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EXISTENCE RESULT FOR HEMIVARIATIONAL INEQUALITY INVOLVING $p(x)$-LAPLACIAN

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Abstract. In this paper we study the nonlinear elliptic problem with $p(x)$-Laplacian (hemivariational inequality). We prove the existence of a nontrivial solution. Our approach is based on critical point theory for locally Lipschitz functionals due to Chang [J. Math. Anal. Appl. **80** (1981), 102–129].

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $\mathcal{C}^2$-boundary $\partial\Omega$ and $N > 2$. In this paper we study the following nonlinear elliptic differential inclusion with $p(x)$-Laplacian

$$\begin{cases} -\Delta_{p(x)}u - \lambda|u(x)|^{p(x)-2}u(x) \in \partial j(x, u(x)) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function satisfying

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < N < \infty \tag{1.2}$$

and

$$p^+ \le \hat{p}^* := \frac{Np^-}{N - p^-}, \tag{1.3}$$

and $j(x, t)$ is a function which is locally Lipschitz in the t-variable (in general it can be nonsmooth) and measurable in x-variable. By $\partial j(x, t)$ we denote the subdifferential with respect to the t-variable in the sense of Clarke [4]. The operator

$$\Delta_{p(x)}u := \operatorname{div}\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big)$$

is the so-called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$. Problems with $p(x)$-Laplacian are more complicated than with p-Laplacian, in particular, they are inhomogeneous and possess “more nonlinearity".

In our problem appears λ, for which we assume that $\lambda < \frac{p^-}{p^+}\lambda_*$, where λ_* is introduced by the following Rayleigh quotient (see Fan-Zhang [10]):

$$\lambda_* = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega |\nabla u(x)|^{p(x)} dx}{\int\limits_\Omega |u(x)|^{p(x)} dx}. \tag{1.4}$$

It may happen that $\lambda_* = 0$ (see Fan-Zhang [10]).

Our starting point is the paper of Gasiński-Papageorgiou [13], where the authors consider a similar problem but with the constant exponent, i.e., when $p(x) \equiv p$. Problems with a constant exponent can be also found in the papers of Gasiński-Papageorgiou [14–16] and Kourogenic-Papageorgiou [20].

More recently, the study of $p(x)$-Laplacian problems has attracted more and more attention. In the papers of Fan-Zhang-Zhao [9] and Fan [6], we can find a theory concerning the eigenvalues of the $p(x)$-Laplacian with both Dirichlet and Neumann boundary conditions. In Fan-Zhang [10] several sufficient conditions are indicated to obtain existence results for a Dirichlet boundary value problem with $p(x)$-Laplacian. In particular the existence of infinitely many solutions is shown. In Fan [7] a multiplicity theorem is proved for the problem with singular coefficients.

Finally we have papers where differential inclusions involving $p(x)$-Laplacian are studied. In Ge-Xue [17] and Qian-Shen [22], a differential inclusion involving $p(x)$-Laplacian and Clarke subdifferential with Dirichlet boundary condition is considered. In the last paper the existence of two solutions of constant sign is proved. Differential inclusions with Neumann boundary conditions were studied in Qian-Shen-Yang [23] and Dai [5]. In Qian-Shen-Yang [23], the inclusions involve a weighted function which is indefinite. In Dai [5], the existence of infinitely many nonnegative solutions is proved. In Ge-Xue-Zhou [18], authors proved sufficient conditions to obtain radial solutions for differential inclusions with $p(x)$-Laplacian. All the above mentioned papers deal with the so called hemivariational inequalities, i.e. the multivalued part is provided by the Clarke subdifferential of the nonsmooth potential (see e.g. Naniewicz-Panagiotopoulos [21]).

The techniques of this paper differ from those used in the above mentioned papers. Our method is more direct and is based on the critical point theory for nonsmooth Lipschitz functionals of Chang [3]. For the convenience of the reader in the next section we briefly present the basic notions and facts from the theory, which will be used in the study of problem (1.1). Moreover, we present the main properties of the general Lebesgue and variable Sobolev spaces.

2. MATHEMATICAL PRELIMINARIES

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|$ we will denote the norm in X and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X, X^*). A function $f : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighbourhood U of x and a constant $K > 0$ depending on U such that $|f(y) - f(z)| \le K\|y - z\|$ for all $y, z \in U$. From convex analysis it is well known that a proper, convex and lower semicontinuous function $g : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its domain $\mathrm{dom} g = \{x \in X : g(x) < \infty\}$.

