

ON THE SOLVABILITY OF DIRICHLET PROBLEM FOR THE WEIGHTED p -LAPLACIAN

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Abstract. The paper investigates the existence and uniqueness of weak solutions for a non-linear boundary value problem involving the weighted p -Laplacian. Our approach is based on variational principles and representation properties of the associated spaces.

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1. INTRODUCTION

In this paper we are concerned with the existence and uniqueness of the weak solution to the boundary value problem

$$D(\Omega) : \begin{cases} -\Delta_{a,p}v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

in which $\Delta_{a,p}$, with $1 < p < \infty$, denotes the p -Laplacian weighted by a vector-valued function $a = (a_1, \dots, a_N)$, that can be (formally) given by

$$\Delta_{a,p}v = \operatorname{div}(a(x)|\nabla v|^{p-2}\overline{\nabla v}), \quad (1.2)$$

where ∇v denotes the weak gradient of a function v and, respectively, div means the divergence operator (also understood in the weak sense). We treat the problem under general conditions on the weight function a , namely, we suppose that the components a_j ($j = 1, \dots, N$) of a are measurable functions on Ω such that

$$a_j(x) \geq 0 \text{ for } x \in \Omega \text{ a.e., } a_j \in L^1_{loc}(\Omega) \text{ and } 1/a_j \in L_\infty(\Omega) \text{ (} j = 1, \dots, N \text{)}. \quad (1.3)$$

Ω is considered an arbitrary open domain in \mathbb{R}^N . We do not assume any smoothness conditions on its boundary $\partial\Omega$, it is not even assumed that the boundary has Lebesgue measure zero.

Boundary value problem involving the p -Laplacian and in general quasi-linear elliptic differential operators were extensively studied by many authors. We restrict ourselves to cite only the works [7, 9, 10] and references therein, and also the survey [4] (see also [3]) for more recent results. The methods used were mostly based on the technique of monotone operators developed by Leray-Lions [14] (see also [7], Section I.1.6, Leray-Lions Theorem, p.31). In this context, it should be noted the variational methods proposed in [2, 13, 15].

Our approach is based on a variational method related to that used in Hilbert spaces case. The essence of it is to interpret the problem as a generalized Dirichlet problem by involving a non-linear form defined on a suitable space which we denote by $W_a^{1,p}(\Omega)$. We prove that for any elements f from the dual space $W_a^{-1,p}(\Omega)$ there exists a uniquely weak solution $v \in W_a^{1,p}(\Omega)$ of the boundary problem (1.1). Moreover the set of all weak solutions of the problem is covered by the entire space $W_a^{1,p}(\Omega)$ whence f runs through on $W_a^{-1,p}(\Omega)$. The main results are presented by Theorem 2.2 in Section 2. The proof of the main results is given in Section 3.

2. DIRICHLET PROBLEM FOR THE WEIGHTED P -LAPLACIAN

The problem (1.1) will be considered as *the generalized Dirichlet problem* written in a variational form, namely, for a given locally sumable function f , we write

$$\int_{\Omega} a(x) \nabla u |\nabla v|^{p-2} \overline{\nabla v} \, dx = \int_{\Omega} u \bar{f} \, dx \quad \text{for all } u \in C_0^{\infty}(\Omega). \quad (2.1)$$

Further on, we assume that the components a_j ($j = 1, \dots, N$) of the vector-valued function a are measurable functions satisfying conditions (1.3).

Under these conditions we consider the following (non-linear) form

$$a[u, v] = \int_{\Omega} a(x) \nabla u |\nabla v|^{p-2} \overline{\nabla v} \, dx \quad (2.2)$$

defined on functions $u, v \in C_0^{\infty}(\Omega)$.

It will need the following auxiliary results.

Lemma 2.1. *Under conditions (1.3) there holds the following inequality*

$$\int_{\Omega} |\nabla u|^p \, dx \leq c \int_{\Omega} |a(x) \nabla u|^p \, dx \quad (2.3)$$

for all $u \in C_0^{\infty}(\Omega)$.

