H}$ solves the Riccati operator equation (2.7). Indeed, eliminating $\mathcal{E}_i$, $i = 1, 2, 3$, we get

$$(\mathcal{H}x)(\theta) = \left[\operatorname{diag}\left\{\frac{\mathfrak{m}^2 - 1 - \theta}{m_1}, \frac{\mathfrak{q}^2 - 2 + \theta}{m_2}\right\}\right] x(\theta),$$

whence

$$\begin{aligned}
x \in D(\mathcal{A}) \Longrightarrow \mathcal{G}x = -\mathcal{D}^*\mathcal{H}\mathcal{A}x + N_-^*\mathcal{C}x = -\mathcal{D}^*[\mathcal{H}\mathcal{A}x + x] = \\
= \frac{1}{1-\alpha\beta}\begin{bmatrix} 1 & \beta \\ \alpha & 1 \end{bmatrix} \int_0^1 \left\{\begin{bmatrix} (1 + \theta - \mathfrak{m}^2)x_1'(\theta) + x_1(\theta) \\ (\theta - 2 + \mathfrak{q}^2)x_2'(\theta) + x_2(\theta) \end{bmatrix}\right\} \mathrm{d}\theta = \\
= \frac{1}{1-\alpha\beta}\begin{bmatrix} 1 & \beta \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} (2 - \mathfrak{m}^2)x_1(1) + (\mathfrak{m}^2 - 1)x_1(0) \\ (\mathfrak{q}^2 - 1)x_2(1) + (2 - \mathfrak{q}^2)x_2(0) \end{bmatrix} = \\
= \frac{1}{1-\alpha\beta}\begin{bmatrix} 1 & \beta \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} (2 - \mathfrak{m}^2)x_1(1) + \alpha(\mathfrak{m}^2 - 1)x_2(0) \\ \beta(\mathfrak{q}^2 - 1)x_1(1) + (2 - \mathfrak{q}^2)x_2(0) \end{bmatrix} = \\
= \frac{1}{1-\alpha\beta}\begin{bmatrix} (2 - \mathfrak{m}^2 + \beta^2\mathfrak{q}^2 - \beta^2)x_1(1) + (2\beta - \mathfrak{q}^2\beta + \alpha\mathfrak{m}^2 - \alpha)x_2(0) \\ (2\alpha - \mathfrak{m}^2\alpha + \beta\mathfrak{q}^2 - \beta)x_1(1) + (2 - \mathfrak{q}^2 + \alpha^2\mathfrak{m}^2 - \alpha^2)x_2(0) \end{bmatrix} = \mathcal{G}_\Lambda x,
\end{aligned}$$

where similar rules of simplification were applied while proving $\mathcal{G} = \mathcal{G}_\Lambda|_{D(\mathcal{A})}$.

Finally, we find the closed-loop system state operator. Since for $x \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1)$ one has:

$$x - \mathcal{D}(\underbrace{R_- + \mathcal{G}_\Lambda\mathcal{D}^{-1}\mathcal{G}_\Lambda}_{=-F_\Lambda})x = \begin{bmatrix} x_1 + \frac{1}{1-\alpha\beta}\,\frac{\alpha\mathfrak{m}-\mathfrak{n}}{\mathfrak{m}}x_2(0)\mathbf{1} + \frac{1}{1-\alpha\beta}\,\frac{\alpha(\mathfrak{q}\beta-\mathfrak{p})}{\mathfrak{q}}x_1(1)\mathbf{1} \\ x_2 + \frac{1}{1-\alpha\beta}\,\frac{\beta(\alpha\mathfrak{m}-\mathfrak{n})}{\mathfrak{m}}x_2(0)\mathbf{1} + \frac{1}{1-\alpha\beta}\,\frac{\mathfrak{q}\beta-\mathfrak{p}}{\mathfrak{q}}x_1(1)\mathbf{1} \end{bmatrix} \in D(\mathcal{A})$$

$$\iff x_1(0) = \frac{\mathfrak{n}}{\mathfrak{m}}x_2(0), \; x_2(1) = \frac{\mathfrak{p}}{\mathfrak{q}}x_1(1),$$

then

$$\mathcal{A}_{\mathrm{opt}}x = \mathcal{A}[x - \mathcal{D}(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x] = \mathcal{A}[x + \mathcal{D}F_\Lambda x] = \mathcal{A}x = \begin{bmatrix} -m_1x_1' \\ m_2x_2' \end{bmatrix},$$

$$D(\mathcal{A}_{\mathrm{opt}}) = \left\{x \in \mathrm{W}^{1,2}(0,1) \oplus \mathrm{W}^{1,2}(0,1) : \; x_1(0) = \frac{\mathfrak{n}}{\mathfrak{m}}x_2(0), \; x_2(1) = \frac{\mathfrak{p}}{\mathfrak{q}}x_1(1)\right\}.$$

Thus $\mathcal{A}_{\text{opt}}$ has the same structure as $\mathcal{A}$ with $\alpha$ and $\beta$ replaced by $\frac{\mathfrak{n}}{\mathfrak{m}}$ and $\frac{\mathfrak{p}}{\mathfrak{q}}$, respectively. Hence the result concerning EXS of the semigroup $\{S(t)\}_{t\geq 0}$ applies to $\{S_{\text{opt}}(t)\}_{t\geq 0}$, i.e., $\{S_{\text{opt}}(t)\}_{t\geq 0}$ is EXS iff $\left(\frac{\mathfrak{n}}{\mathfrak{m}}\frac{\mathfrak{p}}{\mathfrak{q}}\right)^2 < 1$. However, the last inequality was shown to be true – see (6.9), confirming the general EXS result of Theorem 2.3.

Observe that $\mathcal{H}$ is a solution to the Lyapunov/Riccati closed-loop operator equation (2.12) which here reduces to

$$\begin{aligned}&\langle \mathcal{A}_{\text{opt}}x, \mathcal{H}x\rangle_{\text{H}} + \langle x, \mathcal{H}\mathcal{A}_{\text{opt}}x\rangle_{\text{H}} = -\left\|\mathcal{C}x\right\|_{\text{H}}^2 - \left\|F_\Lambda x\right\|_{\text{U}}^2 = \\ &= -\left\|x\right\|_{\text{H}}^2 - \left(\frac{\mathfrak{n}}{\mathfrak{m}} - \alpha\right)^2 x_2^2(0) - \left(\frac{\mathfrak{p}}{\mathfrak{q}} - \beta\right)^2 x_1^2(1), \qquad x \in D(\mathcal{A}_{\text{opt}}).\end{aligned} \tag{6.11}$$

Indeed, for $x \in D(\mathcal{A}_{\text{opt}})$ we have

$$\begin{aligned}&\langle \mathcal{A}_{\text{opt}}x, \mathcal{H}x\rangle_{\text{H}} + \langle x, \mathcal{H}\mathcal{A}_{\text{opt}}x\rangle_{\text{H}} = \\ &= \int_0^1 (1+\theta-\mathfrak{m}^2)\frac{\mathrm{d}x_1^2(\theta)}{\mathrm{d}\theta}\mathrm{d}\theta + \int_0^1 (\mathfrak{q}^2-2+\theta)\frac{\mathrm{d}x_2^2(\theta)}{\mathrm{d}\theta}\mathrm{d}\theta = \\ &= (2-\mathfrak{m}^2)x_1^2(1) - (1-\mathfrak{m}^2)x_1^2(0) + (\mathfrak{q}^2-1)x_2^2(1) - (\mathfrak{q}^2-2)x_2^2(0) - \left\|x\right\|_{\text{H}}^2 = \\ &= \left[2-\mathfrak{m}^2+\frac{\mathfrak{p}^2}{\mathfrak{q}^2}(\mathfrak{q}^2-1)\right]x_1^2(1) + \left[2-\mathfrak{q}^2+\frac{\mathfrak{n}^2}{\mathfrak{m}^2}(\mathfrak{m}^2-1)\right]x_2^2(0) - \left\|x\right\|_{\text{H}}^2\end{aligned}$$

from which (6.11) follows easily by applying (6.4), (6.5), (6.6) and (6.7).

From the Lyapunov characterization of admissibility, applied to (6.11), we conclude that $F_\Lambda$ is admissible with respect to $\{S_{\text{opt}}(t)\}_{t\geq 0}$, whence $u \in \text{L}^2(0,\infty;\text{U})$ and thus $u$ is optimal.

Observe that $F_\Lambda|_{D(\mathcal{A}_{\text{opt}})} = \tau|_{D(\mathcal{A}_{\text{opt}})}$, where $\tau$ is the operator of boundary control given by (6.2). This fact is basic for establishing the structure of optimal control closed-loop system depicted in Figure 2, where the external connections realize the optimal feedback control $u = F_\Lambda x$.

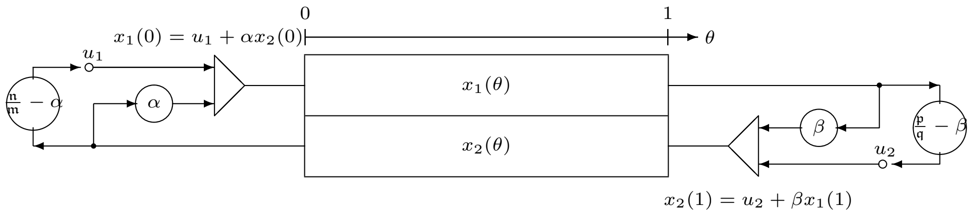


**Fig. 2.** Open/closed-loop control system for the Chapelon-Xu example

**Comment 6.1.** The whole analysis, towards the optimal controller design, presented in [3] ends with finding the spectral factor

$$\Xi(s) = \begin{bmatrix} \sqrt{2} & \frac{\alpha}{\sqrt{2}}e^{-\frac{s}{m_2}} \\ 0 & \sqrt{\frac{\alpha^2+4}{2}} \end{bmatrix}$$

in the case where $\beta = 0$. It coincides with the above computations (here $\mathfrak{m} = \sqrt{2}$, $\mathfrak{n} = \frac{\alpha}{\sqrt{2}}$, $\mathfrak{p} = 0$ and $\mathfrak{q} = \sqrt{\frac{\alpha^2+4}{2}}$). In his discussion, J. Malinen [12] tried to judge what kind of a Riccati equation would be the best to determine the optimal controller, but without any particular conclusion how to complete the unfinished design by A. Chapelon and C.-Z. Xu [3], though his critique qualifies the problem as "mathematically simple" [3, p. 606].

## 7. SOLUTION OF THE LQ-PROBLEM FORMULATED IN [26]. THE SISO CASE

### 7.1. PRELIMINARY CONSIDERATIONS

In [26] the lq-problem has been formulated for the dynamical system modeling propagation of pollutants in a river. In this section we solve the standard lq-problem for a controllable part of this model arising from a general one [26, p. 174] by extracting its second component, describing how the concentration of dissolved oxygen (DO) varies in time. Observe that the second component is affected by the first component but not conversely and the control does not excite the first component, which therefore remains uncontrolled.

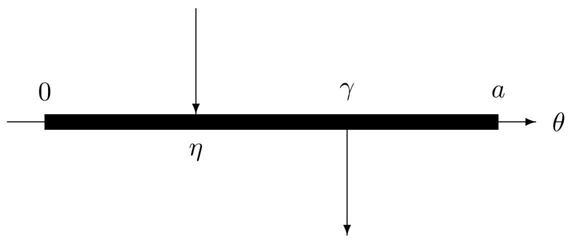


**Fig. 3.** Configuration of measurement and control in the SISO case

Let us consider the SISO case, i.e., the case of a one point control (one aerator) located at $\theta = \eta > 0$ and one output (one sensor measuring DO) located at $\theta = \gamma > \eta$ as depicted in Figure 3. Let $\mathrm{H} = \mathrm{L}^2(0,a)$, $a > 0$ and $\mathrm{U} = \mathrm{Y} = \mathbb{R}$. Then the system dynamics is governed by (1.1) with the following objects. The state operator is

$$\mathcal{A}x = -vx' - K_2 x, \qquad D(\mathcal{A}) = \mathrm{W}_0^{1,2}(0,a), \qquad K_2 > 0$$

and it generates an EXS semigroup on H. The observation functional is given by

$$\mathcal{C}x = x(\gamma), \qquad D(\mathcal{C}) = \{x \in \mathrm{H} : \ x \text{ is continuous at } \theta = \gamma\}.$$

Finally, the factor control vector $d \in \mathrm{H}$ takes the form

$$d(\theta) = -\frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta)} \mathbb{1}(\theta - \eta) = -\frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta)} \chi_{[\eta,a]}(\theta), \quad \theta \in [0,a].$$

Though, from [26, Theorem 3.2] we know that $\mathcal{C}$ is admissible we can strengthen this result by showing that the operator $\mathbf{L}(\mathrm{H}) \ni \mathcal{H}_\Psi = \mathcal{H}_\Psi^* \geq 0$, defined as

$$\left(\mathcal{H}_\Psi x\right)(\theta) := \frac{1}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \chi_{[0,\gamma]}(\theta) x(\theta), \quad x \in \mathrm{H}, \tag{7.1}$$

is the *observability gramian.* Indeed,

$$\langle \mathcal{A}x, \mathcal{H}_\Psi x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}_\Psi \mathcal{A}x\rangle_{\mathrm{H}} = -\int_0^\gamma e^{-\frac{2K_2}{v}(\gamma-\theta)} \tfrac{\mathrm{d}}{\mathrm{d}\theta}\left[x^2(\theta)\right]\mathrm{d}\theta -$$

$$-\tfrac{2K_2}{v}\int_0^\gamma e^{-\frac{2K_2}{v}(\gamma-\theta)} x^2(\theta)\mathrm{d}\theta = -x^2(\gamma) = -\left|\mathcal{C}x\right|^2, \qquad x \in D(\mathcal{A}).$$

Next, $d$ is an admissible factor control vector (this fact has not been examined in [26]). Indeed, by duality, its is enough to show that the observation functional $d^*\mathcal{A}^*$ is admissible with respect to the adjoint semigroup. Here

$$\mathcal{A}^* w = v w' - K_2 w, \qquad D(\mathcal{A}^*) = \left\{ w \in \mathrm{W}^{1,2}(0,a) : \; w(a) = 0 \right\}. \tag{7.2}$$

Observe that

$$d^*\mathcal{A}^* w = \langle \mathcal{A}^* w, d\rangle_{\mathrm{H}} = \int_\eta^a \left[\frac{K_2}{v} w(\theta) - w'(\theta)\right] e^{-\frac{K_2}{v}(\theta-\eta)}\mathrm{d}\theta = w(\eta), \qquad w \in D(\mathcal{A}^*)$$

and because $\mathcal{H}_\Phi \in \mathbf{L}(\mathrm{H})$, $\mathcal{H}_\Phi = \mathcal{H}_\Phi^* \geq 0$

$$\left(\mathcal{H}_\Phi x\right)(\theta) := \frac{1}{v} e^{\frac{2K_2}{v}(\eta-\theta)} \chi_{[\eta,a]}(\theta) x(\theta), \quad x \in \mathrm{H}, \tag{7.3}$$

solves the Lyapunov operator equation ($\mathcal{H}_\Phi$ is the *controllability gramian*)

$$\langle \mathcal{A}^* w, \mathcal{H}_\Phi w\rangle_{\mathrm{H}} + \langle w, \mathcal{H}_\Phi \mathcal{A}^* w\rangle_{\mathrm{H}} = \int_\eta^a e^{\frac{2K_2}{v}(\eta-\theta)} \tfrac{\mathrm{d}}{\mathrm{d}\theta}\left[w^2(\theta)\right]\mathrm{d}\theta -$$

$$-\tfrac{2K_2}{v}\int_\eta^a e^{\frac{2K_2}{v}(\eta-\theta)} w^2(\theta)\mathrm{d}\theta = -w^2(\eta) = -\left|d^*\mathcal{A}^* w\right|^2, \qquad w \in D(\mathcal{A}^*),$$

then the admissibility of $d^*\mathcal{A}^*$ follows from Lyapunov characterization of admissibility. Furthermore, $\ker \mathcal{H}_\Psi$ and $\ker \mathcal{H}_\Phi$ are both nontrivial, whence the system is neither (infinite-time) approximately observable nor approximately controllable, and the method of spectral factorization is not applicable in its full extend.

Since

$$\begin{aligned} \left((sI-\mathcal{A})^{-1}x\right)(\theta) &= \frac{1}{v}\int_0^\theta e^{-\frac{s+K_2}{v}(\theta-\xi)} x(\xi)\mathrm{d}\xi, \\ \left(\mathcal{A}(sI-\mathcal{A})^{-1}x\right)(\theta) &= -x(\theta) + \frac{s}{v}\int_0^\theta e^{-\frac{s+K_2}{v}(\theta-\xi)} x(\xi)\mathrm{d}\xi, \end{aligned} \tag{7.4}$$

then with $\delta := \frac{K_2}{v}(\gamma-\eta) > 0$

$$(\mathcal{A}(sI-\mathcal{A})^{-1}d)(\theta) = \frac{1}{v}\,e^{-\frac{s+K_2}{v}(\theta-\eta)}\mathbb{1}(\theta-\eta),\ \hat{G}(s) = \mathcal{C}\mathcal{A}(sI-\mathcal{A})^{-1}d = \frac{1}{v}e^{-\frac{s}{v}(\gamma-\eta)}e^{-\delta}$$

and $\hat{G} \in \mathrm{H}^{\infty}(\mathbb{C}^{+})$.

Recalling that the resolvent of $\mathcal{A}$ is Laplace transform of the semigroup generated by $\mathcal{A}$ and substituting $t = \frac{\theta-\xi}{v}$ in (7.4) we get

$$(S(t)X)(\theta) = e^{-K_2 t}\left\{\begin{array}{ccc} X(\theta-vt) & \text{if} & a \geq \theta \geq vt \\ 0 & \text{if} & \theta < vt \end{array}\right\}, \qquad t \geq 0,\ \theta \in [0,a], \quad (7.5)$$

whence $\{S(t)\}_{t\geq 0}$ even decays in a finite-time.