For a locally Lipschitz function $f : X \to \mathbb{R}$ we define the generalized directional derivative of f at $x \in X$ in the direction $h \in X$ by

$$f^0(x; h) = \limsup_{x'\to 0,\lambda\to 0} \frac{f(x + x' + \lambda h) - f(x + x')}{\lambda}.$$

The function $h \longmapsto f^0(x, h) \in \mathbb{R}$ is sublinear, continuous so it is the support function of a nonempty, convex and w^*-compact set

$$\partial f(x) = \{x^* \in X^* : \langle x^*, h\rangle \le f^0(x, h) \text{ for all } h \in X\}.$$

The set $\partial f(x)$ is known as the subdifferential of f at x. If $f, g : X \to \mathbb{R}$ are two locally Lipschitz functions, then $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(tf)(x) = t\partial f(x)$ for all $t \in \mathbb{R}$.

A point $x \in X$ is said to be a critical point of the locally Lipschitz function $f : X \to \mathbb{R}$, if $0 \in \partial f(x)$. If $x \in X$ is local minimizer or local maximizer of f, then x is a critical point.

We say that f satisfies the "nonsmooth Palais-Smale condition" (nonsmooth PS-condition for short), if any sequence $\{x_n\}_{n\ge1} \subseteq X$ such that $\{f(x_n)\}_{n\ge1}$ is bounded and $m(x_n) = \min\{\|x^*\|_* : x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

The first theorem is due to Chang [3] and extends to a nonsmooth setting the well known "mountain pass theorem" due to Ambrosetti-Rabinowitz [1].

Theorem 2.1. *If X is a reflexive Banach space, $R : X \to \mathbb{R}$ is a locally Lipschitz functional satisfying the PS-condition and for some $\rho > 0$ and $y \in X$ such that $\|y\| > \rho$, we have*

$$\max\{R(0), R(y)\} < \inf_{\|x\|=\rho} \{R(x)\} =: \eta,$$

then R has a nontrivial critical point $x \in X$ such that the critical value $c = R(x) \ge \eta$ is characterized by the following minimax expression

$$c = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} \{R(\gamma(\tau))\},$$

where $\Gamma = \{\gamma \in \mathcal{C}([0, 1], X) : \gamma(0) = 0, \gamma(1) = y\}$.

In order to discuss problem (1.1), we need to state some properties of the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, which we call generalized Lebesgue-Sobolev spaces (see Fan-Zhao [11, 12]).

Let

$$E(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} : \ u \text{ is measurable}\}.$$

Two functions in $E(\Omega)$ are considered to be one element of $E(\Omega)$, when they are equal almost everywhere. Define

$$L^{p(x)}(\Omega) = \Big\{u \in E(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\},$$

with the norm

$$\|u\|_{p(x)} = \|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \leq 1\Big\}.$$

Then $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space.

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

By $W_0^{1,p(x)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Lemma 2.2 (Fan-Zhao [11]). *If $\Omega \subset \mathbb{R}^N$ is an open domain, then:*

(a) *the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces;*
(b) *the space $L^{p(x)}(\Omega)$ is uniformly convex;*
(c) *if $1 \leq q(x) \in \mathcal{C}(\overline{\Omega})$ and $q(x) \leq p^*(x)$ (respectively $q(x) < p^*(x)$) for any $x \in \overline{\Omega}$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N, \end{cases}$$

then $W^{1,p(x)}(\Omega)$ is embedded continuously (respectively compactly) in $L^{q(x)}(\Omega)$;
(d) *Poincaré inequality holds in $W_0^{1,p(x)}(\Omega)$, i.e., there exists a positive constant c such that*

$$\|u\|_{p(x)} \leq c\|\nabla u\|_{p(x)} \quad \text{for all} \quad u \in W_0^{1,p(x)}(\Omega);$$

(e) *$(L^{p(x)}(\Omega))^* = L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

Lemma 2.3 (Fan-Zhao [11]). *Let $\varphi(u) = \int_\Omega |u(x)|^{p(x)} dx$ for $u \in L^{p(x)}(\Omega)$ and let $\{u_n\}_{n\geq 1} \subseteq L^{p(x)}(\Omega)$. Then:*

(a) *for $u \neq 0$, we have*

$$\|u\|_{p(x)} = a \Longleftrightarrow \varphi\Big(\frac{u}{a}\Big) = 1;$$

(b) *we have*

$$\begin{aligned}\|u\|_{p(x)} < 1 &\iff \varphi(u) < 1,\\ \|u\|_{p(x)} = 1 &\iff \varphi(u) = 1,\\ \|u\|_{p(x)} > 1 &\iff \varphi(u) > 1;\end{aligned}$$

(c) *if $\|u\|_{p(x)} > 1$, then*

$$\|u\|_{p(x)}^{p^-} \le \varphi(u) \le \|u\|_{p(x)}^{p^+};$$

(d) *if $\|u\|_{p(x)} < 1$, then*

$$\|u\|_{p(x)}^{p^+} \le \varphi(u) \le \|u\|_{p(x)}^{p^-};$$

(e) *we have*

$$\lim_{n\to\infty} \|u_n\|_{p(x)} = 0 \iff \lim_{n\to\infty} \varphi(u_n) = 0;$$

(f) *we have*

$$\lim_{n\to\infty} \|u_n\|_{p(x)} = \infty \iff \lim_{n\to\infty} \varphi(u_n) = \infty.$$

Similarly to Lemma 2.3, we have the following result.