Proof. By using the Hölder inequality we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^p dx &\leq \|a^{-1}\|_{L^\infty(\Omega)} \int_{\Omega} |a(x)\nabla u|\nabla u|^{p-2}\overline{\nabla u}| dx \leq \\ &\leq \|a^{-1}\|_{L^\infty(\Omega)} \left(\int_{\Omega} |a(x)\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |\nabla u|^{(p-1)q} dx \right)^{\frac{1}{q}}, \end{aligned}$$

where q is the conjugate number of p . Hence

$$\left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}} \leq \|a^{-1}\|_{L^\infty(\Omega)} \left(\int_{\Omega} |a(x)\nabla u|^p dx \right)^{\frac{1}{p}},$$

that is the desired inequality. □

Next, we consider on $C_0^\infty(\Omega)$ the functional

$$\|u\|_a = \left(\int_{\Omega} |a(x)\nabla u|^p dx \right)^{\frac{1}{p}} \quad \text{for } u \in C_0^\infty(\Omega)$$

which obviously is a norm on $C_0^\infty(\Omega)$. We will denote the completion of $C_0^\infty(\Omega)$ with respect to the metric of this norm $\|\cdot\|_a$ by $W_a^{1,p}(\Omega)$. We will need some properties of the obtained space $W_a^{1,p}(\Omega)$. In this context, note that it is a uniformly convex space (for the concept of the uniformly convex spaces see, for instance, [12]) and therefore there holds a representation theorem for linear continuous functionals defined on it (see [12, Theorem 8.2, p. 288]). Besides, the space $W_a^{1,p}(\Omega)$ can be realized by elements of the Sobolev space $W_0^{1,p}(\Omega)$, more exactly $W_a^{1,p}(\Omega)$ can be embedded continuously in $W_0^{1,p}(\Omega)$. In fact, for any $u \in W_a^{1,p}(\Omega)$ there exists a sequence of elements $u_n \in C_0^\infty(\Omega)$ such that

$$\|u_n - u\|_a \rightarrow 0, \quad n \rightarrow \infty.$$

By Lemma 2.1 one also has

$$\int_{\Omega} |\nabla(u_n - u_m)|^p dx \rightarrow 0, \quad n, m \rightarrow \infty,$$

or, what is the same,

$$\|u_n - u_m\|_{W_0^{1,p}(\Omega)} \rightarrow 0, \quad n, m \rightarrow \infty.$$

Due to the fact that $W_0^{1,p}(\Omega)$ is a complete space there exists an element $v \in W_0^{1,p}(\Omega)$ such that

$$\|u_n - v\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The element v depends only on u and it does not depend on the chosen sequence (u_n) . So, the elements u, v can be identified provided that the norm $\|\cdot\|_a$ is compatible with the Sobolev norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. The compatibility means that if

$$\|u_n\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \quad \text{and} \quad \|u_n - u\|_a \rightarrow 0,$$

then $u = 0$. To prove this fact, without loss generality, we can assume that $\nabla u_n(x) \rightarrow 0$ almost everywhere (otherwise we may pass to a suitable subsequence of (u_n)). For any $\varepsilon > 0$ one has

$$\|u_m - u_n\|_a < \varepsilon$$

for sufficiently large n and m . By applying Fatou's Lemma,

$$\begin{aligned} \|u_n\|_a^p &= \int_{\Omega} |a(x)\nabla u_n|^p dx = \int_{\Omega} \lim_{m \rightarrow \infty} |a(x)\nabla(u_n - u_m)|^p dx \leq \\ &\leq \liminf_{m \rightarrow \infty} \int_{\Omega} |a(x)\nabla(u_n - u_m)|^p dx \leq \varepsilon^p, \end{aligned}$$

we see that $u_n \rightarrow 0$ with respect to the topology norm of $W_a^{1,p}(\Omega)$, and thus $u = 0$.

Next, we denote by $W_a^{-1,p}(\Omega)$ the dual space of $W_a^{1,p}(\Omega)$. Descriptions of dual Sobolev type spaces are found in [1] (see also [11] for related results concerning weighted Sobolev spaces). Besides, we also note the works [5, 6] for some abstract more general results, but for Hilbert spaces case.

Our main result is the following.

Theorem 2.2. *Suppose that the conditions (1.3) are fulfilled. For $f \in W_a^{-1,p}(\Omega)$, the Dirichlet problem (2.1) has a unique weak solution $v \in W_a^{1,p}(\Omega)$, i.e.*

$$\int_{\Omega} a(x)\nabla u |\nabla v|^{p-2} \overline{\nabla v} dx = \langle u, f \rangle$$

for all $u \in C_0^\infty(\Omega)$ (or, equivalently, for any $u \in W_a^{1,p}(\Omega)$). Moreover, the set of all weak solutions, where f runs through $f \in W_a^{-1,p}(\Omega)$ is the entire space $W_a^{1,p}(\Omega)$.

