

TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

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Abstract. A set S of vertices in a graph $G = (V, E)$ is a 2-dominating set if every vertex of $V \setminus S$ is adjacent to at least two vertices of S . The 2-domination number of a graph G , denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G . The 2-domination subdivision number $sd_{\gamma_2}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-domination number. The authors have recently proved that for any tree T of order at least 3, $1 \leq sd_{\gamma_2}(T) \leq 2$. In this paper we provide a constructive characterization of the trees whose 2-domination subdivision number is 2.

Keywords: 2-dominating set, 2-domination number, 2-domination subdivision number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A support vertex is *strong* if it is adjacent to at least two leaves. For a vertex v in a rooted tree T , let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v .

A *2-dominating set* of a graph $G = (V, E)$ is a subset S of vertices where each vertex in $V \setminus S$ is adjacent to at least two vertices of S . The *2-domination number* of a graph G , denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G . A $\gamma_2(G)$ -*set* is a 2-dominating set of G with size $\gamma_2(G)$. The 2-domination numbers have been studied by several authors (see for example [6, 7, 13, 15]).

The 2-*domination subdivision number* $sd_{\gamma_2}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-domination number of G . It is easy to see that [4] the 2-domination number of a graph cannot decrease when an edge of that graph is subdivided. For a more thorough treatment of domination parameters and for terminology not presented here see [12, 16].

Atapour *et al.* [4] showed the following result.

Theorem 1.1. *For any tree T of order $n \geq 3$, $1 \leq sd_{\gamma_2}(T) \leq 2$.*

Hence, trees can be classified as Class 1 or Class 2 depending on whether their 2-domination subdivision numbers are 1 or 2, respectively. In this paper we give a constructive characterization of trees in Class 2. For recent results on the topic “constructive characterization of graphs” the reader may consult [1–3, 9, 11, 14].

We make use of the following observations in this paper.

Theorem 1.2 ([7]). *Every 2-dominating set of a graph G contains every leaf.*

Observation 1.3 ([7]). *Let T be a tree obtained from a nontrivial tree T' by adding a star $K_{1,p}$ with the center vertex v attached by an edge vw at a vertex w of T' . Then $\gamma_2(T') + p \leq \gamma_2(T)$, with equality if $p \geq 2$ or w is a leaf in T' .*

2. TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

In this section we provide a constructive characterization of all trees in Class 2. For this purpose, we describe a procedure to build a family \mathcal{F} of labeled trees that are in Class 2 as follows. The label of a vertex is also called its *status* and denoted $sta(v)$. A labeled P_4 is a P_4 where the two leaves have status A and the other two vertices have status B and status C , respectively. Let \mathcal{F} be the family of labeled trees that: A labeled P_4 is a tree in \mathcal{F} and if T is a tree in \mathcal{F} , then the tree T' obtained from T by the following five operations which extend the tree T by attaching a tree to a vertex $y \in V(T)$, called an *attacher*, is also a tree in \mathcal{F} .

Operation \mathfrak{T}_1 . If $sta(y) = B$ (respectively, C) and y is a support vertex, then \mathfrak{T}_1 adds a vertex x and an edge xy to T with $sta(x) = A$. Moreover, if $\deg(y) = 2$ and y is adjacent to a vertex z of status C (respectively, B), then this operation changes the status of z to C' (respectively, B').

Operation \mathfrak{T}_2 . If $sta(y) = B$ (respectively, C) and y is adjacent to a support vertex z with $\deg(z) = 2$ and $sta(z) = C$ (respectively, B), then \mathfrak{T}_2 adds a vertex x and an edge xy to T with $sta(x) = A$. Moreover, this operation changes the status of z to C' (respectively, B').

Operation \mathfrak{T}_3 . If $sta(y) = A, A', B'$ or C' , then \mathfrak{T}_3 adds a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy to T with $sta(x) = F$ and $sta(x_1) = sta(x_2) = A$. Moreover, this operation changes the status of y from A to A' .

Operation \mathfrak{T}_4 . If $sta(y) = A$, then we have three cases:

Case 1. y is adjacent to a vertex z of status B or B' . Then \mathfrak{T}_4 adds a path yxu to T with $sta(x) = B$, $sta(u) = A$ and changes the status of y from A to C .

Case 2. y is adjacent to a vertex z of status C or C' . Then \mathfrak{T}_4 adds a path yxu to T with $sta(x) = C$, $sta(u) = A$ and changes the status of y from A to B .

Case 3. y is adjacent to a vertex z of status F . Then \mathfrak{T}_4 adds a path yxu to T with $sta(x) = C$, $sta(u) = A$ and changes the status of y from A to B .

Operation \mathfrak{T}_5 . If $sta(y) = F$, then \mathfrak{T}_5 adds a vertex x and an edge xy to the tree T with $sta(x) = A$.