Lemma 2.4 (Fan-Zhao [11])**.** *Let $\Phi(u) = \int\limits_\Omega (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)})dx$ for $u \in W^{1,p(x)}(\Omega)$ and let $\{u_n\}_{n\ge 1} \subseteq W^{1,p(x)}(\Omega)$. Then:*

(a) *for $u \neq 0$, we have*

$$\|u\| = a \iff \Phi\Big(\frac{u}{a}\Big) = 1,$$

(b) *we have*

$$\begin{aligned}\|u\| < 1 &\iff \Phi(u) < 1,\\ \|u\| = 1 &\iff \Phi(u) = 1,\\ \|u\| > 1 &\iff \Phi(u) > 1;\end{aligned}$$

(c) *if $\|u\| > 1$, then*

$$\|u\|^{p^-} \le \Phi(u) \le \|u\|^{p^+};$$

(d) *if $\|u\| < 1$, then*

$$\|u\|^{p^+} \le \Phi(u) \le \|u\|^{p^-};$$

(e) *we have*

$$\lim_{n\to\infty} \|u_n\| = 0 \Longleftrightarrow \lim_{n\to\infty} \Phi(u_n) = 0;$$

(f) *we have*

$$\lim_{n\to\infty} \|u_n\| = \infty \Longleftrightarrow \lim_{n\to\infty} \Phi(u_n) = \infty.$$

In what follows, we make use of the following simple fact.

Lemma 2.5. *Let* $u \in L^{p(x)}(\Omega)$. *Then:*

(a) $|u|^{p(x)-1} \in L^{p'(x)}(\Omega)$;

(b) $\big\||u|^{p(x)-1}\big\|_{p'(x)} \leq 1 + \|u\|_{p(x)}^{p^+}$.

Proof. Part (a) is obvious. To prove part (b), note that if $\| \, |u|^{p(x)-1}\|_{p'(x)} \leq 1$, then the inequality in (b) is evident. So, we can assume that $\| \, |u|^{p(x)-1}\|_{p'(x)} > 1$.

If $\|u\|_{p(x)} > 1$, then from the fact that $p'(x) = \frac{p(x)}{p(x)-1}$ and Lemma 2.3(c), we have

$$\big\||u|^{p(x)-1}\big\|_{p'(x)}^{p'^-} \leq \int\limits_{\Omega} |u(x)|^{(p(x)-1)p'(x)} dx = \int\limits_{\Omega} |u(x)|^{p(x)} dx \leq \|u\|_{p(x)}^{p^+}.$$

Thus, we see that $\| \, |u|^{p(x)-1}\|_{p'(x)} \leq \|u\|_{p(x)}^{\frac{p^+}{p'^-}} \leq 1 + \|u\|_{p(x)}^{p^+}$.

On the other hand, if $\|u\|_{p(x)} < 1$, then in a similar way, we obtain

$$\| \, |u|^{p(x)-1}\|_{p'(x)} \leq \|u\|_{p(x)}^{\frac{p^-}{p'^-}} \leq 1. \qquad \square$$

Consider the following function

$$J(u) = \int\limits_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \quad \text{for all} \quad u \in W_0^{1,p(x)}(\Omega).$$

We know that $J \in \mathcal{C}^1(W_0^{1,p(x)}(\Omega))$ and operator $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the derivative operator of J in the weak sense (see Chang [2]). We denote

$$A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*.$$

Then

$$\langle Au, v\rangle = \int\limits_{\Omega} |\nabla u(x)|^{p(x)-2}(\nabla u(x), \nabla v(x)) dx \quad \text{for all} \quad u, v \in W_0^{1,p(x)}(\Omega). \tag{2.1}$$

Lemma 2.6 (Fan-Zhang [8])**.** *If A is the operator defined above, then A is a continuous, bounded, strictly monotone and maximal monotone operator of type (S_+), i.e., if $u_n \to u$ weakly in $W_0^{1,p(x)}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0,$$

then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

In what follows, for every $r \in \mathbb{R}$, we introduce: $r_+ = \max\{r, 0\}$ and $r_- = \max\{-r, 0\}$.