## 7.2. DIRECT SOLUTION

We seek for a solution of (2.7) in the form $\mathcal{H} = \mathfrak{a}\mathcal{H}_{\Psi} + \mathfrak{b}\mathcal{H}_1$, where

$$(\mathcal{H}_1 x)(\theta) := \frac{1}{v}e^{-\frac{2K_2}{v}(\eta-\theta)}\chi_{[\eta,\gamma]}(\theta)x(\theta) = \frac{e^{2\delta}}{v}e^{-\frac{2K_2}{v}(\gamma-\theta)}\chi_{[\eta,\gamma]}(\theta)x(\theta), \quad x \in \mathrm{H}.$$

Here $N_{-} = -\mathcal{C}d = \frac{1}{v}e^{-\delta}$, $R_{-} := 1 + (\mathcal{C}d)^2 = 1 + \frac{1}{v^2}e^{-2\delta}$ and the Riccati operator equation (2.7) takes the form

$$\langle \mathcal{A}z, \mathcal{H}z\rangle_{\mathrm{H}} + \langle z, \mathcal{H}\mathcal{A}z\rangle_{\mathrm{H}} + (\mathcal{C}z)^2 = R_{-}^{-1}\left[\langle \mathcal{A}z, \mathcal{H}d\rangle_{\mathrm{H}} + (\mathcal{C}d)^{*}\mathcal{C}z\right]^2, \quad z \in D(\mathcal{A}).$$

Its LHS is

$$\begin{aligned}
&\langle \mathcal{A}z, \mathcal{H}z\rangle_{\mathrm{H}} + \langle z, \mathcal{H}\mathcal{A}z\rangle_{\mathrm{H}} + (\mathcal{C}z)^2 = (1-\mathfrak{a})z^2(\gamma) + \langle \mathcal{A}z, \mathfrak{b}\mathcal{H}_1 z\rangle_{\mathrm{H}} + \langle z, \mathfrak{b}\mathcal{H}_1\mathcal{A}z\rangle_{\mathrm{H}} = \\
&= (1-\mathfrak{a})z^2(\gamma) - \mathfrak{b}\int_{\eta}^{\gamma}\left[2z(\theta)z'(\theta) + \frac{2K_2}{v}z^2(\theta)\right]e^{-\frac{2K_2}{v}(\eta-\theta)}\mathrm{d}\theta = \\
&= (1-\mathfrak{a}-e^{2\delta}\mathfrak{b})z^2(\gamma) + \mathfrak{b}z^2(\eta).
\end{aligned}$$

Since

$$\begin{aligned}
&(\mathcal{C}d)^{*}\mathcal{C}z = -\frac{1}{v}e^{-\delta}z(\gamma), \\
&\langle \mathcal{A}z, \mathcal{H}_{\Psi}d\rangle_{\mathrm{H}} = \tfrac{1}{v}e^{-\delta}\int_{\eta}^{\gamma}\left[\tfrac{K_2}{v}z(\theta) + z'(\theta)\right]e^{\frac{K_2}{v}(\theta-\gamma)}\mathrm{d}\theta = \tfrac{1}{v}e^{-\delta}z(\gamma) - \tfrac{1}{v}e^{-2\delta}z(\eta), \\
&\langle \mathcal{A}z, \mathcal{H}_1 d\rangle_{\mathrm{H}} = \frac{1}{v}e^{-\delta}\int_{\eta}^{\gamma}\left[\tfrac{K_2}{v}z(\theta) + z'(\theta)\right]e^{\frac{K_2}{v}(\theta-\eta)}\mathrm{d}\theta = \tfrac{1}{v}e^{\delta}z(\gamma) - \tfrac{1}{v}z(\eta),
\end{aligned}$$

the RHS of our Riccati operator equation reads as

$$\begin{aligned}
&R_{-}^{-1}\left[\langle \mathcal{A}z, \mathcal{H}d\rangle_{\mathrm{H}} + (\mathcal{C}d)^{*}\mathcal{C}z\right]^2 = \\
&= \frac{1}{v^2+e^{-2\delta}}\left\{\left[(\mathfrak{a}-1)e^{-\delta} + \mathfrak{b}e^{\delta}\right]z(\gamma) - \left[\mathfrak{a}e^{-2\delta} + \mathfrak{b}\right]z(\eta)\right\}^2.
\end{aligned}$$

It is not difficult to establish that both sides are equal for

$$\mathfrak{a} = \frac{v^2}{v^2 + e^{-2\delta}}, \qquad \mathfrak{b} = (1-\mathfrak{a})e^{-2\delta} = \frac{e^{-4\delta}}{v^2 + e^{-2\delta}},$$

whence

$$\left(\mathcal{H}x\right)(\theta) = \frac{1}{v} e^{-\frac{2K_2}{v}(\gamma-\theta)} \left\{ \begin{array}{lll} \mathfrak{a} & \text{on} & [0,\eta) \\ 1 & \text{on} & [\eta,\gamma] \\ 0 & \text{on} & (\gamma,a] \end{array} \right\} x(\theta), \tag{7.6}$$

and consequently

$$\mathcal{G}z := [\langle \mathcal{A}z, -\mathcal{H}d\rangle_{\mathrm{H}} - (\mathcal{C}d)^*\mathcal{C}z] = \frac{e^{-2\delta}}{v}\ z(\eta), \qquad z \in D(\mathcal{A}).$$

If $z \in \mathrm{Reg}[0,a]$ – the space of *regulated functions*, i.e., functions having one-sided (finite) limits at every point $\theta \in [0,a]$ then (recall that $z$ is bounded on $[0,a]$), by the Lebesgue dominated convergence theorem:

$$\lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}z = \frac{e^{-2\delta}}{v^2} \lim_{s\to\infty, s\in\mathbb{R}} s \int_0^\eta e^{-\frac{s+K_2}{v}(\eta-\xi)} z(\xi)\mathrm{d}\xi =$$

$$= \tfrac{e^{-2\delta}}{v} \lim_{s\to\infty, s\in\mathbb{R}} \frac{s}{s+K_2} \int_0^\infty e^{-t} z\left(\eta - \frac{v}{s+K_2}t\right) \mathbb{1}\left(\frac{s+K_2}{v}\eta - t\right) \mathrm{d}t = \frac{e^{-2\delta}}{v}\ z(\eta-)$$

and therefore we may take

$$\mathcal{G}_\Lambda z := \frac{e^{-2\delta}}{v}\ z(\eta-), \qquad D(\mathcal{G}_\Lambda) = \mathrm{Reg}[0,a].$$

Now, $d \in D(\mathcal{G}_\Lambda)$ with

$$\mathcal{G}_\Lambda d = \frac{e^{-2\delta}}{v}\ d(\eta-) = 0$$

and (2.11) reads as

$$u = \frac{-ve^{-2\delta}}{v^2+e^{-2\delta}}\ z(\eta-), \qquad z \in D(\mathcal{G}_\Lambda) = \mathrm{Reg}[0,a].$$

Consequently, the closed-loop operator reads as

$$\mathcal{A}_{\mathrm{opt}}x = -vz' - K_2 z, \qquad z(\theta) := x(\theta) + \frac{e^{-2\delta}}{v^2+e^{-2\delta}} x(\eta-) e^{-\frac{K_2}{v}(\theta-\eta)} \chi_{[\eta,a]},$$

$$D(\mathcal{A}_{\mathrm{opt}}) = \left\{ x \in \mathrm{H}:\ z \in \mathrm{W}_0^{1,2}[0,a],\ x(\eta+) = \frac{v^2}{v^2+e^{-2\delta}} x(\eta-) \right\},$$

whence (on $\theta$-intervals $[0,\eta]$, $[\eta,a]$ there holds $(\mathcal{A}_{\rm opt}x)(\theta) = -vx'(\theta) - K_2x(\theta)$)

$$\begin{aligned} x\in D(\mathcal{A}_{\rm opt}) \Longrightarrow & \langle x,\mathcal{A}_{\rm opt}x\rangle_{\rm H} + \langle \mathcal{A}_{\rm opt}x,x\rangle_{\rm H} = \langle x,\mathcal{A}z\rangle_{\rm H} + \langle \mathcal{A}z,x\rangle_{\rm H} = \\ & = -vx^2(\eta-) - vx^2(a) + vx^2(\eta+) - 2K_2\|x\|_{\rm H}^2 = \\ & = -\frac{vc^2(c^2+2)}{(1+c^2)^2}x^2(\eta-) - vx^2(a) - 2K_2\|x\|_{\rm H}^2, \qquad c := \frac{1}{v}e^{-\delta}, \end{aligned}$$

and

$$x(\theta) = \begin{cases} \frac{1}{v}\int\limits_0^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi & \text{if} \quad 0\le\theta<\eta, \\ \blacklozenge + \frac{1}{v}\int\limits_\eta^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi & \text{if} \quad \eta\le\theta\le a, \end{cases} \tag{7.7}$$

where

$$\blacklozenge := \frac{1}{v(1+c^2)}\int\limits_0^\eta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi,$$

solves the resolvent equation $\lambda x - \mathcal{A}_{\rm opt}x = X$ which, by the Lummer-Phillips theorem, implies that $\mathcal{A}_{\rm opt}$ generates an EXS semigroup on H. Moreover, since

$$x\in D(\mathcal{A}_{\rm opt}) \Longrightarrow \langle x,\mathcal{A}_{\rm opt}x\rangle_{\rm H} + \langle \mathcal{A}_{\rm opt}x,x\rangle_{\rm H} \le -\frac{vc^2(c^2+2)}{(1+c^2)^2}x^2(\eta-)$$

then, by Lyapunov characterization of admissibility, the functional $x\longmapsto x(\eta-)$ is admissible with respect to the semigroup generated by $\mathcal{A}_{\rm opt}$, which shows that the control is optimal as it belongs to $\mathrm{L}^2(0,\infty;\mathrm{U})$.

Now (7.7) defines the resolvent of $\mathcal{A}_{\rm opt}$. Thus substituting $t = \frac{\theta-\xi}{v}$ in (7.7) and applying the definition of Laplace transformation, we obtain

$$(S_{\rm opt}(t)X)(\theta) = e^{-K_2t}\begin{cases} X(\theta-vt) & \text{if} \quad 0\le t\le\frac{\theta}{v}, \quad 0\le\theta<\eta, \\ X(\theta-vt) & \text{if} \quad 0\le t\le\frac{\theta-\eta}{v}, \quad \eta\le\theta<a, \\ \frac{1}{1+c^2}X(\theta-vt) & \text{if} \quad \frac{\theta-\eta}{v}\le t\le\frac{\theta}{v}, \quad \eta\le\theta\le a, \\ 0 & \text{elsewhere}, \end{cases}$$

from which we deduce that actually the semigroup $\{S_{\rm opt}(t)\}_{t\ge0}$ decays to 0 in a natural finite time $a/v$. The rate of decaying of $\{S_{\rm opt}(t)\}_{t\ge0}$ is for $\theta\ge\eta$ faster than that of $\{S(t)\}_{t\ge0}$ given by (7.5).

Observe that

$$\mathcal{G}(sI-\mathcal{A})^{-1}d = -\frac{e^{-2\delta}}{v}\frac{1}{v^2}\int\limits_0^\eta e^{-\frac{s+K_2}{v}(\eta-\xi)}e^{-\frac{K_2}{v}(\xi-\eta)}\chi_{[\eta,a]}(\xi)\mathrm{d}\xi \equiv 0$$

and consequently $\Xi(s)\equiv R_-^{1/2}$. This also confirms that our system is not approximate controllable.

## 7.3. OPERATOR-THEORETIC ATTEMPT TO FINDING THE OPTIMAL COST OPERATOR

Since $\mathcal{A}$ can be represented as $\mathcal{A} = v\mathcal{R}_F - K_2 I$, where $\mathcal{R}_F$, stands for the generator of the semigroup of right-shifts on H then the semigroup generated by $\mathcal{A}$ reads as (7.5). Hence the homogeneous part of the output, due to initial condition $x_0 \in \mathrm{H} = \mathrm{L}^2(0,a)$ is

$$\begin{aligned} y_{\mathrm{hom}}(t) &= \left\{ \begin{array}{ccc} e^{-K_2 t} x_0(\gamma - vt) & \text{if} & \left(\frac{a}{v} \geq\right) \frac{\gamma}{v} \geq t \\ 0 & \text{if} & \frac{\gamma}{v} < t \end{array} \right\} = \\ &= \left(\Psi x_0\right)(t) \quad \text{for almost all } t \geq 0, \end{aligned} \tag{7.8}$$

where $\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ denotes the observability map.

Recall that the system transfer function is $\hat{G}(s) = \dfrac{1}{v} e^{-\delta} e^{-s\frac{\delta}{K_2}}$, $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+)$, whence the nonhomogeneous part of the output, due to a control $u \in \mathrm{L}^2(0,\infty)$, takes the form

$$y_{\mathrm{nonhom}}(t) = \left\{ \begin{array}{ccc} \frac{1}{v} e^{-\delta} u\left(t - \frac{\delta}{K_2}\right) & \text{if} & t \geq \frac{\delta}{K_2} \\ 0 & \text{if} & t < \frac{\delta}{K_2} \end{array} \right\} \quad \text{for almost all } t \geq 0, \tag{7.9}$$

and therefore the extended input-output map and its adjoint are

$$\mathbb{F} = \frac{1}{v} e^{-\delta} S_{\mathcal{R}}\left(\frac{\delta}{K_2}\right), \qquad \mathbb{F}^* = \frac{1}{v} e^{-\delta} S_{\mathcal{L}}\left(\frac{\delta}{K_2}\right)$$

and $\mathbb{F}, \mathbb{F}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty))$, where $\{S_{\mathcal{R}}(t)\}_{t\geq 0}$ and $\{S_{\mathcal{L}}(t)\}_{t\geq 0}$ stand for the semigroups of right-, respectively, left-shifts on $\mathrm{L}^2(0,\infty)$.

By (2.5), the optimal control (to be more precise its time-domain form) is

$$u = -\left(\mathbb{F}^*\mathbb{F} + I\right)^{-1} \mathbb{F}^* \Psi x_0. \tag{7.10}$$

But

$$\begin{aligned} \left(\mathbb{F}^* \Psi x_0\right)(t) &= \frac{1}{v} e^{-\delta} \left( S_{\mathcal{L}} \left(\frac{\delta}{K_2}\right) \Psi x_0 \right)(t) = \\ &= \frac{1}{v} e^{-2\delta} \left\{ \begin{array}{ccc} e^{-K_2 t} x_0(\eta - vt) & \text{if} & t \in [0, \frac{\eta}{v}], \\ 0 & \text{if} & t > \frac{\eta}{v}. \end{array} \right. \end{aligned}$$

Since $S_{\mathcal{L}}(t) S_{\mathcal{R}}(t) = I$, then

$$\mathcal{R}^{-1} = \left(\mathbb{F}^*\mathbb{F} + I\right)^{-1} = \frac{v^2}{v^2 + e^{-2\delta}} I$$

and from (7.10) one gets

$$u(t) = \left\{ \begin{array}{ccc} -\dfrac{v e^{-2\delta}}{v^2 + e^{-2\delta}}\, e^{-K_2 t} x_0(\eta - vt) & \text{for almost all} & t \in [0, \frac{\eta}{v}], \\ 0 & \text{for almost all} & t > \frac{\eta}{v}. \end{array} \right. \tag{7.11}$$

From (2.6) we get the optimal cost operator

$$\mathcal{H}x_0 = \left[\Psi^*\Psi - \Psi^*\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^*\Psi\right]x_0 = \mathcal{H}_\Psi x_0 - \Psi^*\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^*\Psi x_0.$$

Directly from definition of the adjoint operator we find the form of $\Psi^*$:

$$\langle\Psi^* f, x_0\rangle_{\mathrm{H}} = \int_0^a \left(\Psi^* f\right)(\theta)x_0(\theta)\mathrm{d}\theta = \langle f, \Psi x_0\rangle_{\mathrm{L}^2(0,\infty)} = \int_0^\infty f(t)\left(\Psi x_0(t)\right)\mathrm{d}t =$$

$$= \int_0^{\gamma/v} f(t)e^{-K_2 t}x_0(\gamma - vt)\mathrm{d}t = \frac{1}{v}\int_0^\gamma f\left(\frac{\gamma-\theta}{v}\right)e^{-\frac{K_2(\gamma-\theta)}{v}}x_0(\theta)\mathrm{d}\theta,$$

which results in

$$\left(\Psi^* f\right)(\theta) = \frac{1}{v}f\left(\frac{\gamma-\theta}{v}\right)e^{-\frac{K_2(\gamma-\theta)}{v}}\chi_{[0,\gamma]}(\theta), \qquad \theta\in[0,a]. \tag{7.12}$$

Since $S_{\mathcal{R}}(t)S_{\mathcal{L}}(t) = \chi_{[t,\infty)}I$, then

$$-\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^* = -\frac{e^{-2\delta}}{v^2+e^{-2\delta}}\chi_{[\frac{\delta}{K_2},\infty)}I,$$

whence

$$-\Psi^*\mathbb{F}\left(\mathbb{F}^*\mathbb{F}+I\right)^{-1}\mathbb{F}^*\Psi x_0 = -\frac{e^{-2\delta}}{v\left(v^2+e^{-2\delta}\right)}e^{-\frac{K_2(\gamma-\theta)}{v}}\chi_{[0,\eta]}(\theta)x_0(\theta)$$

and finally

$$\left(\mathcal{H}x_0\right)(\theta) = \frac{1}{v}e^{-\frac{2K_2}{v}(\gamma-\theta)}\chi_{[0,\gamma]}(\theta)x_0(\theta) - \frac{e^{-2\delta}}{v\left(v^2+e^{-2\delta}\right)}e^{-\frac{2K_2(\gamma-\theta)}{v}}\chi_{[0,\eta]}(\theta)x_0(\theta).$$

The last formula coincides with (7.6) except for one point $\theta=\eta$ what is not essential as the state space is a Lebesgue-type space.

Now, we show that the feedback realization of the optimal control (7.11) is correct. Indeed, an initial condition $x_0$ and a control $u$, not necessarily optimal, give rise to the full state $x(t) = S(t)x_0 + x_{\mathrm{nonhom}}(t)$, $t\geq 0$. Since

$$\hat{x}_{\mathrm{nonhom}}(s)(\theta) = \left(\mathcal{A}(sI-\mathcal{A})^{-1}d\right)(\theta)\hat{u}(s) = \frac{1}{v}\,e^{-\frac{s+K_2}{v}(\theta-\eta)}\mathbb{1}(\theta-\eta)\hat{u}(s) =$$

$$= \frac{1}{v}\,e^{-\frac{K_2}{v}(\theta-\eta)}\chi_{[\eta,a]}(\theta)e^{-\frac{s}{v}(\theta-\eta)}\hat{u}(s),$$

then

$$x_{\mathrm{nonhom}}(t) = \begin{cases} \frac{1}{v}e^{-\frac{K_2}{v}(\theta-\eta)}\chi_{[\eta,a]}(\theta)u\left(t-\frac{\theta-\eta}{v}\right) & \text{if } t\geq\frac{\theta-\eta}{v},\\ 0 & \text{if } t<\frac{\theta-\eta}{v}.\end{cases}$$

Thus if $x_0 \in \mathrm{Reg}[0,a]$ then one has $S(t)x_0 \in \mathrm{Reg}[0,a]$ for every $t \geq 0$ and the optimal feedback controller equation yields

$$u(t) = -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}} \lim_{\theta\to\eta-} x(t)(\theta) = -\tfrac{ve^{-2\delta}}{v^2+e^{-2\delta}} \lim_{\theta\to\eta-} \left(S(t)x_0\right)(\theta) =$$
$$= -\frac{ve^{-2\delta}}{v^2+e^{-2\delta}} \begin{cases} e^{-K_2 t}x_0(\eta - vt-) & \text{if} \quad 0 \leq t < \frac{\eta}{v}, \\ 0 & \text{if} \quad t \geq \frac{\eta}{v}, \end{cases}$$

what agrees with (7.11).