The five operations are shown in Figure 1. Note that operation 3 adds two leaves and all the other operations add one leaf to tree T .

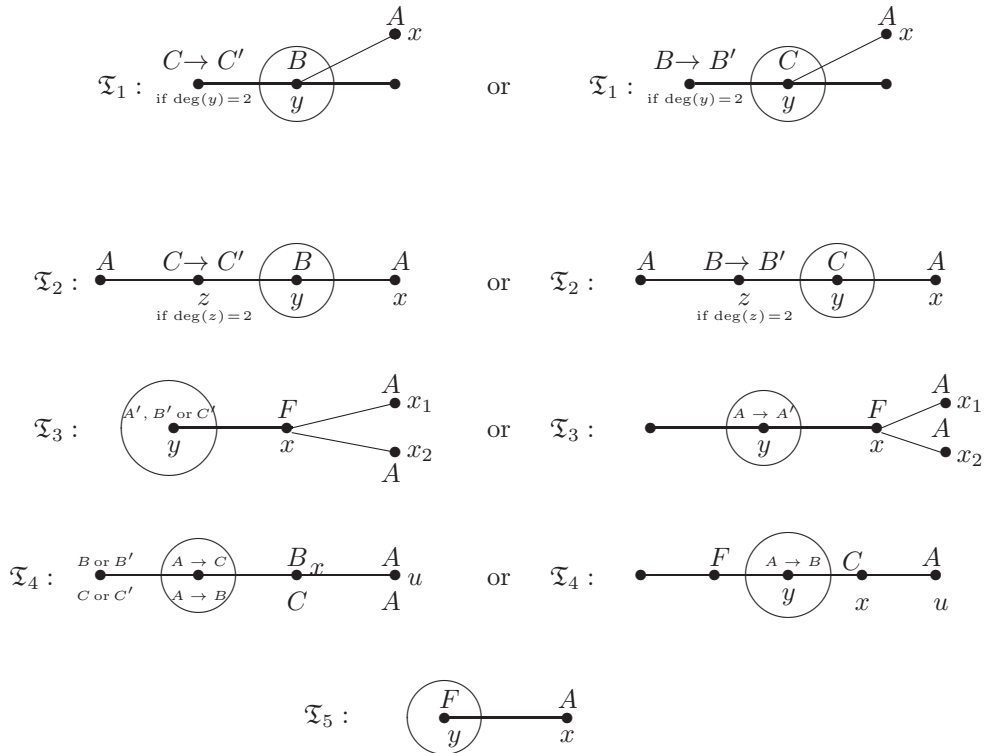


Fig. 1. The five operations

The family \mathcal{F}

If $T \in \mathcal{F}$, we let $A(T)$, $B(T)$, $C(T)$, $F(T)$, $A'(T)$, $B'(T)$ and $C'(T)$ be the set of vertices of status A, B, C, F, A', B' , and C' , respectively, in T . The following observation comes from the way in which each tree in the family \mathcal{F} is constructed.

Observation 2.1. Let $T \in \mathcal{F}$ and $v \in V(T)$.

1. The set of vertices with status A is the set of leaves of tree T .

2. If v is a support vertex, then $sta(v) = B, C, F, B'$ or C' .
3. If $sta(v) = B$ or B' , then v has at least one neighbor y of status C or C' and $N(v) - \{y\} \subset A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$. Thus $A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$ is a 2-dominating set for T .
4. If $sta(v) = C, C'$ or F , then v has at least two neighbors of status A, A', B or B' . Thus $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set for T .

We proceed with the following two propositions.

- Proposition 2.2.** 1. Let T' be a tree of order at least 3 and let y be a leaf of T' . Let T be a tree obtained from T' by adding a path yuv to T' . Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $sd_{\gamma_2}(T) \leq sd_{\gamma_2}(T')$.
2. Let T' be a tree of order at least 3 and let y be a strong support vertex of T' . Let T be a tree obtained from T' by adding a pendant edge yw . Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $sd_{\gamma_2}(T) \leq sd_{\gamma_2}(T')$.
3. Let T' be a tree of order at least 3 and let y be a leaf of T' . Let T be a tree obtained from T' by adding a path yuv to T' and $t(\geq 1)$ pendant edges at y . Then $\gamma_2(T) = \gamma_2(T') + t + 1$. Moreover, $sd_{\gamma_2}(T) \leq sd_{\gamma_2}(T')$.

Proof. (1) By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + 1$. Let F be a set of edges in T' where subdividing the edges in F increases the 2-domination number of T' . Let T_1 and T_2 be the trees obtained from T' and T , respectively, by subdividing the edges in F . Then y is a leaf in T_1 and T_2 is obtained from T_1 by adding a path yuv to T_1 . Thus

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $sd_{\gamma_2}(T) \leq sd_{\gamma_2}(T')$.