3. EXISTENCE OF SOLUTIONS

We start by introducing our hypotheses on the function $j(x,t)$.
$H(j)$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(x,0) = 0$ for almost all $x \in \Omega$ and:

(i) for all $t \in \mathbb{R}$, the function $\Omega \ni x \to j(x,t) \in \mathbb{R}$ is measurable;
(ii) for almost all $x \in \Omega$, the function $\mathbb{R} \ni t \to j(x,t) \in \mathbb{R}$ is locally Lipschitz;
(iii) for almost all $x \in \Omega$ and all $v \in \partial j(x,t)$, we have $|v| \le a(x)$ with $a(x) \in L^\infty_+(\Omega) = \{f \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{x\in\Omega} f(x) > 0\}$;
(iv) there exists $\mu > \frac{p^+\lambda_+}{p^-}$ such that

$$\limsup_{|t|\to 0} \frac{p(x)j(x,t)}{|t|^{p(x)}} < -\mu, \quad \text{uniformly for almost all } x \in \Omega;$$

(v) there exists $\overline{u} \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\overline{c}\|\overline{u}\|^{p^+} \le \int_\Omega j(x,\overline{u}(x))dx, \qquad \text{if } \|\overline{u}\| \ge 1,$$

or

$$\overline{c}\|\overline{u}\|^{p^-} \le \int_\Omega j(x,\overline{u}(x))dx, \qquad \text{if } \|\overline{u}\| < 1,$$

where $\overline{c} := \max\{\frac{1}{p^-}, \frac{\lambda_-}{p^-}\}$.

Remark 3.1. Hypothesis $H(j)$ (v) can be replaced by a less restrictive but "more complicated" one, namely
(v') there exists $\overline{u} \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\frac{1}{p^-}\int_\Omega |\nabla\overline{u}(x)|^{p(x)}dx + \frac{\lambda_-}{p^-}\int_\Omega |\overline{u}(x)|^{p(x)}dx \le \int_\Omega j(x,\overline{u}(x))dx.$$

We introduce two functionals $K, L : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$K(u) = \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx \quad \text{for all} \quad u \in W_0^{1,p(x)}(\Omega)$$

and

$$L(u) = \int_\Omega \frac{\lambda}{p(x)}|u(x)|^{p(x)}dx + \int_\Omega j(x,u(x))dx \quad \text{for all} \quad u \in W_0^{1,p(x)}(\Omega).$$

Functionals K, L are locally Lipschitz. Let us set $R = K - L$. Then $R : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is also locally Lipschitz.

Lemma 3.2. *If hypotheses $H(j)$ hold and $\lambda \in (-\infty, \frac{p^-}{p^+}\lambda_*)$ (see (1.2) and (1.4)), then R satisfies the PS-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ be a sequence such that $\{R(u_n)\}_{n\geq 1}$ is bounded and $m(u_n) \to 0$ as $n \to \infty$. We will show that the sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ is bounded.

Suppose that this is not true. Then, passing to a subsequence if necessary, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$.

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \geq 1$. Then by passing to a further subsequence if necessary, we may also assume that (see Lemma 2.2(c))

$$\begin{array}{ll} y_n \to y & \text{in } L^{p(x)}(\Omega), \\ y_n(x) \to y(x) & \text{for a.a. } x \in \Omega, \\ y_n \to y & \text{weakly in } W_0^{1,p(x)}(\Omega), \end{array} \tag{3.1}$$

as $n \to \infty$. At the beginning, we try establish the asymptotic behaviour of the integral $\int_\Omega \frac{j(x,u_n(x))}{\|u_n\|^\alpha}dx$, where $\alpha > 1$.

By virtue of the Lebourg mean value theorem (see Clarke [4]), we know that for almost all $x \in \Omega$ and for all $n \geq 1$, we can find $v_n(x) \in \partial j(x, k_n u_n(x))$ with $0 < k_n < 1$, such that

$$|j(x,u_n(x)) - j(x,0)| = |\langle v_n(x), u_n(x)\rangle|. \tag{3.2}$$

So, from hypothesis $H(j)$(iii), for almost all $x \in \Omega$, we have

$$|j(x,u_n(x))| \leq |j(x,0)| + a(x)|u_n(x)| \leq a_1 + a_2|u_n(x)|, \tag{3.3}$$

for some $a_1, a_2 > 0$. So for any $\alpha > 1$, we can write that

$$\Big|\int_\Omega \frac{j(x,u_n(x))}{\|u_n\|^\alpha}dx\Big| \leq \int_\Omega \frac{|j(x,u_n(x))|}{\|u_n\|^\alpha}dx \leq \int_\Omega \frac{a_1 + a_2|u_n(x)|}{\|u_n\|^\alpha}dx \leq \frac{a_3}{\|u_n\|^\alpha} + \frac{a_4}{\|u_n\|^{\alpha-1}}$$

for some $a_3, a_4 > 0$. So

$$\frac{j(x,u_n(x))}{\|u_n\|^\alpha}dx \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Because $\|u_n\| \to \infty$ and $|R(u_n)| \leq M$ for all $n \geq 1$, without any loss of generality, we can assume that $\|u_n\| \geq 1$. We have