## 8. SOLUTION OF THE LQ-PROBLEM FORMULATED IN [26]. THE MISO CASE

Combining some ideas of Sections 7.2 and 7.3 one can solve the standard lq-problem in the MISO case of two point controls (two aerators) located at $\theta = \eta_1 > 0$, $\theta = \eta_2 > \eta_1$ and one output (one sensor measuring DO) located at $\theta = \gamma > \eta_2$ as depicted in Figure 4; therefore still we have $\mathrm{Y} = \mathbb{R}$ but now $\mathrm{U} = \mathbb{R}^2$.

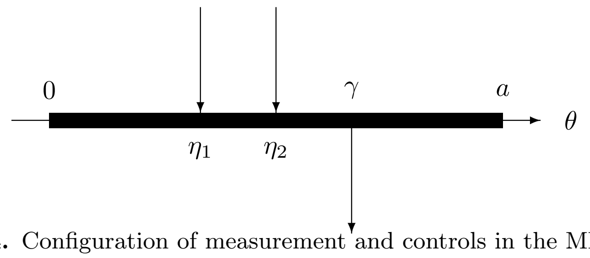


**Fig. 4.** Configuration of measurement and controls in the MISO case

Consequently, the extended observability map $\Psi$ is still given by (7.8) whilst the input-output operator has components somewhat similar to the SISO case:

$$\mathbb{F}u = \begin{bmatrix} \mathbb{F}_1 & \mathbb{F}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$
$$\mathbb{F}_i = c_i S_{\mathcal{R}}\left(\frac{\delta_i}{K_2}\right), \qquad c_i := \frac{1}{v}e^{-\delta_i},\ \delta_i := \frac{K_2(\gamma-\eta_i)}{v}, \quad i = 1,2.$$

Observe that $\eta_1 < \eta_2$ if and only if $\delta_1 > \delta_2$, and

$$\mathbb{F}^* y = \begin{bmatrix} \mathbb{F}_1^* \\ \mathbb{F}_2^* \end{bmatrix} y, \qquad \mathbb{F}_i^* = c_i S_{\mathcal{L}}\left(\frac{\delta_i}{K_2}\right), \quad i = 1,2.$$

Thanks to this

$$I \le \mathcal{R} = I + \mathbb{F}^*\mathbb{F} = \begin{bmatrix} I + \mathbb{F}_1^*\mathbb{F}_1 & \mathbb{F}_1^*\mathbb{F}_2 \\ \mathbb{F}_2^*\mathbb{F}_1 & I + \mathbb{F}_2^*\mathbb{F}_2 \end{bmatrix} =$$

$$= \begin{bmatrix} I + c_1^2 S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_1}{K_2}\right) & c_1 c_2 S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) \\ c_1 c_2 S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_1}{K_2}\right) & I + c_2^2 S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) \end{bmatrix} =$$

$$= \begin{bmatrix} (1+c_1^2)I & c_1 c_2 S_{\mathcal{L}}\left(\frac{\delta_1-\delta_2}{K_2}\right) \\ c_1 c_2 S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) & (1+c_2^2)I \end{bmatrix},$$

because $S_{\mathcal{L}}(t)S_{\mathcal{R}}(t) = I$ and

$$S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) = S_{\mathcal{L}}\left(\frac{\delta_1-\delta_2}{K_2}\right) S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) = S_{\mathcal{L}}\left(\frac{\delta_1-\delta_2}{K_2}\right),$$

$$S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_1}{K_2}\right) = S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) = S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right).$$

It is documented below that knowing explicit forms of $\Psi$, $\Psi^*$, $\mathbb{F}$, $\mathbb{F}^*$ and $(I + \mathbb{F}^*\mathbb{F})^{-1}$, we can explicitly express the optimal control, the optimal cost operator, the optimal controller and even $\{S_{\mathrm{opt}}(t)\}_{t\ge 0}$.

$$(I + \mathbb{F}^*\mathbb{F})^{-1} =$$

$$= \begin{bmatrix} \frac{1+c_2^2}{1+c_1^2+c_2^2} I & -\frac{c_1 c_2}{1+c_1^2+c_2^2} S_{\mathcal{L}}\left(\frac{\delta_1-\delta_2}{K_2}\right) \\ -\frac{c_1 c_2}{1+c_1^2+c_2^2} S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) & \frac{1}{1+c_2^2} I + \frac{c_1^2 c_2^2}{(1+c_1^2+c_2^2)(1+c_2^2)} S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) S_{\mathcal{L}}\left(\frac{\delta_1-\delta_2}{K_2}\right) \end{bmatrix},$$

whence

$$-(I + \mathbb{F}^*\mathbb{F})^{-1}\mathbb{F}^* = \begin{bmatrix} -\frac{c_1}{1+c_1^2+c_2^2} S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) \\ -\frac{c_2}{1+c_2^2} S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) + \frac{c_1^2 c_2}{(1+c_1^2+c_2^2)(1+c_2^2)} S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) \end{bmatrix}$$

and consequently, with $\frac{\delta_1-\delta_2}{K_2} = \frac{\eta_2-\eta_1}{v}$, $\frac{\delta_2}{K_2} < \frac{\delta_1}{K_2} < \frac{\gamma}{v}$, we obtain the optimal control

$$u = -(I + \mathbb{F}^*\mathbb{F})^{-1}\mathbb{F}^*\Psi x_0 = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} =$$

$$= \begin{bmatrix} \left\{ \begin{array}{ll} -\frac{vc_1^2}{1+c_1^2+c_2^2} e^{-K_2 t} x_0(\eta_1 - vt) & \text{if} \quad t \le \frac{\eta_1}{v} \\ 0 & \text{if} \quad t > \frac{\eta_1}{v} \end{array} \right\} \\ \left\{ \begin{array}{ll} -\frac{vc_2^2}{1+c_2^2} e^{-K_2 t} x_0(\eta_2 - vt) & \text{if} \quad t < \frac{\eta_2-\eta_1}{v} \\ -\frac{vc_2^2}{1+c_1^2+c_2^2} e^{-K_2 t} x_0(\eta_2 - vt) & \text{if} \quad t \in \left[\frac{\eta_2-\eta_1}{v}, \frac{\eta_2}{v}\right] \\ 0 & \text{if} \quad t > \frac{\eta_2}{v} \end{array} \right\} \end{bmatrix}. \tag{8.1}$$

Continuing the construction of $\mathcal{H}$, we find using $S_{\mathcal{R}}(t)S_{\mathcal{L}}(t) = \chi_{[t,\infty)}I$,

$$\begin{aligned}
&-\mathbb{F}\left(I+\mathbb{F}^*\mathbb{F}\right)^{-1}\mathbb{F}^* = \\
&= \begin{bmatrix} c_1 S_{\mathcal{R}}\left(\frac{\delta_1}{K_2}\right) & c_2 S_{\mathcal{R}}\left(\frac{\delta_2}{K_2}\right) \end{bmatrix} \times \\
&\times \begin{bmatrix} -\frac{c_1}{1+c_1^2+c_2^2} S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) \\ -\frac{c_2}{1+c_2^2} S_{\mathcal{L}}\left(\frac{\delta_2}{K_2}\right) + \frac{c_1^2 c_2}{(1+c_1^2+c_2^2)(1+c_2^2)} S_{\mathcal{R}}\left(\frac{\delta_1-\delta_2}{K_2}\right) S_{\mathcal{L}}\left(\frac{\delta_1}{K_2}\right) \end{bmatrix} = \\
&= -\frac{c_1^2}{(1+c_1^2+c_2^2)(1+c_2^2)}\chi_{\left[\frac{\delta_1}{K_2},\infty\right)} - \frac{c_2^2}{1+c_2^2}\chi_{\left[\frac{\delta_2}{K_2},\infty\right)},
\end{aligned}$$

where the characteristic functions are taken with respect to time $t \geq 0$. Thanks to this

$$\begin{aligned}
&-\left(\mathbb{F}\left(I+\mathbb{F}^*\mathbb{F}\right)^{-1}\mathbb{F}^*\Psi x_0\right)(t) = \\
&= -\tfrac{c_1^2}{(1+c_1^2+c_2^2)(1+c_2^2)} e^{-K_2 t} x_0(\gamma - vt)\chi_{\left[\frac{\delta_1}{K_2},\frac{\gamma}{v}\right]}(t) - \tfrac{c_2^2}{1+c_2^2} e^{-K_2 t} x_0(\gamma - vt)\chi_{\left[\frac{\delta_2}{K_2},\frac{\gamma}{v}\right]}(t)
\end{aligned}$$

and, by (7.12),

$$\begin{aligned}
-\left(\Psi^*\mathbb{F}\left(I+\mathbb{F}^*\mathbb{F}\right)^{-1}\mathbb{F}^*\Psi x_0\right)(\theta) = &-\frac{c_1^2}{v(1+c_1^2+c_2^2)(1+c_2^2)} e^{-\frac{2K_2}{v}(\gamma-\theta)} x_0(\theta)\chi_{[0,\eta_1]}(\theta) - \\
&- \frac{c_2^2}{v(1+c_2^2)} e^{\frac{-2K_2}{v}(\gamma-\theta)} x_0(\theta)\chi_{[0,\eta_2]}(\theta).
\end{aligned}$$

Adding the observability gramian $\Psi^*\Psi x_0 = \mathcal{H}_\Psi x_0$, given by (7.1), we get

$$\begin{aligned}
\left(\mathcal{H}x_0\right)(\theta) &= \left(\mathcal{H}_\Psi x_0\right)(\theta) - \left(\Psi^*\mathbb{F}\left(I+\mathbb{F}^*\mathbb{F}\right)^{-1}\mathbb{F}^*\Psi x_0\right)(\theta) = \\
&= \frac{1}{v}\, e^{\frac{-2K_2}{v}(\gamma-\theta)} \left\{ \begin{array}{ccc} \frac{1}{1+c_1^2+c_2^2} & \text{on} & [0,\eta_1] \\ \frac{1}{1+c_2^2} & \text{on} & (\eta_1,\eta_2] \\ 1 & \text{on} & (\eta_2,\gamma] \\ 0 & \text{on} & (\gamma,a] \end{array} \right\} x_0(\theta).
\end{aligned}$$

Passing to computations for $\mathcal{G}$ we observe that

$$\mathcal{G}z := -\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z = \begin{bmatrix} \langle \mathcal{A}z, -\mathcal{H}d_1\rangle_{\mathrm{H}} - (\mathcal{C}d_1)\mathcal{C}z \\ \langle \mathcal{A}z, -\mathcal{H}d_2\rangle_{\mathrm{H}} - (\mathcal{C}d_2)\mathcal{C}z \end{bmatrix}, \qquad z \in D(\mathcal{A}),$$

because here

$$\mathcal{D} = \begin{bmatrix} d_1 & d_2 \end{bmatrix},\ \mathcal{D}^* = \begin{bmatrix} d_1^* \\ d_2^* \end{bmatrix};\ d_i(\theta) = -\frac{1}{v}e^{-\frac{K_2}{v}(\theta-\eta_i)}\chi_{[\eta_i,a]}(\theta),\ \theta \in [0,a],\ i = 1,2.$$

A slight modification of (7.3) (two components instead of one, each associated with a separate control) shows that $\mathcal{H}_\Phi \in \mathbf{L}(\mathrm{H})$, $\mathcal{H}_\Phi = \mathcal{H}_\Phi^* \geq 0$,

$$(\mathcal{H}_\Phi x)(\theta) := \frac{1}{v}\left[e^{\frac{2K_2}{v}(\eta_1-\theta)}\chi_{[\eta_1,a]}(\theta) + e^{\frac{2K_2}{v}(\eta_2-\theta)}\chi_{[\eta_2,a]}(\theta)\right]x(\theta), \quad x \in \mathrm{H},$$

solves the Lyapunov operator equation ($\mathcal{H}_\Phi$ is the *controllability gramian*)

$$\begin{aligned}
&\langle \mathcal{A}^* w, \mathcal{H}_\Phi w\rangle_{\mathrm{H}} + \langle w, \mathcal{H}_\Phi \mathcal{A}^* w\rangle_{\mathrm{H}} = \\
&= \int_{\eta_1}^{\eta_2} e^{\frac{2K_2}{v}(\eta_1-\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta}\left[w^2(\theta)\right]\mathrm{d}\theta - \frac{2K_2}{v}\int_{\eta_1}^{\eta_2} e^{\frac{2K_2}{v}(\eta_1-\theta)} w^2(\theta)\mathrm{d}\theta + \\
&\quad + \int_{\eta_2}^{a}\left[e^{\frac{2K_2}{v}(\eta_1-\theta)} + e^{\frac{2K_2}{v}(\eta_2-\theta)}\right]\frac{\mathrm{d}}{\mathrm{d}\theta}\left[w^2(\theta)\right]\mathrm{d}\theta - \\
&\quad - \frac{2K_2}{v}\int_{\eta_1}^{a}\left[e^{\frac{2K_2}{v}(\eta_1-\theta)} + e^{\frac{2K_2}{v}(\eta_2-\theta)}\right] w^2(\theta)\mathrm{d}\theta = \\
&= -w^2(\eta_1) - w^2(\eta_2) = -\left\|\mathcal{D}^*\mathcal{A}^* w\right\|_{\mathrm{U}}^2, \qquad w \in D(\mathcal{A}^*),
\end{aligned}$$

whence $\mathcal{D}$ is admissible, though the system is not (infinite-time) approximately controllable as $\ker \mathcal{H}_\Phi \neq \{0\}$.

Since

$$\begin{aligned}
&(\mathcal{C}d_i)^*\mathcal{C}z = -\,c_i z(\gamma), \quad i = 1, 2, \\
&\langle \mathcal{A}z, \mathcal{H}d_1\rangle_{\mathrm{H}} = \frac{c_1}{1+c_2^2}\int_{\eta_1}^{\eta_2}\left[\frac{K_2}{v}z(\theta) + z'(\theta)\right]e^{\frac{K_2}{v}(\theta-\gamma)}\mathrm{d}\theta + \\
&\qquad + c_1\int_{\eta_2}^{\gamma}\left[\frac{K_2}{v}z(\theta) + z'(\theta)\right]e^{\frac{K_2}{v}(\theta-\gamma)}\mathrm{d}\theta = \\
&\qquad = \frac{vc_1}{1+c_2^2}\left[c_2 z(\eta_2) - c_1 z(\eta_1)\right] + c_1\left[z(\gamma) - vc_2 z(\eta_2)\right], \\
&\langle \mathcal{A}z, \mathcal{H}d_2\rangle_{\mathrm{H}} = c_2\int_{\eta_2}^{\gamma}\left[\frac{K_2}{v}z(\theta) + z'(\theta)\right]e^{\frac{K_2}{v}(\theta-\gamma)}\mathrm{d}\theta = c_2\left[z(\gamma) - vc_2 z(\eta_2)\right],
\end{aligned}$$

then

$$\mathcal{G}z = \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}\, z(\eta_1) + \frac{vc_1c_2^3}{1+c_2^2}\, z(\eta_2) \\ vc_2^2\, z(\eta_2) \end{bmatrix}, \qquad z \in D(\mathcal{A}) = \mathrm{W}_0^{1,2}(0,a). \tag{8.2}$$

The last step is to determine the optimal feedback controller using (2.11). For that we need

$$R_- = I + (\mathcal{C}\mathcal{D})^*(\mathcal{C}\mathcal{D}) = \begin{bmatrix} 1+c_1^2 & c_1c_2 \\ c_1c_2 & 1+c_2^2 \end{bmatrix} \geq I$$

and an extension $\mathcal{G}_\Lambda$. It is enough to determine an extension onto $\mathrm{Reg}[0,a]$. Let $z \in \mathrm{Reg}[0,a]$. Then, by (7.4), (8.2) and the Lebesgue dominated convergence theorem:

$$\begin{aligned}
&\lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI-\mathcal{A})^{-1}z = \\
&= \lim_{s\to\infty, s\in\mathbb{R}} s \begin{bmatrix} \frac{c_1^2}{1+c_2^2}\int\limits_0^{\eta_1} e^{-\frac{s+K_2}{v}(\eta_1-\xi)}z(\xi)\mathrm{d}\xi + \frac{c_1c_2^3}{1+c_2^2}\int\limits_0^{\eta_2} e^{-\frac{s+K_2}{v}(\eta_2-\xi)}z(\xi)\mathrm{d}\xi \\ c_2^2\int\limits_0^{\eta_2} e^{-\frac{s+K_2}{v}(\eta_2-\xi)}z(\xi)\mathrm{d}\xi \end{bmatrix} = \\
&= \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}\lim\limits_{s\to\infty, s\in\mathbb{R}} I_1(s) + \frac{vc_1c_2^3}{1+c_2^2}\lim\limits_{s\to\infty, s\in\mathbb{R}} I_2(s) \\ vc_2^2 \lim\limits_{s\to\infty, s\in\mathbb{R}} I_2(s) \end{bmatrix} = \\
&= \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}z(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}z(\eta_2-) \\ vc_2^2 z(\eta_2-) \end{bmatrix} := \mathcal{G}_\Lambda z,
\end{aligned} \tag{8.3}$$

where

$$I_i(s) := \frac{s}{s+K_2}\int\limits_0^\infty e^{-t} z\left(\eta_i - \tfrac{vt}{s+K_2}\right)\mathbb{1}\left(\tfrac{\eta_i(s+K_2)}{v} - t\right)\mathrm{d}t, \qquad i=1,2.$$

Since $d_i \in \mathrm{Reg}[0,a]$, $i=1,2$, then

$$\begin{aligned}
\mathcal{G}_\Lambda\mathcal{D} &= \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}d_1(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}d_1(\eta_2-) & \frac{vc_1^2}{1+c_2^2}d_2(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}d_2(\eta_2-) \\ vc_2^2 d_1(\eta_2-) & vc_2^2 d_2(\eta_2-) \end{bmatrix} = \\
&= \begin{bmatrix} -\frac{c_1^2c_2^2}{1+c_2^2} & 0 \\ -c_1c_2 & 0 \end{bmatrix}
\end{aligned}$$

and, by (2.11),

$$\begin{aligned}
&u = -(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda x = \\
&= -\begin{bmatrix} \frac{1+c_1^2+c_2^2}{1+c_2^2} & c_1c_2 \\ 0 & 1+c_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}x(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}x(\eta_2-) \\ vc_2^2 x(\eta_2-) \end{bmatrix} = \\
&= -\frac{1}{1+c_1^2+c_2^2}\begin{bmatrix} 1+c_2^2 & -c_1c_2 \\ 0 & \frac{1+c_1^2+c_2^2}{1+c_2^2} \end{bmatrix} \begin{bmatrix} \frac{vc_1^2}{1+c_2^2}x(\eta_1-) + \frac{vc_1c_2^3}{1+c_2^2}x(\eta_2-) \\ vc_2^2 x(\eta_2-) \end{bmatrix} = \\
&= -\begin{bmatrix} \frac{vc_1^2}{1+c_1^2+c_2^2}\ x(\eta_1-) \\ \frac{vc_2^2}{1+c_2^2}\ x(\eta_2-) \end{bmatrix}.
\end{aligned} \tag{8.4}$$

This controller has astonishingly simple realization, depicted in Figure 5, in the

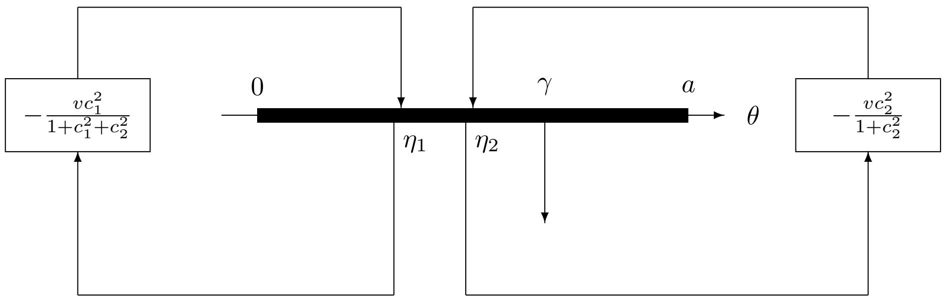


**Fig. 5.** Optimal feedback controller realization in the MISO case

form of two separated loops of negative proportional feedback control, though the gain coefficient of the first controller depends on $\eta_2$, i.e., the position around which the second loop operates.