Next, for $p < N$ we let $p^* = Np/(N-p)$ for the critical Sobolev exponent, and denote s' for the conjugate number of any $s \in [p, p^*]$. It follows from the Sobolev embedding theorems that any element $f \in L_{s'}(\Omega)$ can be viewed as an element in $W_0^{-1,p}(\Omega)$ (see, for instance, [8, Theorem 3.7, p. 230]). Thus, from the fact that $W_a^{1,p}(\Omega)$ is embedded continuously in $W_0^{1,p}(\Omega)$, f can be treated as an element of the space $W_a^{-1,p}(\Omega)$. Taking into account this fact, we can formulate.

Corollary 2.3. *Under conditions (1.3) for every $f \in L_{s'}(\Omega)$ there exists a unique weak solution $v \in W_a^{1,p}(\Omega)$ solving problem (2.1).*

3. PROOF OF THEOREM 2.2

Due to the estimate (2.3) in Lemma 2.1 the form $a[u, v]$ defined by (2.2) can be extended on elements of the space $W_a^{1,p}(\Omega)$. To this form we can associate an operator A from $W_a^{1,p}(\Omega)$ into $W_a^{-1,p}(\Omega)$ as follows. For any $v \in W_a^{1,p}(\Omega)$, we consider the functional f defined on $W_a^{1,p}(\Omega)$ by

$$\langle u, f \rangle = \int_{\Omega} a(x) \nabla u |\nabla v|^{p-2} \overline{\nabla v} \, dx, \quad u \in W_a^{1,p}(\Omega).$$

This functional is linear and bounded, hence $f \in W_a^{-1,p}(\Omega)$. Moreover the operator A satisfies the following variational equation

$$\langle u, Av \rangle = a[u, v] \quad \text{for all } u, v \in W_a^{1,p}(\Omega), \tag{3.1}$$

where $\langle u, v \rangle = \int_{\Omega} u \overline{v} \, dx$.

Next, we change the topology of the space $W_a^{1,p}(\Omega)$ by defining the following metric on this space

$$d_a(u, v) = \sup_{\|w\|_a=1} |a[w, u] - a[w, v]|, \tag{3.2}$$

i.e.,

$$d_a(u, v) = \sup_{\|w\|_a=1} \left| \int_{\Omega} a(x) \nabla w \left(|\nabla u|^{p-2} \overline{\nabla u} - |\nabla v|^{p-2} \overline{\nabla v} \right) \, dx \right|, \quad u, v \in W_a^{1,p}(\Omega).$$

$W_a^{1,p}(\Omega)$ equipped with the metric d_a becomes a complete metric space. Moreover, for any $u, v \in W_a^{1,p}(\Omega)$ we have

$$\|Au - Av\|_{W_a^{-1,p}(\Omega)} = \sup_{\|w\|_a=1} |\langle w, Au - Av \rangle| = \sup_{\|w\|_a=1} |a[w, u] - a[w, v]| = d_a(u, v),$$

hence A is an isometry viewed as an operator from the metric space $(W_a^{1,p}(\Omega), d_a)$ into $W_a^{-1,p}(\Omega)$. Due to the fact that the space $W_a^{1,p}(\Omega)$ is a uniformly convex space (that implies it is a strictly convex space) the operator A is injective. Besides, A is surjective from the representation theorem, i.e. to every continuous linear functional $f \in W_a^{-1,p}(\Omega)$ there exists a unique element $v \in W_a^{1,p}(\Omega)$ such that

$$\langle u, f \rangle = a[u, v] \quad \text{for all } u \in W_a^{1,p}(\Omega).$$

Hence, A is a bijection between the space $W_a^{1,p}(\Omega)$ and $W_a^{-1,p}(\Omega)$ and there exists the inverse operator A^{-1} . Therefore, for $f \in W_a^{-1,p}(\Omega)$ the element $v = A^{-1}f$ is a weak solution of the generalized Dirichlet problem and the set of those solutions covers the entire space $W_a^{1,p}(\Omega)$ whence f runs through the dual space $W_a^{-1,p}(\Omega)$. The proof of Theorem 2.2 is complete. □

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