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n(x)|^{p(x)}dx - \int_\Omega \frac{\lambda}{p(x)}|u_n(x)|^{p(x)}dx - \int_\Omega j(x,u_n(x))dx \leq M. \tag{3.5}$$

Let us consider two cases.

Case 1. Let us assume that $\lambda = \lambda_+ > 0$.

So, in particular

$$\int_\Omega \frac{1}{p^+}|\nabla u_n(x)|^{p(x)}dx - \int_\Omega \frac{\lambda_+}{p^-}|u_n(x)|^{p(x)}dx - \int_\Omega j(x, u_n(x))dx \le M. \tag{3.6}$$

From the definition of λ_* (see (1.4)), we have

$$\lambda_* \int_\Omega |u_n(x)|^{p(x)}dx \le \int_\Omega |\nabla u_n(x)|^{p(x)}dx \quad \text{for all} \quad n \ge 1. \tag{3.7}$$

Using (3.7) in (3.6), we get

$$\Big(\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-}\Big) \int_\Omega |\nabla u_n(x)|^{p(x)}dx - \int_\Omega j(x, u_n(x))dx \le M. \tag{3.8}$$

Let us consider two subcases.

Subcase 1.1. We can choose a subsequence $\{u_n\}_{n\ge1} \subseteq L^{p(x)}(\Omega)$ such that

$$\|\nabla u_n\|_{p(x)} \le 1 \quad \text{for all } n \ge 1.$$

Then using Lemma 2.3(d) in (3.8), we have

$$\Big(\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-}\Big)\|\nabla u_n\|_{p(x)}^{p^+} - \int_\Omega j(x, u_n(x))dx \le M.$$

Dividing the last inequality by $\|u_n\|^{p^+}$, we obtain

$$\Big(\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-}\Big)\|\nabla y_n\|_{p(x)}^{p^+} - \int_\Omega \frac{j(x, u_n(x))}{\|u_n\|^{p^+}}dx \le \frac{M}{\|u_n\|^{p^+}}. \tag{3.9}$$

We know that $\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-} > 0$. From this fact and (3.4), if we pass to the limit as $n \to \infty$ in (3.9), we obtain

$$\nabla y_n \to 0 \quad \text{in} \quad L^{p(x)}(\Omega; \mathbb{R}^N).$$

Subcase 1.2. If Subcase 1.1. does not hold, then we can choose a subsequence $\{u_n\}_{n\ge1} \subseteq L^{p(x)}(\Omega)$ such that

$$\|\nabla u_n\|_{p(x)} > 1 \quad \text{for all} \quad n \ge 1.$$

Then using Lemma 2.3(c) in (3.8), we have

$$\Big(\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-}\Big)\|\nabla u_n\|_{p(x)}^{p^-} - \int_\Omega j(x, u_n(x))dx \le M.$$

Dividing the last inequality by $\|u_n\|^{p^-}$, we obtain

$$\Big(\frac{1}{p^+} - \frac{\lambda_+}{\lambda_* p^-}\Big)\|\nabla y_n\|^{p^-}_{p(x)} - \int\limits_\Omega \frac{j(x,u_n(x))}{\|u_n\|^{p^-}}dx \le \frac{M}{\|u_n\|^{p^-}}. \tag{3.10}$$

So again, if we pass to the limit as $n \to \infty$ in (3.10) and use (3.4), we get that

$$\nabla y_n \to 0 \quad \text{in} \quad L^{p(x)}(\Omega;\mathbb{R}^N).$$

Thus in both subcases, we obtained that

$$\nabla y_n \to 0 \quad \text{in} \quad L^{p(x)}(\Omega;\mathbb{R}^N). \tag{3.11}$$

Case 2. Now, we assume that $\lambda \le 0$.

From (3.5), we have

$$\int\limits_\Omega \frac{1}{p^+}|\nabla u_n(x)|^{p(x)}dx - \int\limits_\Omega j(x,u_n(x))dx \le M. \tag{3.12}$$

Again, let us consider two subcases.

Subcase 2.1. We can choose a subsequence $\{u_n\}_{n\ge1} \subseteq L^{p(x)}(\Omega)$ such that $\|\nabla u_n\|_{p(x)} \le 1$ for all $n \ge 1$.