To verify that our feedback controller is correctly constructed we determine the full state

$$x(t)(\theta) = \left\{ \begin{array}{cl} e^{-K_2 t} x_0(\theta - vt) & \text{if} \quad \frac{a}{v} \geq \frac{\theta}{v} \geq t \\ 0 & \text{if} \quad \frac{\theta}{v} < t \end{array} \right\} +$$

$$+ \left\{ \begin{array}{cl} \frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta_1)} \chi_{[\eta_1,a]}(\theta) u_1\left(t - \frac{\theta-\eta_1}{v}\right) & \text{if} \quad t \geq \frac{\theta-\eta_1}{v} \\ 0 & \text{if} \quad t < \frac{\theta-\eta_1}{v} \end{array} \right\} +$$

$$+ \left\{ \begin{array}{cl} \frac{1}{v} e^{-\frac{K_2}{v}(\theta-\eta_2)} \chi_{[\eta_2,a]}(\theta) u_2\left(t - \frac{\theta-\eta_2}{v}\right) & \text{if} \quad t \geq \frac{\theta-\eta_2}{v}, \\ 0 & \text{if} \quad t < \frac{\theta-\eta_2}{v}. \end{array} \right.$$

This is because the first components represents $(S(t)x_0)(\theta)$ while the remaining terms, representing the non-homogeneous part of the state have the Laplace transform

$$(\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D})(\theta)\hat{u}(s) = \frac{1}{v}\, e^{-\frac{K_2}{v}(\theta-\eta_1)} \chi_{[\eta_1,a]}(\theta) e^{-\frac{s}{v}(\theta-\eta_1)} \hat{u}_1(s) + \\ + \frac{1}{v}\, e^{-\frac{K_2}{v}(\theta-\eta_2)} \chi_{[\eta_2,a]}(\theta) e^{-\frac{s}{v}(\theta-\eta_2)} \hat{u}_2(s),$$

Thus if $x_0 \in \operatorname{Reg}[0,a]$ then one has $S(t)x_0 \in \operatorname{Reg}[0,a]$ for every $t \geq 0$ and the optimal feedback controller equation (8.4) yields

$$u = \left[ \begin{array}{c} -\frac{vc_1^2}{1+c_1^2+c_2^2} \left\{ \begin{array}{cl} e^{-K_2 t} x_0(\eta_1 - vt) & \text{if} \quad \frac{\eta_1}{v} \geq t \\ 0 & \text{if} \quad \frac{\eta_1}{v} < t \end{array} \right\} \\ -\frac{vc_2^2}{1+c_2^2} [❶ + ❷] \end{array} \right].$$

$$❶ := \begin{cases} e^{-K_2 t} x_0(\eta_2 - vt) & \text{if} \quad \frac{\eta_2}{v} \geq t, \\ 0 & \text{if} \quad \frac{\eta_2}{v} < t, \end{cases}$$

$$❷ := \begin{cases} \frac{1}{v} e^{-\frac{K_2}{v}(\eta_2-\eta_1)} u_1\left(t - \frac{\eta_2-\eta_1}{v}\right) & \text{if} \quad t \geq \frac{\eta_2-\eta_1}{v}, \\ 0 & \text{if} \quad t < \frac{\eta_2-\eta_1}{v}, \end{cases}$$

where $u_1$ denotes the first component of $u$, we already determined, whence

$$u_1\left(t - \frac{\eta_2 - \eta_1}{v}\right) = -\frac{vc_1^2}{1 + c_1^2 + c_2^2} \begin{cases} e^{\frac{K_2}{v}(\eta_2-\eta_1)} e^{-K_2 t} & \text{if} \quad \frac{\eta_2}{v} \geq t, \\ 0 & \text{if} \quad \frac{\eta_2}{v} < t, \end{cases}$$

and short calculations confirm (8.1).

**Remark 8.1** (Limit passage from MISO to SISO case)**.** Taking $\eta_1 \to -\infty$, which implies $\delta_1 \to \infty$, $c_1 \to 0$ and fixing $\eta_2 = \eta$, $\delta_2 = \delta$, $c_2 = c = \frac{1}{v}e^{-\delta}$ we get $u_1 = 0$ and $u_2 = u$, where $u$ is the optimal control or controller in the SISO case.

Let $x \in \mathrm{Reg}[0, a]$ and $u$ be the optimal controller (8.4). A discussion of conditions ensuring that $z = x + \mathcal{D}u \in D(\mathcal{A})$ (in particular, $z = x + \mathcal{D}u$ must be continuous on [0,a]), leads to the following form of the closed-loop state operator:

$$\begin{aligned} \mathcal{A}_{\mathrm{opt}} x &= -vz' - K_2 z, \\ z(\theta) &:= x(\theta) + \frac{c_1^2}{1 + c_1^2 + c_2^2} x(\eta_1 -) e^{-\frac{K_2}{v}(\theta-\eta_1)} \chi_{[\eta_1, a]} + \\ &+ \frac{c_2^2}{1 + c_2^2} x(\eta_2 -) e^{-\frac{K_2}{v}(\theta-\eta_2)} \chi_{[\eta_2, a]}, \\ D(\mathcal{A}_{\mathrm{opt}}) &= \Big\{ x \in \mathrm{H} : z \in \mathrm{W}_0^{1,2}[0, a], \ \ x(\eta_1 +) = \frac{1 + c_2^2}{1 + c_1^2 + c_2^2} x(\eta_1 -), \\ &\qquad x(\eta_2 +) = \frac{1}{1 + c_2^2} x(\eta_2 -) \Big\}. \end{aligned}$$

Hence (on $\theta$-intervals $[0, \eta_1]$, $[\eta_1, \eta_2]$, $[\eta_2, a]$ there holds $(\mathcal{A}_{\mathrm{opt}} x)(\theta) = = -vx'(\theta) - k_2 x(\theta))$

$$\begin{aligned} & x \in D(\mathcal{A}_{\mathrm{opt}}) \Longrightarrow \langle x, \mathcal{A}_{\mathrm{opt}} x \rangle_{\mathrm{H}} + \langle \mathcal{A}_{\mathrm{opt}} x, x \rangle_{\mathrm{H}} = \langle x, \mathcal{A} z \rangle_{\mathrm{H}} + \langle \mathcal{A} z, x \rangle_{\mathrm{H}} = \\ & = -vx^2(\eta_1 -) - vx^2(\eta_2 -) - vx^2(a) + vx^2(\eta_1 +) + vx^2(\eta_2 +) - 2K_2 \|x\|_{\mathrm{H}}^2 = \\ & = -\tfrac{vc_2^2(c_2^2+2)}{(1+c_2^2)^2} x^2(\eta_1 -) - \tfrac{vc_1^2(c_1^2+2c_2^2+2)}{(1+c_1^2+c_2^2)^2} x^2(\eta_2 -) - vx^2(a) - 2K_2 \|x\|_{\mathrm{H}}^2, \end{aligned}$$

and

$$x(\theta) = \begin{cases} \dfrac{1}{v} \int\limits_0^\theta e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}} X(\xi) \mathrm{d}\xi & \text{if} \quad 0 \leq \theta < \eta_1, \\ \spadesuit & \text{if} \quad \eta_1 \leq \theta < \eta_2, \\ \clubsuit & \text{if} \quad \eta_2 \leq \theta \leq a, \end{cases} \tag{8.5}$$

where

$$\spadesuit \;=\; \frac{1+c_2^2}{v(1+c_1^2+c_2^2)}\int\limits_0^{\eta_1} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi+\frac{1}{v}\int\limits_{\eta_1}^{\theta} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi,$$

$$\clubsuit \;=\; \frac{1}{v(1+c_1^2+c_2^2)}\int\limits_0^{\eta_1} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi+\frac{1}{v(1+c_2^2)}\int\limits_{\eta_1}^{\eta_2} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi+ +\frac{1}{v}\int\limits_{\eta_2}^{\theta} e^{-\frac{(K_2+\lambda)(\theta-\xi)}{v}}X(\xi)\mathrm{d}\xi,$$

solves the resolvent equation $\lambda x - \mathcal{A}_{\mathrm{opt}}x = X$ which, by the Lummer-Phillips theorem, implies that $\mathcal{A}_{\mathrm{opt}}$ generates an EXS semigroup on H. Moreover, since

$$x \in D(\mathcal{A}_{\mathrm{opt}}) \Longrightarrow$$
$$\Longrightarrow \langle x, \mathcal{A}_{\mathrm{opt}}x\rangle_{\mathrm{H}} + \langle \mathcal{A}_{\mathrm{opt}}x, x\rangle_{\mathrm{H}} \le -\frac{vc_2^2(c_2^2+2)}{(1+c_2^2)^2}x^2(\eta_1-) - \frac{vc_1^2(c_1^2+2c_2^2+2)}{(1+c_1^2+c_2^2)^2}x^2(\eta_2-)$$

then, by Lyapunov characterization of admissibility, the functionals $x \longmapsto x(\eta_1-)$, $x \longmapsto x(\eta_2-)$ are admissible with respect to the semigroup genarated by $\mathcal{A}_{\mathrm{opt}}$, which confirms that the optimal control is in $\mathrm{L}^2(0,\infty;\mathrm{U})$.

Now (8.5) defines the resolvent of $\mathcal{A}_{\mathrm{opt}}$. Thus substituting $t = \frac{\theta-\xi}{v}$ in (8.5) and applying the definition of Laplace transformation, we obtain

$$(S_{\mathrm{opt}}(t)X)(\theta) = e^{-K_2 t}\begin{cases} X(\theta-vt) \text{ if } 0 \le t \le \frac{\theta}{v}, & 0 \le \theta < \eta_1,\\ \frac{1+c_2^2}{1+c_1^2+c_2^2}X(\theta-vt) \text{ if } \frac{\theta-\eta_1}{v} \le t \le \frac{\theta}{v}, & \eta_1 \le \theta < \eta_2,\\ X(\theta-vt) \text{ if } 0 \le t \le \frac{\theta-\eta_1}{v}, & \eta_1 \le \theta < \eta_2,\\ \frac{1}{(1+c_2^2)(1+c_1^2+c_2^2)}X(\theta-vt) \text{ if } \frac{\theta-\eta_1}{v} \le t \le \frac{\theta}{v}, & \eta_2 \le \theta \le a,\\ \frac{1}{1+c_2^2}X(\theta-vt) \text{ if } \frac{\theta-\eta_2}{v} \le t \le \frac{\theta-\eta_1}{v}, & \eta_2 \le \theta \le a,\\ X(\theta-vt) \text{ if } 0 \le t \le \frac{\theta-\eta_2}{v}, & \eta_2 \le \theta \le a,\\ 0 \quad \text{elsewhere}, \end{cases}$$

from which we deduce that actually the semigroup $\{S_{\mathrm{opt}}(t)\}_{t\ge 0}$ decays to 0 is a natural finite time $a/v$. The rate of decaying of $\{S_{\mathrm{opt}}(t)\}_{t\ge 0}$ is for $\theta \ge \eta$ faster than that of $\{S(t)\}_{t\ge 0}$ given by (7.5).

Finally, we shall find $\mathcal{G}_\Lambda$ using the method of spectral factorization and compare it with (8.3). Here $\hat{G}(s) = \begin{bmatrix} c_1 e^{-s\frac{\delta_1}{K_2}} & c_2 e^{-s\frac{\delta_2}{K_2}} \end{bmatrix}$ and therefore the Popov spectral function reads as

$$\Pi(j\omega) = I + \hat{G}^*(-j\omega)\hat{G}(j\omega) = \begin{bmatrix} 1+c_1^2 & c_1c_2e^{j\omega\frac{\delta_1-\delta_2}{K_2}} \\ c_1c_2e^{-j\omega\frac{\delta_1-\delta_2}{K_2}} & 1+c_2^2 \end{bmatrix} \ge I.$$

The lower triangular matrix

$$\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U})) \ni \Xi(s) = \begin{bmatrix} a & 0 \\ be^{-s\frac{\delta_1-\delta_2}{K_2}} & c \end{bmatrix} \longrightarrow \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} := D \text{ as } s \to \infty, s \in \mathbb{R},$$

is a spectral factor of $\Pi(j\omega)$, i.e., $\Xi^T(-j\omega)\Xi(-j\omega) = \Pi(j\omega)$ iff $(a,b,c)$ solves the system of equations:

$$a^2 + b^2 = 1 + c_1^2, \quad bc = c_1 c_2, \quad c^2 = 1 + c_2^2,$$

and moreover, then

$$\Xi^{-1}(s) = \frac{1}{ac}\begin{bmatrix} c & 0 \\ -be^{-s\frac{\delta_1-\delta_2}{K_2}} & a \end{bmatrix} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U})).$$

Assuming $\mathcal{G}_\Lambda = \begin{bmatrix} \mathcal{G}_\Lambda^1 \\ \mathcal{G}_\Lambda^2 \end{bmatrix}$ we establish that here the realization equation (4.1) is

$$\begin{aligned}\hat{G}_{\mathcal{G}}(s) &= \mathcal{G}_\Lambda \mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D} = \\ &= \begin{bmatrix} \mathcal{G}_\Lambda^1 \\ \mathcal{G}_\Lambda^2 \end{bmatrix} \begin{bmatrix} \frac{1}{v} e^{-\frac{s+K_2}{v}(\theta-\eta_1)}\chi_{[\eta_1,a]} & \frac{1}{v} e^{-\frac{s+K_2}{v}(\theta-\eta_2)}\chi_{[\eta_2,a]} \end{bmatrix} = \\ &= \Xi^*(0)\left[\Xi(s) - D\right] = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ be^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \end{bmatrix} = \begin{bmatrix} \frac{c_1^2 c_2^2}{1+c_2^2} e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \\ c_1 c_2 e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \end{bmatrix}.\end{aligned}$$

For $\mathcal{G}_\Lambda^1 = m_1 x(\eta_2-)$, $\mathcal{G}_\Lambda^2 = m_2 x(\eta_2-)$ we obtain

$$\begin{bmatrix} \frac{m_1 c_1}{v c_2} e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \\ \frac{m_2 c_1}{v c_2} e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \end{bmatrix} = \begin{bmatrix} \frac{c_1^2 c_2^2}{1+c_2^2} e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \\ c_1 c_2 e^{-s\frac{\delta_1-\delta_2}{K_2}} & 0 \end{bmatrix},$$

whence

$$m_1 = \frac{v c_1 c_2^3}{1 + c_2^2}, \quad m_2 = v c_2^2, \qquad \mathcal{G}_\Lambda = \begin{bmatrix} \frac{v c_1 c_2^3}{1+c_2^2} x(\eta_2-) \\ v c_2^2 x(\eta_2-) \end{bmatrix},$$

which differs from (8.3). The difference is caused by the fact that the pair $(\mathcal{A}, \mathcal{D})$ is not *approximately controllable* and therefore the realization equation returns $\mathcal{G}_\Lambda$ only partially. Nevertheless, the approximately controllable part of (8.3) coincides with $\mathcal{G}_\Lambda$ found via the method of spectral factorization.

## 9. CONCLUSIONS AND DISCUSSION

In this paper we have presented a solution of the optimal control/controller synthesis to a rather general lq-problem with infinite-time horizon for the class of infinite-dimensional systems in factor form, governed by (1.1). Complete solutions of two exemplary standard lq-problems, having physical meaning, met in the literature, have been provided. The problem of calculating optimal control/controller in the MIMO case of two measurements and two aerators has been addressed to and solved by my Ph.D. student Elżbieta Żołopa.

Basic assumptions where EXS of the semigroup generated by the state operator $\mathcal{A}$ and (**A1**), (**A2**) and (**A3**). There still exist systems governed by (1.1) to which our theory does not apply, e.g., to the electric $\mathfrak{RC}$-transmission line for which $\mathrm{H} = \mathrm{L}^2(0,\infty)$, $\mathrm{U} = \mathbb{R} = \mathrm{Y}$, the state operator reads as

$$\mathcal{A}x = x'', \qquad D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1) :\ x'(1) = 0, x(0) = 0\},$$

$\mathcal{A} = \mathcal{A}^* < 0$, so $\mathcal{A}$ generates an EXS analytic semigroup on H; the observation functional $\mathcal{C}x = x(1)$ is admissible, so (**A1**) holds. It is proved in [9, Lemma 5.2, p. 27] that the factor control vector $\mathcal{D} = d = \mathbf{1}$ is not admissible, so (**A2**) is not satisfied, though the system transfer function $\hat{G}(s) = \mathcal{C}\mathcal{D} - s\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{D} = \frac{1}{\cosh\sqrt{s}}$ is in $\mathrm{H}^\infty(\mathbb{C}^+)$ with $\|\hat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+)} = 1$.

Our solution of the lq-problem is given separately in time-domain and frequency-domain domains, contrary to [24] where a mixture of these both aspects has been used. Moreover, our method of derivation of the solution to lq-problems is different and is close to Bellman's dynamical programming whilst in [24] an idea of Pontryagin's maximum principle has been exploited – see Comments B.2 in Appendix B for more details.