Then using Lemma 2.3(d) in (3.12), we have

$$\frac{1}{p^+}\|\nabla u_n\|^{p^+}_{p(x)} - \int\limits_\Omega j(x,u_n(x))dx \le M.$$

Dividing the last inequality by $\|u_n\|^{p^+}$, we obtain

$$\frac{1}{p^+}\|\nabla y_n\|^{p^+}_{p(x)} - \int\limits_\Omega \frac{j(x,u_n(x))}{\|u_n\|^{p^+}}dx \le \frac{M}{\|u_n\|^{p^+}}. \tag{3.13}$$

We know that $\frac{1}{p^+} > 0$. From this fact and (3.4), if we pass to the limit as $n \to \infty$ in (3.13), we obtain

$$\nabla y_n \to 0 \quad \text{in} \quad L^{p(x)}(\Omega;\mathbb{R}^N).$$

Subcase 2.2. If Subcase 2.1 does not hold, so we can choose a subsequence $\{u_n\}_{n\ge1} \subseteq L^{p(x)}(\Omega)$ such that $\|\nabla u_n\|_{p(x)} > 1$ for all $n \ge 1$.

Then using Lemma 2.3(c) in (3.12), we have

$$\frac{1}{p^+}\|\nabla u_n\|^{p^-}_{p(x)} - \int\limits_\Omega j(x,u_n(x))dx \le M.$$

In a similar way like in Subcase 2.1, we obtain

$$\nabla y_n \to 0 \quad \text{in} \quad L^{p(x)}(\Omega;\mathbb{R}^N).$$

Thus in both subcases, we obtained that

$$\nabla y_n \to 0 \qquad \text{in} \quad L^{p(x)}(\Omega;\mathbb{R}^N). \tag{3.14}$$

Using again (3.7) in (3.6) in another way, we get

$$\Big(\frac{\lambda_*}{p^+}-\frac{\lambda_+}{p^-}\Big)\int\limits_\Omega |u_n(x)|^{p(x)}dx-\int\limits_\Omega j(x,u_n(x))dx\le M. \tag{3.15}$$

In a similar way, considering two cases (depending on whether we choose a subsequence $\{u_n\}_{n\ge1}\subseteq L^{p(x)}(\Omega)$ for which $\|u_n\|_{p(x)}>1$ or $\|u_n\|_{p(x)}<1$ for all $n\ge1$) and using Lemma 2.3(c), (d) and the fact that $\frac{\lambda_*}{p^+}-\frac{\lambda_+}{p^-}>0$, we conclude that

$$y_n \to 0 \qquad \text{in} \quad L^{p(x)}(\Omega). \tag{3.16}$$

From (3.11), (3.14) and (3.16), we get

$$y_n \to 0 \qquad \text{in} \quad W_0^{1,p(x)}(\Omega). \tag{3.17}$$

But on the other hand, from the definition of y_n, we know that $\|y_n\|=1$ for all $n\ge1$, a contradiction. Thus the sequence $\{u_n\}_{n\ge1}\subseteq W_0^{1,p(x)}(\Omega)$ is bounded.

Hence, by passing to a subsequence if necessary, we may assume that (see Lemma 2.2(c))

$$\begin{aligned} &u_n \to u \quad \text{weakly in } W_0^{1,p(x)}(\Omega),\\ &u_n \to u \quad \text{in } L^{r(x)}(\Omega), \end{aligned} \tag{3.18}$$

for any $r\in\mathcal{C}(\overline{\Omega})$, with $r^+=\max\limits_{x\in\Omega} r(x)<\hat{p}^*:=\frac{Np^-}{N-p^-}$.

Since $\partial R(u_n)\subseteq (W_0^{1,p(x)}(\Omega))^*$ is weakly compact, nonempty and the norm functional is weakly lower semicontinuous in a Banach space, then we can find $u_n^*\in\partial R(u_n)$ such that $\|u_n^*\|_*=m(u_n)$, for $n\ge1$.

Consider the operator $A: W_0^{1,p(x)}(\Omega)\to (W_0^{1,p(x)}(\Omega))^*$, defined by (2.1). In particular, we know that A is maximal monotone (see Lemma 2.6). Then, for every $n\ge1$, we have

$$u_n^* = Au_n-\lambda|u_n|^{p(x)-2}u_n-v_n^*, \tag{3.19}$$

where $v_n^*\in\partial\psi(u_n)\subseteq L^{p'(x)}(\Omega)$, for $n\ge1$, with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ and $\psi: W_0^{1,p(x)}(\Omega)\to\mathbb{R}$ is defined by

$$\psi(u_n)=\int\limits_\Omega j(x,u_n(x))dx.$$

We know that if $v_n^*\in\partial\psi(u_n)$, then $v_n^*(x)\in\partial j(x,u_n(x))$ (see Clarke [4]).

From the choice of the sequence $\{u_n\}_{n\ge1}\subseteq W_0^{1,p(x)}(\Omega)$, at least for a subsequence, we have

$$|\langle u_n^*,w\rangle|\le\varepsilon_n\|w\| \quad \text{for all } w\in W_0^{1,p(x)}(\Omega), \tag{3.20}$$

with $\varepsilon_n\searrow 0$.

Putting $w = u_n - u$ in (3.20) and using (3.19), we obtain

$$\begin{aligned}\langle Au_n, u_n - u\rangle - \lambda \int_\Omega |u_n(x)|^{p(x)-2} u_n(x)(u_n - u)(x)dx - \\ - \int_\Omega v_n^*(x)(u_n - u)(x)dx \leq \varepsilon_n \|u_n - u\|.\end{aligned} \tag{3.21}$$

Using Lemma 2.2(e), we see that

$$\lambda \int_\Omega |u_n(x)|^{p(x)-2} u_n(x)(u_n - u)(x)dx \leq \lambda\Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\| \, |u_n|^{p(x)-1}\|_{p'(x)} \|u_n - u\|_{p(x)},$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

We know that $\{u_n\}_{n\geq 1} \subseteq L^{p(x)}(\Omega)$ is bounded, so using (3.18) and Lemma 2.5, we can conclude that

$$\lambda \int_\Omega |u_n(x)|^{p(x)-2} u_n(x)(u_n - u)(x)dx \to 0 \quad \text{as } n \to \infty$$

and

$$\int_\Omega v_n^*(x)(u_n - u)(x)dx \to 0 \quad \text{as } n \to \infty.$$

So from (3.21), if we pass to the limit as $n \to \infty$, we have

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0. \tag{3.22}$$

Thus from Lemma 2.6, we have that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ as $n \to \infty$. So, we have proved that R satisfies the PS-condition. □

Lemma 3.3. *If hypotheses $H(j)$ holds and $\lambda < \frac{p^-}{p^+}\lambda_*$, then there exists $\beta_1, \beta_2 > 0$ such that for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| < 1$, we have*

$$R(u) \geq \beta_1 \|u\|^{p^+} - \beta_2 \|u\|^{\theta},$$

with $p^+ < \theta \leq \hat{p}^* := \frac{Np^-}{N-p^-}$.

Proof. Let $\varepsilon > 0$ be such that $\frac{p^+\lambda_+}{p^-} + \varepsilon < \mu$. From hypothesis $H(j)$(iv), we can find $\delta > 0$, such that for almost all $x \in \Omega$ and all t such that $|t| \leq \delta$, we have

$$j(x,t) \leq \frac{1}{p(x)}(-\mu + \varepsilon)|t|^{p(x)}.$$

On the other hand, from the proof of Lemma 3.2 (see (3.3)), we know that for almost all $x \in \Omega$ and all t such that $|t| > \delta$, we have

$$|j(x,t)| \leq a_1 + a_2|t|,$$

for some $a_1, a_2 > 0$. Thus for almost all $x \in \Omega$ and all $t \in \mathbb{R}$ we have

$$j(x,t) \leq \frac{1}{p(x)}(-\mu + \varepsilon)|t|^{p(x)} + \gamma |t|^{\theta},$$

with some $\gamma > 0$ and $p^+ < \theta < \hat{p}^*$. Using this, we obtain that

$$\begin{aligned} R(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx - \int_\Omega \frac{\lambda}{p(x)} |u(x)|^{p(x)} dx - \int_\Omega j(x, u(x)) dx \geq \\ &\geq \int_\Omega \frac{1}{p^+} |\nabla u(x)|^{p(x)} dx - \int_\Omega \frac{\lambda_+}{p^-} |u(x)|^{p(x)} dx + \\ &\quad + \frac{1}{p^+} \int_\Omega (\mu - \varepsilon) |u(x)|^{p(x)} dx - \gamma \int_\Omega |u(x)|^{\theta} dx = \\ &= \frac{1}{p^+} \int_\Omega |\nabla u(x)|^{p(x)} dx + \Big(\frac{\mu - \varepsilon}{p^+} - \frac{\lambda_+}{p^-} \Big) \int_\Omega |u(x)|^{p(x)} dx - \gamma \|u\|_\theta^\theta. \end{aligned}$$

From the choice of ε, we have

$$\frac{\mu - \varepsilon}{p^+} - \frac{\lambda_+}{p^-} > 0,$$

so

$$R(u) \geq \beta_1 \Big[\int_\Omega |\nabla u(x)|^{p(x)} dx + \int_\Omega |u(x)|^{p(x)} dx \Big] - \gamma \|u\|_\theta^\theta,$$

where $\beta_1 := \min\{\frac{1}{p^+}, \frac{\mu-\varepsilon}{p^+} - \frac{\lambda_+}{p^-}\}$.