If the state operator $\mathcal{A}$ in (1.1) does not generate an EXS $\mathrm{C}_0$-semigroup then one may look for an output static negative feedback exponentially stabilizing control $u = -Ky$, dictated by an operator $K \in \mathbf{L}(\mathrm{Y},\mathrm{U})$ such that

$$\begin{aligned}\dot{x} &= \mathcal{A}[x + \mathcal{D}u] = \mathcal{A}[(x - \mathcal{D}K\mathcal{C}x) + (\mathcal{D}u + \mathcal{D}K\mathcal{C}x)] = \\ &= \underbrace{\mathcal{A}(I - \mathcal{D}K\mathcal{C})}_{:=\mathcal{A}_{\mathrm{new}}}[x + \underbrace{(I - \mathcal{D}K\mathcal{C})^{-1}\mathcal{D}}_{:=\mathcal{D}_{\mathrm{new}}}\underbrace{(u + K\mathcal{C}x)}_{:=u_{\mathrm{new}}}],\end{aligned}$$

where now $\mathcal{A}_{\mathrm{new}}$ generates an EXS $\mathrm{C}_0$-semigroup on H and the triple $(\mathcal{A}_{\mathrm{new}}, \mathcal{D}_{\mathrm{new}}, \mathcal{C})$ satifies the assumptions (**A1**), (**A2**) and (**A3**). EXS implies $\ker(I - \mathcal{D}K\mathcal{C}) = \{0\}$, so $I - \mathcal{D}K\mathcal{C}$ is (unboundedly) invertible. Next, if $\ker(I - \mathcal{C}\mathcal{D}K) = \{0\}$ then $\mathcal{D}_{\mathrm{new}}$ is well defined, and

$$\mathcal{D}_{\mathrm{new}}v = \mathcal{D}v + \mathcal{D}K(I - \mathcal{C}\mathcal{D}K)^{-1}\mathcal{C}\mathcal{D}v.$$

Since the state $x$ and, consequently, the output $y = \mathcal{C}x$ of the original and the transformed system are the same, it is clear that the problem of minimizing $J$ for the original system reduces to the problem of minimizing the performance index

$$\begin{aligned}&J(x_0, u_{\mathrm{new}}) = \\ &= \int_0^\infty \begin{bmatrix} y(t) \\ u_{\mathrm{new}}(t) - Ky(t)\end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R\end{bmatrix} \begin{bmatrix} y(t) \\ u_{\mathrm{new}}(t) - Ky(t)\end{bmatrix} \mathrm{d}t = \\ &= \int_0^\infty \begin{bmatrix} y(t) \\ u_{\mathrm{new}}(t)\end{bmatrix}^* \begin{bmatrix} Q - NK - K^*N^* + K^*RK & N - K^*R \\ N^* - RK & R\end{bmatrix} \begin{bmatrix} y(t) \\ u_{\mathrm{new}}(t)\end{bmatrix} \mathrm{d}t\end{aligned}$$

for the transformed system.

The proposed approach can be illustrated by the following example. Let $\mathrm{H} = \mathrm{L}^2(0,1)$, $\mathcal{A}x = -x'$, $D(\mathcal{A}) = \{x \in \mathrm{W}^{1,2}(0,1) : \ x(0) = 2x(1)\}$. This operator is $\omega = \ln 2 (> 0)$-dissipative in the equivalent scalar product $\langle x_1, x_2 \rangle_{\mathrm{e}} = \langle x_1, \mathcal{M}x_2 \rangle_{\mathrm{H}}$ induced by the operator of multiplication $(\mathcal{M}x)(\theta) := e^{2\theta \ln 2} x(\theta)$, because

$$\langle x, \mathcal{A}x \rangle_{\mathrm{e}} + \langle \mathcal{A}x, x \rangle_{\mathrm{e}} = \langle x, \mathcal{M}\mathcal{A}x \rangle_{\mathrm{H}} + \langle \mathcal{A}x, \mathcal{M}x \rangle_{\mathrm{H}} = 2 \ln 2 \, \|x\|_{\mathrm{e}}^2, \qquad \forall x \in D(\mathcal{A}).$$

Furthermore, $R(\lambda I - \mathcal{A}) = \mathrm{H}$ for $\mathrm{Re}\, \lambda > \ln 2$, and by the Lumer-Phillips theorem, $\mathcal{A}$ generates a $\mathrm{C}_0$-semigroup on H. This semigroup is neither EXS nor uniformly bounded as $\ln 2$ is an eigenvalue of $\mathcal{A}$, corresponding to its eigenvector $e^{\theta \ln 2}$. $\mathrm{Y} = \mathbb{R} = \mathrm{U}$, so the observation operator $\mathcal{C}$ is a functional, i.e., $\mathcal{C} = c^{\#}$, and here $c^{\#}x = x(1)$, $D(c^{\#}) = \{x \in \mathrm{H} : \ x \text{ is continuous at } \theta = 1\}$; the factor control operator $\mathcal{D}$ is a vector, i.e., $\mathcal{D} = d$, and here $d = \mathbf{1} \in \mathrm{H}$.

Thus for $K = 2$ one has $(I - \mathcal{D}K\mathcal{C})x = x - Kdc^{\#}x = x - 2 \cdot \mathbf{1}x(1)$, $(I - \mathcal{C}\mathcal{D}K)^{-1} = (1 - Kc^{\#}d)^{-1} = -1$, $d_{\mathrm{new}} = -\mathbf{1}$ and $\mathcal{A}_{\mathrm{new}}$ can equivalently be written as $\mathcal{A}_{\mathrm{new}}x = -x'$, $D(\mathcal{A}_{\mathrm{new}}) = \mathrm{W}_0^{1,2}(0,1)$. Now, the results of Section 7.1 apply with $a = 1$, $v = 1$, $K_2 = 0$, $\gamma = 1$ and $\eta = 0$ to show that $\mathcal{A}_{\mathrm{new}}$ generates an EXS semigroup of right-shifts (7.5) (recall that this semigroup decays to zero in a finite time) and (**A1**), (**A2**), (**A3**) hold, where the gramians are $\mathcal{H}_\Psi = \mathcal{H}_\Phi = I$ and the system transfer functions is $\hat{G}(s) = e^{-s}$. The standard lq-problem for the original system reduces to the lq-problem with $Q = 5$, $N = -2$ and $R = 1$, whence the Popov spectral function is

$$\Pi(j\omega) = 1 - 4\, \mathrm{Re}\, \hat{G}(j\omega) + 5|\hat{G}(j\omega)|^2 = 6 - 4 \cos \omega \geq 2.$$

Observe that, contrary to Section 7.1, the system gramians are coercive, whence in particular, the pair $(\mathcal{A}_{\mathrm{new}}, d)$ is approximately controllable, so the method of spectral factorization is applicable. Take its spectral factor

$$\Xi(s) = \frac{\sqrt{2} + \sqrt{10}}{2} + \frac{\sqrt{2} - \sqrt{10}}{2}\, e^{-s}, \qquad s \mapsto \Xi(s),\ s \mapsto \Xi^{-1}(s) \in \mathrm{H}^\infty(\mathbb{C}^+).$$

Then $V = \Xi(0) = \sqrt{2}$, and (3.4) yields

$$\mathcal{G}x = (1 - \sqrt{5})x(1), \qquad \forall x \in D(\mathcal{A}_{\mathrm{new}}).$$

It is enough to consider its extension

$$\mathcal{G}_\Lambda x = (1 - \sqrt{5})x(1) \Longrightarrow \mathcal{F}_\Lambda x = \frac{3 - \sqrt{5}}{2} x(1), \qquad x \in D(c^{\#}),$$

and the optimal controller for the transformed system is $u = \mathcal{F}_\Lambda x$ (whence the optimal controller for the original system is $u = -\frac{1+\sqrt{5}}{2} x(1)$), whilst the optimal closed-loop system reads as

$$\dot{x} = \mathcal{A}_{\mathrm{opt}} x = \mathcal{A}_{\mathrm{new}}(x + d_{\mathrm{new}} \mathcal{F}_\Lambda x) = -\left[ x - \mathbf{1} \frac{3 - \sqrt{5}}{2} x(1) \right]' = -x',$$

$$D(\mathcal{A}_{\mathrm{opt}}) = \left\{ x \in \mathrm{W}^{1,2}(0,1) : \ x(0) = \frac{3 - \sqrt{5}}{2} x(1) \right\},$$

and again $\mathcal{A}_{\rm opt}$ is $\omega = \ln\frac{3-\sqrt{5}}{2}(<0)$-dissipative with respect to the equivalent scalar product $\langle x_1, x_2\rangle_{\rm E} = \langle x_1, \mathcal{M}x_2\rangle_{\rm H}$, but now induced by the operator of multiplication $(\mathcal{M}x)(\theta) := e^{2\theta\ln\frac{3-\sqrt{5}}{2}}x(\theta)$.

Part (I) of Theorem 2.3 can also be proved using the reciprocality approach. This requires some modifications of the reasoning used in [11, Proof of Lemma 5.1, pp. 3074–3077].

A derivation of the closed-loop system Lyapunov/Riccati operator equation in the general case of implicitly given optimal control remains an open problem.

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(1) *focused the author's attention on the right coprime representation of the transfer function factorization applied in Section 6 to arrange the spectral factorization,*
(2) *encouraged the author to solve the example of Chapelon and Xu in the general case by completing the author's solution of the case* $\beta = 0$ *with a more informative case of* $\alpha = \beta$ *(here* $\mathfrak{m} = \mathfrak{m}_B$*, where* $\mathfrak{m}_B = \sqrt{\frac{2+\alpha^2+\sqrt{4+\alpha^4}}{2}}$, $\mathfrak{n} = \frac{\alpha}{\mathfrak{m}}$, $\mathfrak{p} = \mathfrak{n}$ *and* $\mathfrak{q} = \mathfrak{m}$*),*
(3) *sent his full solution of the problem, confirming the results presented in Section 6. In Callier's solution* $\mathfrak{mq}$ *is in an elegant way expressed by* $\det[I + \mathbf{M}_*(0)\mathbf{M}(0)] = \mu - 2\alpha(\alpha+\beta)$ *rather than* $\mu$ *and* $\Delta$*. By inverting the Laplace transformations dictated by the resolvents of* $\mathcal{A}$ *and* $\mathcal{A}^*$ *he established explicit forms of* $\{S(t)\}_{t\geq 0}$, $\{S^*(t)\}_{t\geq 0}$ *and verified their EXS. Finally, by direct calculations, i.e., without invoking the general result of Theorem 2.3, he showed that the closed-loop operator* $\mathcal{A}_{\rm opt}$ *generates also an EXS semigroup* $\{S_{\rm opt}(t)\}_{t\geq 0}$*.*

## A. AUXILIARY THEOREMS

**Theorem A.1** (Paley-Wiener)**.** *Let* X *be a Hilbert space. Then the map* $f \longmapsto \hat{f}\big|_{\mathbb{C}^+}$ *is isometric isomorphism of* $\mathrm{L}^2(0,\infty;\mathrm{X})$ *onto* $\mathrm{H}^2(\mathbb{C}^+,\mathrm{X})$*. Moreover, for* $f \in \mathrm{L}^2(0,\infty;\mathrm{X})$*,*

$$\hat{f}(\alpha + js) = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{(\mathcal{F}f)(r)}{\alpha^2+(s-r)^2}\mathrm{d}r.$$

*As* $\alpha \searrow 0$, $\left\|\hat{f}(\alpha+js) - (\mathcal{F}f)(s)\right\| \to 0$ *(s)-almost everywhere, and*

$$\int_{-\infty}^{\infty}\left\|\hat{f}(\alpha+js) - (\mathcal{F}f)(s)\right\|_{\mathrm{X}}^2 \mathrm{d}s \to 0.$$

*Here $\hat{f}$ is the Laplace transform of $f$ and $\mathcal{F}f$ is its Fourier transform, whilst* $\mathrm{H}^2(\mathbb{C}^+, \mathrm{X})$ *stands for the space of holomorphic functions* $g : \mathbb{C}_+ \ni s \longmapsto g(s) \in \mathrm{X}$ *such that*

$$\|g\|^2_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{X})} = \sup_{\alpha>0} \int_{-\infty}^{\infty} \left\|\hat{f}(\alpha + js)\right\|^2_{\mathrm{X}} \mathrm{d}s < \infty.$$

**Theorem A.2** (Phragmén-Lindelöf)**.** *Let* $\varphi \in (0, \frac{\pi}{2}]$, $\Sigma_\varphi := \{z \in \mathbb{C} : |\arg z| < \varphi\}$ *and let* $h : \overline{\Sigma}_\varphi \longrightarrow \mathrm{X}$ *be continuous on* $\overline{\Sigma}_\varphi$ *and holomorphic in* $\Sigma_\varphi$*. Set* $\alpha := \frac{\pi}{2\varphi}$*. Assume that for all* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon$ *such that*

$$\|h(z)\| \leq C_\varepsilon e^{\varepsilon|z|^\alpha} \qquad (z \in \Sigma_\varphi).$$

*If* $\left\|h(re^{\pm j\varphi})\right\| \leq M$ *for all* $r > 0$*, then* $\|h(z)\| \leq M$ *for all* $z \in \Sigma_\varphi$*.*

**Theorem A.3** (Devinatz-Shinbrot)**.** *Let* $\mathfrak{H}$ *be any Hilbert space and* $\mathfrak{P}$ *any closed, linear subspace. If* $A \in \mathbf{L}(\mathfrak{H})$ *and* $P$ *is orthogonal projection on* $\mathfrak{P}$*, then* $T_P(A) := PA|_{\mathfrak{P}}$ *is a Toeplitz operator. Let* $A$ *be invertible. Then,* $T_P(A)$ *is invertible iff* $A$ *can be factored in the form* $A = A_-A_+$*, where* $A_-$*,* $A_+$ *are bounded invertible operators,* $A_-$ *takes* $\mathfrak{P}^\perp$ *onto itself,* $A_+$ *takes* $\mathfrak{P}$ *onto itself. Moreover, if* $T_P(A)$ *is invertible, then* $[T_P(A)]^{-1} = A_+^{-1}PA_-^{-1}|_{\mathfrak{P}}$*.*

## B. PROOFS OF THE MAIN RESULTS

*Proof of Lemma 2.1. Part 1.* Indeed, $x$ is uniformly continuous because $t \longmapsto R_t u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ is uniformly continuous [10, p. 1394] and, by (**A2**), $\Phi \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H})$. Now, if $w \in D(\mathcal{A}^*)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle w, x(t)\rangle_{\mathrm{H}} =&\frac{\mathrm{d}}{\mathrm{d}t}\langle w, S(t)x_0\rangle_{\mathrm{H}} + \frac{\mathrm{d}}{\mathrm{d}t}\langle w, \mathcal{A}\mathcal{W}R_t u\rangle_{\mathrm{H}} = \\ =&\langle S^*(t)\mathcal{A}^*w, x_0\rangle_{\mathrm{H}} + \left\langle \mathcal{A}^*w, \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}R_t u\right\rangle_{\mathrm{H}} = \\ =&\langle \mathcal{A}^*w, S(t)x_0\rangle_{\mathrm{H}} + \langle \mathcal{A}^*w, \mathcal{A}\mathcal{W}R_t u + \mathcal{D}u(t)\rangle_{\mathrm{H}},\end{aligned}$$

where the last equality is met thanks to (**A2**) as then $\mathcal{W}R_t u$ is a *strong* solution of $\dot{x} = \mathcal{A}x + \mathcal{D}u$ with null initial condition [15, Theorem 2.9/(ii), p. 109].

*Part 2.* The first assertion is proved in [10, pp. 1394 - 1395]. For the second observe that

$$\begin{aligned}\langle z, x(t)\rangle_{\mathrm{H}} =&\langle z, S(t)x_0\rangle_{\mathrm{H}} + \langle z, \Phi R_t u\rangle_{\mathrm{H}} = \langle z, S(t)x_0\rangle_{\mathrm{H}} + \langle \Phi^* z, R_t u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \\ =&\langle z, S(t)x_0\rangle_{\mathrm{H}} + \int_0^t \langle(\Phi^* z)(t-\tau), u(\tau)\rangle_{\mathrm{U}}\mathrm{d}\tau.\end{aligned}$$

Now by (**A2**) and EXS, we also have $\Phi^* \in \mathbf{L}(\mathrm{H}, \mathrm{L}^1(0,\infty;\mathrm{U}))$ – see [8, Appendix C, Lemma C1 with $c^{\#}$ replaced by $\mathcal{D}^*\mathcal{A}^*$ and $S(t)$ replaced by $S^*(t)$], whence by standard convolution result

$$\begin{aligned} \|\langle z, x(\cdot)\rangle_{\mathrm{H}}\|_{\mathrm{L}^2(0,\infty)} &\le \|\langle z, S(\cdot)x_0\rangle_{\mathrm{H}}\|_{\mathrm{L}^2(0,\infty)} + \|\Phi^* z\|_{\mathrm{L}^1(0,\infty;\mathrm{U})}\, \|u\|_{\mathrm{L}^2(0,\infty;\mathrm{U})} \le \\ &\le \|z\|_{\mathrm{H}}\, \|x_0\|_{\mathrm{H}} \frac{M}{\sqrt{2\alpha}} + \|\Phi^*\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^1(0,\infty;\mathrm{U}))}\, \|z\|_{\mathrm{H}}\, \|u\|_{\mathrm{L}^2(0,\infty;\mathrm{U})}. \end{aligned} \tag{B.1}$$

Notice that substituting $z = \mathcal{A}^* w$, $w \in D(\mathcal{A}^*)$ we get $\langle w, x(\cdot)\rangle_{\mathrm{H}} \in \mathrm{W}^{1,2}(0,\infty)$. □

**Comment B.1.** It follows from the above proof that the weak solution (2.1) satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{A}^{-1}x(t)\right] = x(t) + \mathcal{D}u(t), \\ \qquad\qquad x(0) = x_0 \end{cases}$$

is a strong sense.

*Proof of Lemma 2.2.* If $u \in \mathrm{W}^{1,2}([0,\infty);\mathrm{U})$ then $\mathcal{W}R_t u$ is a classical solution of $\dot{x} = \mathcal{A}x + \mathcal{D}u$ with null initial condition [7, Appendix A]; see also [11, Remark 2.1]. Hence,

$$\begin{aligned} x(t) &= S(t)[x_0 + \mathcal{D}u(0)] - S(t)\mathcal{D}u(0) + \mathcal{A}\mathcal{W}R_t u = \\ &= S(t)[x_0 + \mathcal{D}u(0)] - S(t)\mathcal{D}u(0) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}R_t u - \mathcal{D}u(t) = \\ &= S(t)[x_0 + \mathcal{D}u(0)] - S(t)\mathcal{D}u(0) + \mathcal{W}R_t\dot{u} + S(t)\mathcal{D}u(0) - \mathcal{D}u(t) = \\ &= S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{W}R_t\dot{u} - \mathcal{D}u(t). \end{aligned}$$

Now, by the admissibility and since $x_0 + \mathcal{D}u(0) \in D(\mathcal{A})$,

$$x(t) + \mathcal{D}u(t) = S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{W}R_t\dot{u} \in D(\mathcal{A}), \qquad \forall t \ge 0$$

and

$$\mathcal{A}[x(t) + \mathcal{D}u(t)] = \mathcal{A}S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{A}\mathcal{W}R_t\dot{u}.$$

On comparison, by the admissibility and [15, Theorem 2.4/(ii), p. 107], $\mathcal{W}R_t\dot{u}$ is a classical solution of $\dot{z} = \mathcal{A}z + \mathcal{D}\dot{u}$ with null initial condition, whence

$$\begin{aligned} \mathcal{A}S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{A}\mathcal{W}R_t\dot{u} &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{S(t)[x_0 + \mathcal{D}u(0)]\right\} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}R_t\dot{u} - \mathcal{D}\dot{u}(t) = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{W}R_t\dot{u} - \mathcal{D}u(t)\right\} = \dot{x}(t) \end{aligned}$$

and $x$ satisfies (1.1) in classical sense.

Observe that for the classical solution $x$, the resolution $x(t) = [x(t) + \mathcal{D}u(t)] - \mathcal{D}u(t)$ implies

$$\begin{aligned} y(t) = \mathcal{C}x(t) &= \mathcal{C}[x(t) + \mathcal{D}u(t)] - \mathcal{C}\mathcal{D}u(t) = (\mathcal{C}\mathcal{A}^{-1})\mathcal{A}[x(t) + \mathcal{D}u(t)] - \mathcal{C}\mathcal{D}u(t) = \\ &= (\mathcal{C}\mathcal{A}^{-1})\dot{x}(t) - \mathcal{C}\mathcal{D}u(t) = \\ &= (\mathcal{C}\mathcal{A}^{-1})\frac{\mathrm{d}}{\mathrm{d}t}\left\{S(t)[x_0 + \mathcal{D}u(0)] + \mathcal{W}R_t\dot{u} - \mathcal{D}u(t)\right\} - \mathcal{C}\mathcal{D}u(t) = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{(\mathcal{C}\mathcal{A}^{-1})S(t)[x_0 + \mathcal{D}u(0)] + (\mathcal{C}\mathcal{A}^{-1})\mathcal{W}R_t\dot{u} - (\mathcal{C}\mathcal{A}^{-1})\mathcal{D}u(t)\right\} - \mathcal{C}\mathcal{D}u(t), \end{aligned}$$

but, by (**A2**) and (**A1**),

$$
\begin{aligned}
&(\mathcal{C}\mathcal{A}^{-1})\mathcal{W}R_t\dot{u} = (\mathcal{C}\mathcal{A}^{-1})\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}R_t u - S(t)\mathcal{D}u(0)\right] = \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{C}\mathcal{A}^{-1})\mathcal{W}R_t u - (\mathcal{C}\mathcal{A}^{-1})S(t)\mathcal{D}u(0) = \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}(t-\tau)}\left\{(\mathcal{Z}[Du(\tau)])\,(t-\tau)\right\}\mathrm{d}\tau + (\mathcal{C}\mathcal{A}^{-1})\mathcal{D}u(t) - (\mathcal{C}\mathcal{A}^{-1})S(t)\mathcal{D}u(0)
\end{aligned}
$$

giving

$$
y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{C}\mathcal{A}^{-1}S(t)x_0 + \int_0^t (\Psi[\mathcal{D}u(\tau)])\,(t-\tau)\mathrm{d}\tau\right] - \mathcal{C}\mathcal{D}u(t),
$$

from which (2.3) follows by definition of $\Psi$.

The Laplace transform of $t \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{C}\mathcal{A}^{-1})\mathcal{W}R_t\dot{u}$ equals

$$
s^2(\mathcal{C}\mathcal{A}^{-1})(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s) - s(\mathcal{C}\mathcal{A}^{-1})(sI-\mathcal{A})^{-1}\mathcal{D}u(0),
$$

whence

$$
x_0 = 0 \Longrightarrow \hat{y}(s) = \left[s^2(\mathcal{C}\mathcal{A}^{-1})(sI-\mathcal{A})^{-1}\mathcal{D} - s(\mathcal{C}\mathcal{A}^{-1})\mathcal{D} - \mathcal{C}\mathcal{D}\right]\hat{u}(s) = \hat{G}(s)\hat{u}(s).
$$

Now, if (**A3**) holds too, then (2.3) extends to (2.4). □

*Proof of Theorem 2.3.* Since (**A1**), (**A2**) and (**A3**) hold then for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ the output $y$ is in $\mathrm{L}^2(0,\infty;\mathrm{Y})$ and is given by (2.4). Hence the performance index $J$ is a continuous functional of $(x_0,u) \in \mathrm{H}\times\mathrm{L}^2(0,\infty;\mathrm{U})$ and reads as

$$
\begin{aligned}
J(x_0,u) = \langle u, (R + N^*\mathbb{F} + \mathbb{F}^*Q\mathbb{F} + \mathbb{F}^*N)u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} + \langle x_0, \Psi^*Q\Psi x_0\rangle_{\mathrm{H}} + \\
+ \langle u, 2(\mathbb{F}^*Q + N^*)\Psi x_0\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})}.
\end{aligned} \tag{B.2}
$$

Since $\mathcal{R}$ is coercive then $\mathcal{R}^{-1} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$ and the optimal control exists, it is unique and equals (2.5). On this control $J$ achieves its minimal value (2.6).

*Proof of* (I). By Lemma 2.2, for every $u \in \mathrm{W}^{1,2}([0,\infty);\mathrm{U})$ and $x_0 \in \mathrm{H}$ such that $x_0 + \mathcal{D}u(0) \in D(\mathcal{A})$, the first equation in (1.1) has a unique classical solution (2.1), and the quadratic form $\mathcal{V}(x) := \langle x, \mathcal{H}x\rangle_{\mathrm{H}}$, dictated by an operator $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$, can be differentiated along the solution of (1.1) giving

$$
\begin{aligned}
\dot{\mathcal{V}}(x,u) =& \langle \dot{x}, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}\dot{x}\rangle_{\mathrm{H}} = \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} = \\
=& \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}(x+\mathcal{D}u)\rangle_{\mathrm{H}} - \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}\mathcal{D}u\rangle_{\mathrm{H}} + \\
&+ \langle (x+\mathcal{D}u), \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} - \langle \mathcal{D}u, \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}}.
\end{aligned} \tag{B.3}
$$

Recall the output representation (2.2). To take into account the performance index, we add and subtract its integrand in the RHS of (B.3), which yields

$$\begin{aligned}
\dot{\mathcal{V}}(x,u) &= \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}(x+\mathcal{D}u)\rangle_{\mathrm{H}} - \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}\mathcal{D}u\rangle_{\mathrm{H}} + \\
&\quad + \langle (x+\mathcal{D}u), \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} - \langle \mathcal{D}u, \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} + \\
&\quad + \langle Q\mathcal{C}x, \mathcal{C}x\rangle_{\mathrm{Y}} + \langle \mathcal{C}x, Nu\rangle_{\mathrm{Y}} + \langle Nu, \mathcal{C}x\rangle_{\mathrm{Y}} + \langle u, Ru\rangle_{\mathrm{U}} - \\
&\quad - \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \\
&= \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}(x+\mathcal{D}u)\rangle_{\mathrm{H}} + \langle (x+\mathcal{D}u), \mathcal{H}\mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} + \\
&\quad + \langle Q\mathcal{C}(x+\mathcal{D}u), \mathcal{C}(x+\mathcal{D}u)\rangle_{\mathrm{Y}} - \\
&\quad - \langle \mathcal{A}(x+\mathcal{D}u), \mathcal{H}\mathcal{D}u\rangle_{\mathrm{H}} - \langle \mathcal{H}\mathcal{D}u, \mathcal{A}(x+\mathcal{D}u)\rangle_{\mathrm{H}} - \\
&\quad - \langle Q\mathcal{C}(x+\mathcal{D}u), \mathcal{C}\mathcal{D}u\rangle_{\mathrm{Y}} - \langle Q\mathcal{C}\mathcal{D}u, \mathcal{C}(x+\mathcal{D}u)\rangle_{\mathrm{Y}} + \\
&\quad + \langle Nu, \mathcal{C}(x+\mathcal{D}u)\rangle_{\mathrm{Y}} + \langle \mathcal{C}(x+\mathcal{D}u), Nu\rangle_{\mathrm{Y}} + \\
&\quad + \langle Q\mathcal{C}\mathcal{D}u, \mathcal{C}\mathcal{D}u\rangle_{\mathrm{Y}} + \langle u, Ru\rangle_{\mathrm{U}} - \langle \mathcal{C}\mathcal{D}u, Nu\rangle_{\mathrm{Y}} - \langle Nu, \mathcal{C}\mathcal{D}u\rangle_{\mathrm{Y}} - \\
&\quad - \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}.
\end{aligned}$$

Assume that $R_-$ is coercive, whence invertible. Factorize $R_-$ as $R_- = V^*V$ ($V, V^{-1} \in \mathbf{L}(\mathrm{U})$). Such a factorization exists and is determined up to a unitary operator: $V = UR_-^{\frac{1}{2}}$, where $U$ is a unitary operator and $RR_-^{\frac{1}{2}}$ stands for the (unique) square root of $R_-$. Now, if $\mathcal{H}$ solves the Riccati operator equation (2.7) we can represent $\dot{\mathcal{V}}(x,u)$ as

$$\dot{\mathcal{V}}(x,u) = \left\|V^{-*}\mathcal{G}(x+\mathcal{D}u) + Vu\right\|_{\mathrm{U}}^2 - \begin{bmatrix} y \\ u \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}.$$

Consequently,

$$\begin{aligned}
&\mathcal{V}[x(t)] - \mathcal{V}(x_0) = \\
&= \int_0^t \|V^{-*}\mathcal{G}\left[x(\tau)+\mathcal{D}u(\tau)\right] + Vu(\tau)\|_{\mathrm{U}}^2 \,\mathrm{d}\tau - \int_0^t \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y(\tau) \\ u(\tau) \end{bmatrix} \mathrm{d}\tau\ .
\end{aligned}$$

By Lemma 2.1, we have $x \in \mathrm{BUC}_0(0,\infty;\mathrm{H})$, whence $\lim_{t\to\infty} \mathcal{V}[x(t)] = 0$ and

$$\begin{aligned}
0 \le \int_0^\infty &\left\|V\left\{R_-^{-1}\mathcal{G}\left[x(t)+\mathcal{D}u(t)\right] + u(t)\right\}\right\|_{\mathrm{U}}^2 \mathrm{d}t = J(x_0,u) - \mathcal{V}(x_0) \le \\
&\le \gamma(\|u\|_{\mathrm{L}^2(0,\infty;\mathrm{U})}^2 + \|x_0\|_{\mathrm{H}}^2), \qquad \forall (x_0,u) \in D(\mathcal{A}) \times \mathrm{W}_0^{1,2}([0,\infty);\mathrm{U}),
\end{aligned}$$

where the existence of a positive constant $\gamma$ follows from (B.2) and definition of $\mathcal{V}$. Since $(x_0,u) \in D(\mathcal{A}) \times \mathrm{W}_0^{1,2}([0,\infty);\mathrm{U})$ is a dense subspace of $\mathrm{H} \times \mathrm{L}^2(0,\infty;\mathrm{U})$ this, in particular, implies that the mapping

$$\mathrm{H} \times \mathrm{L}^2(0,\infty;\mathrm{U}) \ni (x_0,u) \longmapsto \mathcal{G}\left[x(t)+\mathcal{D}u(t)\right] = \mathcal{G}\mathcal{A}^{-1}\dot{x}(t) \in \mathrm{L}^2(0,\infty;\mathrm{U})$$

is densely defined and bounded; here $\mathcal{G}\mathcal{A}^{-1} = N_-^*\mathcal{C}\mathcal{A}^{-1} - \mathcal{D}^*\mathcal{H}$. The proof of Lemma 2.2 suggests that its closure is given by $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}\mathcal{A}^{-1}x(t)\right]$, what can be confirmed using the Laplace transformation

$$\begin{aligned}\mathcal{G}\left[\hat{x}(s)+\mathcal{D}\hat{u}(s)\right] &= \mathcal{G}\left[(sI-\mathcal{A})^{-1}x_0+\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s)+\mathcal{D}\hat{u}(s)\right] = \\ &= \mathcal{G}\left[(sI-\mathcal{A})^{-1}x_0+s(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s)\right] = \\ &= \mathcal{G}(sI-\mathcal{A})^{-1}x_0+s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s) = \\ &= s\mathcal{G}\mathcal{A}^{-1}\hat{x}(s)-\mathcal{G}\mathcal{A}^{-1}x_0.\end{aligned}$$

If $\hat{u}_n \longrightarrow \hat{u}$ in $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$ and $\mathcal{M}\hat{u}_n \longrightarrow \hat{v}$ in $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$ as $n\to\infty$, where $\mathcal{M}$ is the operator of multiplication of $\hat{f}$ by $s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}$, i.e., $(\mathcal{M}\hat{f})(s) := s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{f}(s)$, then $\hat{u}_n(s) \longrightarrow \hat{u}(s)$ and $(\mathcal{M}\hat{u}_n)(s) \longrightarrow \hat{v}(s)$ in U as $n\to\infty$ thanks to the inequality $\|\hat{f}(s)\|_{\mathrm{U}} \le \frac{1}{\sqrt{2\operatorname{Re}s}}\|\hat{f}\|_{\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})}$, which holds for $s\in\mathbb{C}^+$. Hence $(\mathcal{M}\hat{u}_n)(s) \longrightarrow (\mathcal{M}\hat{u})(s)$ in U because $s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\in\mathbf{L}(\mathrm{U})$ and thus $\mathcal{M}\hat{u}=\hat{v}$, which means that $\mathcal{M}$ is closed. By the closed-graph theorem $\mathcal{M}\in\mathbf{L}(\mathrm{H}^2(\mathbb{C}^+;\mathrm{U}))$. Applying Lemma D.1 of Appendix D we get $s\longmapsto s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\in\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U}))$. Somewhat similar arguments, applied to the operator $x_0\longmapsto\mathcal{G}(sI-\mathcal{A})^{-1}x_0$, prove that it belongs to $\mathbf{L}(\mathrm{H},\mathrm{H}^2(\mathbb{C}^+,\mathrm{U}))$.

It is straightforward to check that the Laplace transform of $t\longmapsto\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}\mathcal{A}^{-1}x(t)\right]$ is $s\longmapsto\mathcal{G}(sI-\mathcal{A})^{-1}x_0+s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s)$.

Now

$$0\le\int_0^\infty\left\|V\left\{R_-^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}\mathcal{A}^{-1}x(t)\right]+u(t)\right\}\right\|_{\mathrm{U}}^2\mathrm{d}t = J(x_0,u)-\mathcal{V}(x_0),$$
$$\forall(x_0,u)\in\mathrm{H}\times\mathrm{L}^2(0,\infty;\mathrm{U}).$$

Suppose that the control $u$ given by (2.9), where $x$ satisfies (2.10), is in $\mathrm{L}^2(0,\infty;\mathrm{U})$. Then

$$J(x_0,u)=\mathcal{V}(x_0)=\langle x_0,\mathcal{H}x_0\rangle_{\mathrm{H}}\le J(x_0,u_{\mathrm{opt}})=\langle x_0,\mathcal{H}_{\mathrm{opt}}x_0\rangle_{\mathrm{H}},$$

where $u_{\mathrm{opt}}$, $\mathcal{H}_{\mathrm{opt}}$ are, respectively, the optimal control given by (2.5), and the optimal cost operator, given by (2.6). By the uniqueness of optimal control we then have $u=u_{\mathrm{opt}}$ and, consequently $\mathcal{H}=\mathcal{H}_{\mathrm{opt}}$ (in particular, this means that the optimal cost operator (2.6) is the *maximal* solution to the Riccati operator equation (2.7); others solution of (2.7) are merely *lower bounds* of the performance index), $\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{G}_{\mathrm{opt}}\mathcal{A}^{-1}x(t)\right]+R_-u_{\mathrm{opt}}(t)=0$ or, equivalently,

$$\left[R_-+s\mathcal{G}_{\mathrm{opt}}(sI-\mathcal{A})^{-1}\mathcal{D}\right]\hat{u}_{\mathrm{opt}}(s)=-\mathcal{G}_{\mathrm{opt}}(sI-\mathcal{A})^{-1}x_0.$$

It remains to examine the closed-loop system. By Lemma 2.1, the natural candidate for $S_{\mathrm{opt}}(t)$ operator is

$$S_{\mathrm{opt}}(t)x_0=[S(t)+\Phi R_t\mathfrak{M}]x_0,\qquad\{S_{\mathrm{opt}}(t)\}_{t\ge0}\subset\mathbf{L}(\mathrm{H}),\quad S_{\mathrm{opt}}(0)=I,$$

and $t \longmapsto S_{\mathrm{opt}}(t)x_0$ is in $\mathrm{BUC}_0([0,\infty),\mathrm{H})$. To show that $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$ is a $\mathrm{C}_0$-semigroup we have to verify the semigroup property. For that we have

$$\begin{aligned}
S_{\mathrm{opt}}(t+\tau)x_0 &= S(t+\tau)x_0 + \mathcal{A}\int_0^{t+\tau} S(t+\tau-r)\mathcal{D}(\mathfrak{M}x_0)(r)\mathrm{d}r = \\
&= S(\tau)\left[S(t)x_0 + \mathcal{A}\int_0^{t} S(t-r)\mathcal{D}(\mathfrak{M}x_0)(r)\mathrm{d}r\right] + \\
&\quad + \mathcal{A}\int_0^{\tau} S(t-\xi)\mathcal{D}(\mathfrak{M}x_0)(t+\xi)\mathrm{d}\xi = \\
&= S(\tau)[S_{\mathrm{opt}}(t)x_0] + \mathcal{A}\int_0^{\tau} S(t-\xi)\mathcal{D}\left(S_{\mathcal{L}_{\mathrm{U}}}(t)\mathfrak{M}x_0\right)(\xi)\mathrm{d}\xi,
\end{aligned}$$

where $\{S_{\mathcal{L}_{\mathrm{U}}}(t)\}_{t\geq 0}$ stands for the semigroup of left-shifts generated by $\mathcal{L}_{\mathrm{U}}$. Eliminating $S(\tau)$ from the first component we obtain

$$S_{\mathrm{opt}}(t+\tau)x_0 = S_{\mathrm{opt}}(\tau)[S_{\mathrm{opt}}(t)x_0] - \Phi R_\tau \mathfrak{M} S_{\mathrm{opt}}(t)x_0 + \Phi R_\tau S_{\mathcal{L}_{\mathrm{U}}}(t)\mathfrak{M}x_0,$$

whence the semigroup property is met if $\mathfrak{M}[S_{\mathrm{opt}}(t)x_0] = S_{\mathcal{L}_{\mathrm{U}}}(t)\mathfrak{M}x_0$, i.e., when the optimal control for initial state $S_{\mathrm{opt}}(t)x_0$ coincides with the left translation by $t$ of the optimal control for initial state $x_0$. For the latter observe that:

$$J(x_0, \mathfrak{M}x_0 \underset{t}{\Diamond} v) = J(x_0,\mathfrak{M}x_0) - J(S_{\mathrm{opt}}(t)x_0, S_{\mathcal{L}_U}(t)\mathfrak{M}x_0) + J(S_{\mathrm{opt}}(t)x_0, v),$$

where the *concatenation* of functions at $t \geq 0$ is defined as

$$(f \underset{t}{\Diamond} g)(\tau) := \begin{cases} f(\tau) & \text{if} \quad 0 \leq \tau < t, \\ g(\tau - t) & \text{if} \quad \tau \geq t. \end{cases}$$

The state at $t$ equals $S_{\mathrm{opt}}(t)x_0$, while the control on $[t,\infty)$ equals $S_{\mathscr{R}_U}(t)\nu$. Hence, shifting the initial time to 0, we conclude that a part of the value of $J(x_0, \mathfrak{M}x_0 \underset{t}{\Diamond} v)$ due to integration on $[t,\infty)$ is $J(S_{\mathrm{opt}}(t)x_0, v)$. On the other side, a part of the value $J(x_0, \mathfrak{M}x_0 \underset{t}{\Diamond} v)$ due to integration on $[0,t)$ equals $J(x_0,\mathfrak{M}x_0)$ minus a part of the value of $J(x_0,\mathfrak{M}x_0)$ due to integration on $[t,\infty)$. Shifting, in the last component, the initial time to 0 we find that it equals $J(S_{\mathrm{opt}}(t)x_0, S_{\mathcal{L}_U}(t)\mathfrak{M}x_0)$. Minimizing with respect to $v$ we get $J(S_{\mathrm{opt}}(t)x_0, S_{\mathcal{L}_U}(t)\mathfrak{M}x_0) = J(S_{\mathrm{opt}}(t)x_0, \mathfrak{M}S_{\mathrm{opt}}(t)x_0)$, from which we get the desired identity $\mathfrak{M}[S_{\mathrm{opt}}(t)x_0] = S_{\mathcal{L}_{\mathrm{U}}}(t)\mathfrak{M}x_0$.