As $\theta \leq p^*(x) = \frac{Np(x)}{N-p(x)}$, then $W_0^{1,p(x)}(\Omega)$ is embedded continuously in $L^\theta(\Omega)$ (see Lemma 2.2(c)). So, there exists $c > 0$ such that

$$\|u\|_\theta \leq c\|u\| \qquad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{3.23}$$

Using (3.23) and Lemma 2.4(d), for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| < 1$, we have

$$R(u) \geq \beta_1 \|u\|^{p^+} - \beta_2 \|u\|^\theta,$$

where $\beta_2 = \gamma c^\theta$. □

Using Lemmas 3.2 and 3.3, we can prove the following existence theorem for problem (1.1).

Theorem 3.4. *If hypotheses $H(j)$ holds and $\lambda < \frac{p^-}{p^+}\lambda_*$, then problem (1.1) has a nontrivial solution.*

Proof. From Lemma 3.3 we know that there exist $\beta_1, \beta_2 > 0$, such that for all $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| < 1$, we have

$$R(u) \geq \beta_1 \|u\|^{p^+} - \beta_2 \|u\|^{\theta} = \beta_1 \|u\|^{p^+} \Big(1 - \frac{\beta_2}{\beta_1} \|u\|^{\theta - p^+}\Big).$$

Since $p^+ < \theta$, if we choose $\rho > 0$ small enough, we will have that $R(u) \geq L > 0$, for all $u \in W_0^{1,p(x)}(\Omega)$, with $\|u\| = \rho$ and some $L > 0$.

Now, let $\overline{u} \in W_0^{1,p(x)}(\Omega)$ and $\overline{c} > 0$ be as in hypothesis $H(j)(v)$. We have

$$\begin{aligned} R(\overline{u}) &= \int_\Omega \frac{1}{p(x)} |\nabla \overline{u}(x)|^{p(x)} dx - \int_\Omega \frac{\lambda}{p(x)} |\overline{u}(x)|^{p(x)} dx - \int_\Omega j(x, \overline{u}(x)) dx \leq \\ &\leq \frac{1}{p^-} \int_\Omega |\nabla \overline{u}(x)|^{p(x)} dx + \frac{\lambda_-}{p^-} \int_\Omega |\overline{u}(x)|^{p(x)} dx - \int_\Omega j(x, \overline{u}(x)) dx \leq \\ &\leq \overline{c} \int_\Omega (|\nabla \overline{u}(x)|^{p(x)} + |\overline{u}(x)|^{p(x)}) dx - \int_\Omega j(x, \overline{u}(x)) dx, \end{aligned}$$

where $\overline{c} = \max\{\frac{1}{p^-}, \frac{\lambda_-}{p^-}\}$.

Using Lemma 2.4(c) or (d) and hyphothesis $H(j)$(v), we get $R(\overline{u}) \leq 0$. This permits the use of Theorem 2.1, which gives us $u \in W_0^{1,p(x)}(\Omega)$ such that $R(u) > 0 \geq R(0)$ and $0 \in \partial R(u)$. From the last inclusion we obtain

$$0 = Au - \lambda |u|^{p(x)-2} u - v^*,$$

where $v^* \in \partial \psi(u)$. Hence

$$Au = \lambda |u|^{p(x)-2} u + v^*,$$

so for all $v \in \mathcal{C}_0^\infty(\Omega)$, we have $\langle Au, v\rangle = \lambda \langle |u|^{p(x)-2} u, v\rangle + \langle v^*, v\rangle$ and thus

$$\int_\Omega |\nabla u(x)|^{p(x)-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^N} dx = \int_\Omega \lambda |u(x)|^{p(x)-2} u(x) v(x) dx + \int_\Omega v^*(x) v(x) dx$$

for all $v \in \mathcal{C}_0^\infty(\Omega)$.

From the definition of the distributional derivative we have

$$\begin{cases} -\operatorname{div}\big(|\nabla u(x)|^{p(x)-2} \nabla u(x)\big) = \lambda |u(x)|^{p(x)-2} u(x) + v(x) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

so

$$\begin{cases} -\Delta_{p(x)} u - \lambda |u(x)|^{p(x)-2} u(x) \in \partial j(x, u(x)) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Therefore, $u \in W_0^{1,p(x)}(\Omega)$ is a nontrivial solution of (1.1). □

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