For $u = \mathfrak{M}x_0$ we have $x(t) = S_{\mathrm{opt}}(t)x_0$ in Lemma 2.1, whence using (B.1) we get

$$\|\langle z, S_{\mathrm{opt}}(\cdot)x_0\rangle_{\mathrm{H}}\|_{\mathrm{L}^2(0,\infty)} \leq \|z\|_{\mathrm{H}}\,\|x_0\|_{\mathrm{H}}\left[\frac{M}{\sqrt{2\alpha}} + \|\Phi^*\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^1(0,\infty;\mathrm{U}))}\,\|\mathfrak{M}\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{U}))}\right],$$

from which EXS follows by the result of [23, with $p = 2$].

The identity $x_{\rm opt}(t) = S_{\rm opt}(t)x_0$ follows from comparing their Laplace transforms. Indeed, we the aid of (2.10), we get $s\mathcal{A}^{-1}\hat{x}_{\rm opt}(s) - \mathcal{A}^{-1}x_0 = \hat{x}_{\rm opt}(s) + \mathcal{D}\hat{u}_{\rm opt}(s)$, whence

$$\begin{aligned}\hat{x}_{\rm opt}(s) &= (s\mathcal{A}^{-1} - I)^{-1}\mathcal{A}^{-1}x_0 + (s\mathcal{A}^{-1} - I)^{-1}\mathcal{D}\hat{u}_{\rm opt}(s) = \\ &= (sI - \mathcal{A})^{-1}x_0 + \mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}\hat{u}_{\rm opt}(s).\end{aligned}$$

But, directly by the definition of $S_{\rm opt}(t)x_0$, the last expression is readily seen to be the Laplace transform of $[S(t) + \Phi R_t\mathfrak{M}]x_0$. We also have

$$\begin{aligned}\hat{x}_{\rm opt}(s) &= \left[s(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}_{\rm opt}\mathcal{A}^{-1}) - I\right]^{-1}(\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}_{\rm opt}\mathcal{A}^{-1})x_0 = \\ &= \left[sI - (\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}_{\rm opt}\mathcal{A}^{-1})^{-1}\right]^{-1}x_0.\end{aligned}$$

This is the resolvent of a closed densely defined state operator of the closed-loop system:

$$\begin{aligned}&\mathcal{A}_{\rm opt}x := (\mathcal{A}^{-1} + \mathcal{D}R_-^{-1}\mathcal{G}_{\rm opt}\mathcal{A}^{-1})^{-1}x, \\ &D(\mathcal{A}_{\rm opt}) = \{x \in {\rm H}:\ \exists! z \in D(\mathcal{A}),\ z \text{ solves the equation } z + \mathcal{D}R_-^{-1}\mathcal{G}_{\rm opt}z = x\}.\end{aligned}$$

*Proof of* (II). It is enough to observe that if for some $\mathcal{G}$, tied with $\mathcal{H}$ by (2.8), the operator-valued function $s \longmapsto [R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}]^{-1}$ belongs to ${\rm H}^\infty(\mathbb{C}^+, \mathbf{L}({\rm U}))$ then the equation

$$\left[R_- + s\mathcal{G}(sI - \mathcal{A})^{-1}\mathcal{D}\right]\hat{u}(s) = -\mathcal{G}(sI - \mathcal{A})^{-1}x_0$$

has a unique solution $\hat{u} \in {\rm H}^2(\mathbb{C}^+, {\rm U})$, so $\hat{u} = \hat{u}_{\rm opt}$ thanks to (I). □

**Comment B.2.** The fact $\mathfrak{M}[S_{\rm opt}(t)x_0] = S_{\mathcal{L}_{\rm U}}(t)\mathfrak{M}x_0$ is closely related to the *principle of optimality*, accordingly to which an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Furthermore, formally the *Hamilton-Jacobi-Bellman equation* for our lq-problem is

$$0 = \min_u \left\{\dot{\mathcal{V}}(x,u) + \begin{bmatrix} y \\ u\end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R\end{bmatrix}\begin{bmatrix} y \\ u\end{bmatrix}\right\} = \min_u \left\|V^{-*}\mathcal{G}(x + \mathcal{D}u) + Vu\right\|^2_{\rm U},$$

where the second equality holds when $\mathcal{H}$ solves to the Riccati operator equation (2.7) and $\mathcal{G}$ is given by (2.8). We have established that the optimal cost operator induces $\mathcal{V}$ which is a *maximal semiconcave* solution to this Hamilton-Jacobi-Bellman equation.

In [24] an idea of *Pontryagin's maximum principle* has been employed to get Riccati operator equation and to characterize the optimal controller.

## C. PROOFS OF MAIN RESULTS OF SECTION 3

*Proof of Theorem 3.1. Part* (I). If $\Pi(j\omega) \geq \varepsilon I$ for all $\omega \in \mathbb{R}$ and some $\varepsilon > 0$ then $J$ is being minimized by $\hat{u} \in {\rm L}^2(j\mathbb{R}, {\rm U})$ satisfying

$$\Pi\hat{u} = -\left[\hat{G}^*Q + N^*\right]\widehat{\Psi x_0} = -\left[Q\hat{G} + N\right]^*\widehat{\Psi x_0} = \mathcal{F}\left[(\mathbb{F}^*Q + N^*)\Psi x_0\right], \qquad \text{(C.1)}$$

where $\mathcal{F}$ stands for the Fourier transform. Observe that

$$\begin{aligned}&\langle u, \mathcal{R}u\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \langle \mathcal{F}u, \mathcal{F}\mathcal{R}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{U})} = \\ &= \langle \mathcal{F}u, R\mathcal{F}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{U})} + \langle N\mathcal{F}u, \hat{G}\mathcal{F}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{Y})} + \langle \hat{G}\mathcal{F}u, N\mathcal{F}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{Y})} + \\ &\quad + \langle \hat{G}\mathcal{F}u, Q\hat{G}\mathcal{F}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{Y})} = \langle \mathcal{F}u, \Pi\mathcal{F}u\rangle_{\mathrm{L}^2(j\mathbb{R};\mathrm{U})} \geq \varepsilon \left\|\mathcal{F}u\right\|^2_{\mathrm{L}^2(j\mathbb{R};\mathrm{U})} = \\ &= \varepsilon \left\|\hat{u}\right\|^2_{\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})} = \varepsilon \left\|u\right\|^2_{\mathrm{L}^2(0,\infty;\mathrm{U})},\end{aligned}$$

whence $\mathcal{R}$ is coercive too. The converse may not hold because the Fourier transform $\mathcal{F}$ is a unitary isomorphism between $\mathrm{L}^2(\mathbb{R};\mathrm{U})$ and $\mathrm{L}^2(j\mathbb{R};\mathrm{U})$, while controls are in $\mathrm{L}^2(0,\infty;\mathrm{U})$. Nevertheless, the existence of $\mathrm{L}^2(j\mathbb{R},\mathrm{U})$-minimizer implies the existence of $\mathrm{L}^2(0,\infty;\mathrm{U})$-minimizer, or equivalently, $\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$-minimizer and all these minimizers are equal.

We need to express $\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$-minimizer rather than $\mathrm{L}^2(j\mathbb{R},\mathrm{U})$-minimizer. For that the *Wiener - Hopf projection method* is adequate. From [5, Lemma 2, p. 475 with: $\mathfrak{H} = \mathrm{L}^2(j\mathbb{R},\mathrm{U})$, $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$, $P$ - the projection from $\mathfrak{H}$ onto its closed subspace $\mathfrak{P}$, $(Af)(j\omega) := \Pi(j\omega)f(j\omega)$ (here we employ the facts that $A$ is bounded and has a strongly positive real part) and with the *Toeplitz operator* $T_P(A) := PA|_{\mathfrak{P}}$], we know that $T_{P_+}(\Pi) := P_+\Pi|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})}$, where $P_+$ stands for the projection from $\mathrm{L}^2(j\mathbb{R};\mathrm{U})$ onto its closed subspace $\mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$, is invertible. Hence (C.1) has a unique solution $\hat{u}$ in $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$.

This $\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$-minimizer is more precisely characterized by [5, Theorem 5, p. 478 with $\mathfrak{H} = \mathrm{L}^2(j\mathbb{R},\mathrm{U})$, $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$, $\mathfrak{Q} = \mathfrak{P}^\perp = \mathrm{H}^2(\mathbb{C}^-,\mathrm{U})$, $(Af)(j\omega) := \Pi(j\omega)f(j\omega)$ (here we employ, in addition, the facts that $A$ is invertible and $A \geq 0$) and because $T_P(A) = PA|_{\mathfrak{P}} = T_{P_+}(\Pi) := P_+\Pi|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})}$ is invertible] (this theorem is recalled in Appendix A). Under these hypotheses, there exists a bounded and invertible $A_+$ (on $\mathrm{L}^2(j\mathbb{R},\mathrm{U})$) which takes $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$ onto itself such that

$$A = A_+^* A_+$$

(a factor $A_-$ appearing in the basic statement can be taken to be $A_- = A_+^*$, i.e., its $\mathfrak{H} = \mathrm{L}^2(j\mathbb{R},\mathrm{U})$-adjoint operator (as justified by the remark on p. 482, first line from the top)), and

$$[T_P(A)]^{-1}f = A_+^{-1}PA_+^{-*}f, \qquad f \in \mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U}). \tag{C.2}$$

Since $A_+$ maps $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$ onto itself, the boudedness of $A_+$ on $\mathfrak{H} = \mathrm{L}^2(j\mathbb{R},\mathrm{U})$ implies the boundedness of its restriction $A_+|_{\mathfrak{P}}$ to $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$. Furthermore, $A_+|_{\mathfrak{P}}$ is injective because $A_+$ is injective as a boundedly invertible operator on $\mathfrak{H} = \mathrm{L}^2(j\mathbb{R},\mathrm{U})$. Thus $A_+|_{\mathfrak{P}}$ is boundedly invertible on $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$.

Let $S$ be the *canonical shift* on $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+,\mathrm{U})$ [16, p. 95, where the Hardy classes have been defined for the upper complex half-plane rather than $\mathbb{C}^+$ – so one has to replace $z$ in (5-4) by $js$],

$$(Sf)(s) := \frac{s-1}{s+1}f(s),\ f \in \mathrm{H}^2(\mathbb{C}^+,\mathrm{U});\ (S^*g)(s) = g(s) + 2\frac{g(s)-g(1)}{s-1},\ g \in \mathrm{H}^2(\mathbb{C}^+,\mathrm{U}),$$

then $A_+|_{\mathfrak{P}}$ clearly commutes with $S$, and in accordance with [16, Definition 1.6 (i), p. 6] $A_+|_{\mathfrak{P}}$ is $S$-*analytic*. Recall that the canonical shift $S$ is, modulo the conformal mapping $z = \frac{s-1}{s+1}$ of $\mathbb{C}^+$ onto the unit disc $\mathbb{D}$, multiplication by $z$ of the *Taylor transform* (*expansion*), which corresponds, in terms of the Taylor transform, to the (discrete-time) right-shift on $\ell^2(\mathbb{Z}^+)$, and that an operator $Z \in \mathbf{L}(\mathrm{H})$ is $S$-analytic if $SZ = ZS$.

Now, by [16, Theorem C (i), p. 96 with $\Omega = \mathbb{C}^+$, $\mathcal{C} = \mathrm{U}$], the operator $A_+|_{\mathfrak{P}}$ must be of the form $(A_+|_{\mathfrak{P}} f)(s) = \Xi(s)f(s)$, with $\Xi \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$. By similar arguments applied to $(A_+|_{\mathfrak{P}})^{-1}$, we obtain $((A_+|_{\mathfrak{P}})^{-1} f)(s) = \Xi^{-1}(s)f(s)$, with $\Xi^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$. Consequently $(A_+ f)(j\omega) = \Xi(j\omega)f(j\omega)$, $(A_+^{-1} f)(j\omega) = \Xi^{-1}(j\omega)f(j\omega)$ and we get the factorization $\Pi(j\omega) = \Xi^*(j\omega)\Xi(j\omega)$, whilst the formula for $[T_P(A)]^{-1}$ yields the $\mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$-minimizer (3.3). Furthermore, in accordance with [16, Definition 1.6 (iii), p. 6], the operators $A_+|_{\mathfrak{P}}$, $(A_+|_{\mathfrak{P}})^{-1}$ are $S$-*outer*. Recall that an operator $Z \in \mathbf{L}(\mathrm{H})$ is $S$-outer if $Z$ is $S$-analytic and $\overline{Z\mathrm{H}}$ reduces $S$, i.e., $\overline{Z\mathrm{H}}$ is an invariant subspace for both $S$ and $S^*$. Here the range of $Z = A_+|_{\mathfrak{P}} \in \mathbf{L}(\mathrm{H}^2(\mathbb{C}^+, \mathrm{U}))$ clearly equals $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$.

Suppose that there exists a spectral factorization

$$A = A_+^* A_+ = C_+^* C_+,$$

where $(C_+|_{\mathfrak{P}} f)(s) = \Sigma_C(s)f(s)$, with $\Sigma_C$, $\Sigma_C^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$, whence $C_+|_{\mathfrak{P}}$ is $S$-*outer*. Now, by [16, Corollary, p. 52] there exists an operator $B \in \mathbf{L}(\mathrm{H}^2(\mathbb{C}^+, \mathrm{U}))$ such that $A_+|_{\mathfrak{P}} = B\, C_+|_{\mathfrak{P}}$ which is an $S$-*constant inner* operator with *initial* and *final* spaces $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$. From [16, Theorem C (iv), p. 96] we conclude that $B$ is an operator of multiplication by an independent of $s$ operator which is a partial isometry with *initial* and *final* spaces $\mathfrak{P} = \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$. Actually we have more, since $A_+|_{\mathfrak{P}}$, $C_+|_{\mathfrak{P}}$ are both boundedly invertible – the operator $B$ is a bounded and invertible partial isometry, so $BB^*B = B$, whence $BB^* = I = B^*B$ and $B$ is a constant unitary operator. The whole discussion above shows that the spectral factorization $\Pi(j\omega) = \Xi^*(j\omega)\Xi(j\omega)$ associated with $\Xi$ such that $\Xi$, $\Xi^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ is determined uniquely up to a $s$-independent unitary operator multiplier which belongs to $\mathbf{L}(\mathrm{U})$.

*Part* (II). Since $\hat{G}(0) = -\mathcal{C}\mathcal{D}$, then

$$\Pi(j0) = R + 2\,\mathrm{Re}[N^*\hat{G}(0)] + [\hat{G}(0)]^* Q \hat{G}(0) = R_-$$

and $R_-$ is coercive, whence the Riccati operator equation (2.7) is meaningful.

Complexifying H, Y, U and substituting $z := j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u \in D(\mathcal{A})$ into (2.7) we get

$$\begin{aligned}\left\|V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\right\|_{\mathrm{U}}^2 = {} & \langle j\omega\mathcal{A}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u, j\omega\mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\rangle_{\mathrm{H}} + \\ & + \langle j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u, j\omega\mathcal{H}\mathcal{A}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\rangle_{\mathrm{H}} + \\ & + \langle Q\mathcal{C}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u, \mathcal{C}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\rangle_{\mathrm{Y}} =\end{aligned}$$

$$= \omega^2 \Bigg[ \langle \underbrace{\mathcal{A}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u}_{= j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u - \mathcal{D}u}, \mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\rangle_{\mathrm{H}} +$$

$$+ \langle (j\omega I - \mathcal{A})^{-1}\mathcal{D}u, \mathcal{H} \underbrace{\mathcal{A}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u}_{= j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u - \mathcal{D}u} \rangle_{\mathrm{H}} \Bigg] +$$

$$+ \langle Q[\hat{G}(j\omega) + \mathcal{C}\mathcal{D}]u, [\hat{G}(j\omega) + \mathcal{C}\mathcal{D}]u\rangle_{\mathrm{Y}} =$$

$$= \omega^2 \left[ \langle u, -\mathcal{D}^*\mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u\rangle_{\mathrm{H}} + \langle (j\omega I - \mathcal{A})^{-1}\mathcal{D}u, -\mathcal{H}\mathcal{D}u\rangle_{\mathrm{H}} \right] +$$

$$+ \langle Q\hat{G}(j\omega)u, \hat{G}(j\omega)u\rangle_{\mathrm{Y}} + \langle Q\mathcal{C}\mathcal{D}u, \hat{G}(j\omega)u\rangle_{\mathrm{Y}} + \langle Q\hat{G}(j\omega)u, \mathcal{C}\mathcal{D}u\rangle_{\mathrm{Y}} + \langle Q\mathcal{C}\mathcal{D}u, \mathcal{C}\mathcal{D}u\rangle_{\mathrm{Y}}.$$

Next, using again the notation $2\,\mathrm{Re}\, Z := Z + Z^*$, $Z \in \mathbf{L}(\mathrm{U})$, the last equation reads as

$$\begin{aligned} &\left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]^* \left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] - [\hat{G}(j\omega)]^* Q\hat{G}(j\omega) - \\ &-2\,\mathrm{Re}[(\mathcal{C}\mathcal{D})^* Q\hat{G}(j\omega)] - (\mathcal{C}\mathcal{D})^* Q(\mathcal{C}\mathcal{D}) = \omega^2 \; 2\,\mathrm{Re}\left[-\mathcal{D}^*\mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]. \end{aligned} \tag{C.3}$$

By (2.8), still with $z := j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u \in D(\mathcal{A})$, we have

$$\begin{aligned} &-\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u + N_-^*\mathcal{C}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u = \mathcal{D}^*\mathcal{H}\mathcal{A}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}u = \\ &= \mathcal{D}^*\mathcal{H}\left\{ j\omega \left[ j\omega(j\omega I - \mathcal{A})^{-1} - I \right] \right\} \mathcal{D}u = -\omega^2\mathcal{D}^*\mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}u - j\omega\mathcal{D}^*\mathcal{H}\mathcal{D}u, \end{aligned}$$

whence

$$\begin{aligned} &\omega^2 \; 2\,\mathrm{Re}\left[-\mathcal{D}^*\mathcal{H}(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] = \\ &= 2\,\mathrm{Re}\left[N_-^*\mathcal{C}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] - 2\,\mathrm{Re}\left[\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] = \\ &= 2\,\mathrm{Re}\left[N_-^*\hat{G}(j\omega)\right] + 2\,\mathrm{Re}\left[N_-^*(\mathcal{C}\mathcal{D})\right] - 2\,\mathrm{Re}\left[\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] = \\ &= 2\,\mathrm{Re}\left[N^*\hat{G}(j\omega)\right] - 2\,\mathrm{Re}\left[(\mathcal{C}\mathcal{D})^* Q\hat{G}(j\omega)\right] + 2\,\mathrm{Re}\left[N^*(\mathcal{C}\mathcal{D})\right] - 2(\mathcal{C}\mathcal{D})^* Q(\mathcal{C}\mathcal{D}) - \\ &-2\,\mathrm{Re}\left[\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]. \end{aligned} \tag{C.4}$$

(C.3) and (C.4) yield

$$\begin{aligned} &\left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]^* \left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] - [\hat{G}(j\omega)]^* Q\hat{G}(j\omega) - \\ &-2\,\mathrm{Re}[(\mathcal{C}\mathcal{D})^* Q\hat{G}(j\omega)] - (\mathcal{C}\mathcal{D})^* Q(\mathcal{C}\mathcal{D}) = 2\,\mathrm{Re}[N^*\hat{G}(j\omega)] - 2\,\mathrm{Re}[(\mathcal{C}\mathcal{D})^* Q\hat{G}(j\omega)] + \\ &+2\,\mathrm{Re}[N^*(\mathcal{C}\mathcal{D})] - 2(\mathcal{C}\mathcal{D})^* Q(\mathcal{C}\mathcal{D}) - 2\,\mathrm{Re}[\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}], \end{aligned}$$

or equivalently,

$$\begin{aligned} &\left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]^* \left[V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] + 2\,\mathrm{Re}\left[\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] = \\ &= \hat{G}^*(j\omega)Q\hat{G}(j\omega) + 2\,\mathrm{Re}[N^*\hat{G}(j\omega)] + 2\,\mathrm{Re}[N^*(\mathcal{C}\mathcal{D})] - (\mathcal{C}\mathcal{D})^* Q(\mathcal{C}\mathcal{D}). \end{aligned}$$

Adding $R_-$ to both sides and applying its definition we get

$$\begin{aligned} &\left[V + V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right]^* \left[V + V^{-*}\mathcal{G}j\omega(j\omega I - \mathcal{A})^{-1}\mathcal{D}\right] = \\ &= \hat{G}^*(j\omega)Q\hat{G}(j\omega) + 2\,\mathrm{Re}[N^*\hat{G}(j\omega)] + R, \end{aligned} \tag{C.5}$$

This means that $\Pi(j\omega)$ is nonnegative and has the spectral factorization (3.2), (3.4).

By (3.2), $\|\Pi(j\omega)\|_{\mathbf{L}(\mathrm{U})} = \|\Xi^*(j\omega)\Xi(j\omega)\|_{\mathbf{L}(\mathrm{U})} = \|\Xi(j\omega)\|^2_{\mathbf{L}(\mathrm{U})}$, whence $\Xi$ is bounded on the imaginary axis, analytic on $\overline{\mathbb{C}^+}$, where it grows as a trinomial in $|s|$,

$$\begin{aligned}
\|\Xi(s)\|_{\mathbf{L}(\mathrm{U})} &\le \|V\| + \left\|V^{-1}\right\| \left\|\mathcal{G}s(sI-\mathcal{A})^{-1}\mathcal{D}\right\| \le \\
&\le \|V\| + \left\|V^{-1}\right\| \left\|\mathcal{G}\mathcal{A}^{-1}\right\| \left\|s\mathcal{A}(sI-\mathcal{A})^{-1}\right\| \|\mathcal{D}\| \le \\
&\le \|V\| + \left\|V^{-1}\right\| \left\|\mathcal{G}\mathcal{A}^{-1}\right\| \left(|s|^2 \left\|(sI-\mathcal{A})^{-1}\right\| + |s|\right) \|\mathcal{D}\| \le \\
&\underbrace{\le}_{(1.2)} \|V\| + \left\|V^{-1}\right\| \left\|\mathcal{G}\mathcal{A}^{-1}\right\| \left(|s|^2 \tfrac{M}{\alpha} + |s|\right) \|\mathcal{D}\| .
\end{aligned}$$

Now, one can apply a vector version of the *Phragmén-Lindelöf theorem* [1, Theorem 3.9.8, p. 179 with a Banach space $X = \mathbf{L}(\mathrm{U})$ and an opening angle $\varphi = \pi/2$] (recalled in Appendix A) to conclude that $s \longmapsto \Xi(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$.

Inserting (2.8) into (2.7) we get the *Lyapunov operator equation*

$$\langle \mathcal{A}z, \mathcal{H}z\rangle_{\mathrm{H}} + \langle z, \mathcal{H}\mathcal{A}z\rangle_{\mathrm{H}} = -\langle Q\mathcal{C}z, \mathcal{C}z\rangle_{\mathrm{Y}} + \left\|V^{-*}\mathcal{G}z\right\|^2_{\mathrm{U}}, \quad z \in D(\mathcal{A}).$$

Substituting $z = S(t)x_0$, $x_0 \in D(\mathcal{A})$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S(t)x_0, \mathcal{H}S(t)x_0\rangle_{\mathrm{H}} = -\langle Q\mathcal{C}S(t)x_0, \mathcal{C}S(t)x_0\rangle_{\mathrm{Y}} + \left\|V^{-*}\mathcal{G}S(t)x_0\right\|^2_{\mathrm{U}}.$$

Integrating both sides from 0 to $t$ and employing EXS we can pass to the limit $t \to \infty$, which yields

$$\langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}} + \int_0^\infty \langle Q\mathcal{C}S(t)x_0, \mathcal{C}S(t)x_0\rangle_{\mathrm{Y}}\mathrm{d}t = \int_0^\infty \left\|V^{-*}\mathcal{G}S(t)x_0\right\|^2_{\mathrm{U}} \mathrm{d}t.$$

Due to $\mathcal{H} \in \mathbf{L}(\mathrm{H})$ and (**A1**), the validity of this formula extends to all $x_0 \in \mathrm{H}$ giving, by the Paley-Wiener theorem, $s \longmapsto V^{-*}\mathcal{G}(sI-\mathcal{A})^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+; \mathrm{U})$.

Let $u$ be a solution giving rise to classical solution $x$ satisfying the implicit feedback control formula (2.9). Then

$$\begin{aligned}
R_-u(t) &= -\mathcal{G}[x(t) + \mathcal{D}u(t)] = -[-\mathcal{D}^*\mathcal{H} + N_-^*(\mathcal{C}\mathcal{A})^{-1}]\mathcal{A}[x(t) + \mathcal{D}u(t)] = \\
&= -[-\mathcal{D}^*\mathcal{H} + N_-^*(\mathcal{C}\mathcal{A})^{-1}]\dot{x}(t).
\end{aligned}$$

Since

$$\begin{aligned}
\hat{\dot{x}}(s) &= s\hat{x}(s) - x_0 = s(sI-\mathcal{A})^{-1}x_0 - x_0 + s\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s) = \\
&= \mathcal{A}(sI-\mathcal{A})^{-1}x_0 + s\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s),
\end{aligned}$$

then

$$\left[R_- + s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D}\right]\hat{u}(s) = V^*\Xi(s)\hat{u}(s) = -\mathcal{G}(sI-\mathcal{A})^{-1}x_0.$$

Now, if $s \longmapsto \Xi^{-1}(s) \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ then, by admissibility of $V^{-*}\mathcal{G}$ one obtains $\hat{u} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$ or, equivalently, $u \in \mathrm{L}^2(0, \infty; \mathrm{U})$ and, by Theorem 2.3, $u$ is optimal.

In accordance with the terminology used in [24], this spectral factor is *regular* if the limit

$$\lim_{s\to\infty, s\in\mathbb{R}} \Xi(s)u = Du, \qquad D \in \mathbf{L}(\mathrm{U}),$$

exists and $D^{-1} \in \mathbf{L}(\mathrm{U})$. In the context of (3.4) this means that $\mathcal{G}_\Lambda$ is a restriction of the Yosida approximation of $\mathcal{G}$ to, at least, $\mathcal{R}(\mathcal{D})$,

$$Du := \lim_{s\to\infty, s\in\mathbb{R}} \Xi(s)u = Vu + V^{-*}\mathcal{G}_\Lambda\mathcal{D}u = V^{-*}(R_- + \mathcal{G}_\Lambda\mathcal{D})u.$$

Hence $R_- + \mathcal{G}_\Lambda\mathcal{D}$ is boundedly invertible iff the limit operator $D$ is boundedly invertible and then (2.11) reads as

$$u = -D^{-1}V^{-*}\mathcal{G}_\Lambda x \quad \text{for} \quad x \in D(\mathcal{G}_\Lambda). \tag{C.6}$$

Introducing $\hat{G}_\mathcal{G}(s) := s\mathcal{G}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{G}_\Lambda\mathcal{D} = \mathcal{G}_\Lambda\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}$, we also have

$$\left[R_- + \mathcal{G}_\Lambda\mathcal{D} + \hat{G}_\mathcal{G}(j\omega)\right]^* R_-^{-1}\left[R_- + \mathcal{G}_\Lambda\mathcal{D} + \hat{G}_\mathcal{G}(j\omega)\right] = \Pi(j\omega)$$

and

$$\Xi(s) = V + V^{-*}\left[\mathcal{G}_\Lambda\mathcal{D} + \hat{G}_\mathcal{G}(s)\right] = V^{-*}\left[R_- + \mathcal{G}_\Lambda\mathcal{D} + \hat{G}_\mathcal{G}(s)\right].$$

□

**Comment C.1.** An important item of the proof above was to show that the feedback control (2.11) gives rise, in the time-domain, to a control $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$, which ensures its optimality. For that, we can also adapt the arguments from [11, Proof of Theorem 5.1]. The Laplace transform of the control given by (2.11) is

$$\hat{u} = -(R_- + \mathcal{G}_\Lambda\mathcal{D})^{-1}\mathcal{G}_\Lambda(sI - \mathcal{A}_{\mathrm{opt}})^{-1}x_0,$$

and we wish to show that $\hat{u} \in \mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$. Premultiplying the resolvent equation for the closed-loop state operator $\mathcal{A}_{\mathrm{opt}}$:

$$sx - \mathcal{A}_{\mathrm{opt}}x = sx - \mathcal{A}\left[x - \mathcal{D}\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda x\right] = x_0 \in \mathrm{H}, \qquad s \in \mathbb{C}^+$$

by $\mathcal{G}_\Lambda(sI-\mathcal{A})^{-1}$ one obtains

$$\left[I + \mathcal{G}_\Lambda\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\right]\mathcal{G}_\Lambda(sI-\mathcal{A}_{\mathrm{opt}})^{-1}x_0 = \mathcal{G}_\Lambda(sI-\mathcal{A})^{-1}x_0.$$

Now, thanks to (3.4):

$$\Xi(s)\underbrace{\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda(sI-\mathcal{A}_{\mathrm{opt}})^{-1}x_0}_{=-\hat{u}(s)} = V^{-*}\mathcal{G}(sI-\mathcal{A})^{-1}x_0,$$

or, since $\Xi^{-1}(s) \in \mathbf{L}(\mathrm{U})$ and $s \longmapsto \mathcal{G}_\Lambda(sI-\mathcal{A})^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+;\mathrm{U})$, we have

$$\hat{u}(s) = -\Xi^{-1}(s)V^{-*}\mathcal{G}_\Lambda(sI-\mathcal{A})^{-1}x_0, \qquad \hat{u} \in \mathrm{H}^2(\mathbb{C}^+;\mathrm{U}).$$

In particular, this means that the observation operator given by the RHS of (2.11) is admissible with respect to $\{S_{\mathrm{opt}}(t)\}_{t\geq 0}$.

## D. PROOF OF A RESULT BY ROSENBLUM AND ROVNYAK

For the sake of clarity here we prove the result of Rosenblum and Rovnyak [16, Theorem B, pp. 15–16 and Theorem C (i), p. 96 with $\Omega = \mathbb{C}^+$, $\mathcal{C} = \mathrm{U}$] stated therein mainly for Hardy spaces on the unit disk.

**Lemma D.1.** *If the operator of multiplication* $(\mathcal{M}\hat{f})(s) = \Omega(s)\hat{f}(s)$ *by an analytic operator multiplier* $\mathbb{C}^+ \ni s \longmapsto \Omega(s) \in \mathbf{L}(\mathrm{U})$*, belongs to* $\mathbf{L}(\mathrm{H}^2(\mathbb{C}^+, \mathrm{L}(\mathrm{U})))$*, then* $\Omega \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$.

*Proof.* Let $S$ be the *canonical shift* on $\mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$ [16, p. 95, where the Hardy classes have been defined for the upper complex half-plane rather than $\mathbb{C}^+$ – so one has to replace $z$ in (5-4) by $js$],

$$(S\hat{f})(s) := \frac{s-1}{s+1}\hat{f}(s),\ \hat{f} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U});\ (S^*\hat{g})(s) = \hat{g}(s) + 2\frac{\hat{g}(s) - \hat{g}(1)}{s-1},\ \hat{g} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U}).$$

It is straightforward to establish that $\frac{1}{s+\overline{\lambda}}c$, where $c \in \mathrm{U}$ is an eigenvector of $S^*$, corresponding to its eigenvalue $\frac{\overline{\lambda}-1}{\overline{\lambda}+1}$. By the Paley-Wiener theorem, ($\hat{f} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U})$ iff $f \in \mathrm{L}^2(0, \infty; \mathrm{U})$), there holds with $e_\lambda(t) := e^{-\overline{\lambda}t}$, $\hat{e}_\lambda(s) = \frac{1}{s+\overline{\lambda}}$:

$$\left\langle \hat{f}, \hat{e}_\lambda c \right\rangle_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \langle f, e_\lambda c\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} = \int_0^\infty \langle f(t), e^{-\overline{\lambda}t}c\rangle_{\mathrm{U}}\mathrm{d}t =$$
$$= \int_0^\infty \langle e^{-\lambda t}f(t), c\rangle_{\mathrm{U}}\mathrm{d}t = \left\langle \int_0^\infty e^{-\lambda t}f(t)\mathrm{d}t, c \right\rangle_{\mathrm{U}} = \langle \hat{f}(\lambda), c\rangle_{\mathrm{U}},$$

whence, replacing $\hat{f}$ by $\mathcal{M}\hat{f}$ one obtains

$$\left\langle \hat{f}, \mathcal{M}^*\hat{e}_\lambda c \right\rangle_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \left\langle \mathcal{M}\hat{f}, \hat{e}_\lambda c \right\rangle_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \langle (\mathcal{M}\hat{f})(\lambda), c\rangle_{\mathrm{U}} =$$
$$= \langle \Omega(\lambda)\hat{f}(\lambda), c\rangle_{\mathrm{U}} = \langle \hat{f}(\lambda), [\Omega(\lambda)]^*c\rangle_{\mathrm{U}} = \langle f, e_\lambda[\Omega(\lambda)]^*c\rangle_{\mathrm{L}^2(0,\infty;\mathrm{U})} =$$
$$= \left\langle \hat{f}, \hat{e}_\lambda[\Omega(\lambda)]^*c \right\rangle_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})}, \qquad \forall \hat{f} \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{U}), \quad \forall c \in \mathrm{U},$$

or equivalently

$$\hat{e}_\lambda[\Omega(\lambda)]^*c = \mathcal{M}^*\hat{e}_\lambda c, \qquad \forall c \in \mathrm{U}.$$

Hence

$$\frac{1}{\sqrt{2\,\mathrm{Re}\,\lambda}}\,\|[\Omega(\lambda)]^*c\|_{\mathrm{U}} = \|\hat{e}_\lambda[\Omega(\lambda)]^*c\|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \|\mathcal{M}^*\hat{e}_\lambda c\|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} \le$$
$$\le \|\mathcal{M}\|_{\mathbf{L}(\mathrm{H}^2(\mathbb{C}^+,\mathrm{L}(\mathrm{U})))}\,\|\hat{e}_\lambda c\|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \|\mathcal{M}\|_{\mathbf{L}(\mathrm{H}^2(\mathbb{C}^+,\mathrm{L}(\mathrm{U})))}\frac{1}{\sqrt{2\,\mathrm{Re}\,\lambda}}\|c\|_{\mathrm{U}}, \quad \forall c \in \mathrm{U},$$

yielding $\|\Omega(\lambda)\|_{\mathbf{L}(\mathrm{U})} \le \|\mathcal{M}\|_{\mathbf{L}(\mathrm{H}^2(\mathbb{C}^+,\mathrm{L}(\mathrm{U})))}$. This estimate is valid for any $\lambda \in \mathbb{C}^+$, or equivalently for any $\mu = \frac{\lambda-1}{\lambda+1}$, $|\mu| < 1$, because the open unit disk is filled up by eigenvalues of $S^*$. □

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Piotr Grabowski
pgrab@agh.edu.pl

AGH University of Science and Technology
Faculty of Electrical Engineering, Automatics, Biomedical Engineering
and Computer Science
Institute of Control and Biomedical Engineering
al. Mickiewicza 30, 30-059 Krakow, Poland

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