(2) Let u, v be the two leaves of T' adjacent to y in T' . Then u, v, w are leaves in T . It is easy to see that for every $\gamma_2(T')$ -set S , $S \cup \{w\}$ is a 2-dominating set of T . It follows that $\gamma_2(T) \leq \gamma_2(T') + 1$. Now if S' is a $\gamma_2(T)$ -set, then $\{u, v, w\} \subseteq S'$. Hence, $S' - \{w\}$ is a 2-dominating set of T' . Thus $\gamma_2(T) = \gamma_2(T') + 1$.

Let F be a set of edges in T' where subdividing the edges in F increases the 2-domination number of T' . Let T_1 and T_2 be the trees obtained from T' and T , respectively, by subdividing the edges in F . Then T_2 is obtained from T_1 by adding the pendant edge yw . If $F \cap \{yu, yv\} = \emptyset$, then, as before, we have $\gamma_2(T_2) = \gamma_2(T_1) + 1$ and so

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

Now suppose that $|F \cap \{yu, yv\}| \geq 1$. We may assume the edge yu is subdivided by inserting a vertex x . Obviously, for every $\gamma_2(T_1)$ -set S , $S \cup \{w\}$ is a 2-dominating set of T and so $\gamma_2(T_2) \leq \gamma_2(T_1) + 1$. Now if D is a $\gamma_2(T_2)$ -set, then by Theorem 1.2, $w \in D$ and to dominate x twice we must have $x \in D$ or $y \in D$. In each case $(D - \{x\}) \cup \{y\}$ is a 2-dominating set for T_1 . It follows that $\gamma_2(T_2) = \gamma_2(T_1) + 1$. As before, we have

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $sd_{\gamma_2}(T) \leq sd_{\gamma_2}(T')$.

- (3) The proof is similar to (1) and (2) and therefore omitted. \square

Proposition 2.3. *Let T be a tree obtained from a tree T' of order at least 3 by attaching a star $K_{1,t}$ ($t \geq 2$) with center x and joining x to a vertex y of T' . Then $\gamma_2(T) = \gamma_2(T') + t$. Moreover, $\text{sd}_{\gamma_2}(T) \leq \text{sd}_{\gamma_2}(T')$.*

Proof. By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + t$. An argument similar to that described in Proposition 2.2 (Part 1) shows that $\text{sd}_{\gamma_2}(T) \leq \text{sd}_{\gamma_2}(T')$. □

Reordering a set of operations with respect to a subset of $\{\mathfrak{T}_i\}_{i=1}^5$

Let T be a tree obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$. Let $J \subseteq \{1, 2, 3, 4, 5\}$ and $\mathfrak{T}_j \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $j \in J$. The following algorithm reorders operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ with respect to $\mathfrak{T}_j, j \in J$. It is easy to see that if we apply operations $\mathfrak{T}^i, 1 \leq i \leq m$ on a labeled P_4 , according to the new ordering, we obtain T .

Algorithm

1. Set $k = 0$.
2. Add one to k . If $k > m$, stop.
3. If $\mathfrak{T}^k \notin \{\mathfrak{T}_j \mid j \in J\}$, go to Step 2. If $\mathfrak{T}^k = \mathfrak{T}_j$ for some $j \in J$, proceed as follows. Find the smallest $\ell \in \{1, 2, \dots, k - 1\}$ such that applying \mathfrak{T}_j before \mathfrak{T}^ℓ does not lead to a different tree from T . If such an ℓ does not exist, go to Step 2, otherwise apply \mathfrak{T}_j before \mathfrak{T}^ℓ .

Note that for given successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ there exists a unique reordering with respect to a given subset of $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$.

Example 2.4. Let T (Figure 2) be obtained by applying the sequence $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4$ on the initial path $x_1x_2x_3x_4$. We see that \mathfrak{T}_3 adds the star with center x_5 and the leaves x_6 and x_7 to x_4 (Figure 3), \mathfrak{T}_5 adds x_8 to x_5 (Figure 4), \mathfrak{T}_1 adds x_9 to x_2 , \mathfrak{T}_4 adds $x_{10}x_{11}$ to x_8 , \mathfrak{T}_1 adds x_{12} to x_{10} (Figure 5), \mathfrak{T}_4 adds $x_{13}x_{14}$ to x_{11} , \mathfrak{T}_3 adds the star with center x_{15} and the leaves x_{16} and x_{17} to x_3 (Figure 6), \mathfrak{T}_5 adds x_{18} to x_{15} and \mathfrak{T}_4 adds the path $x_{19}x_{20}$ to x_{17} (Figure 7). Then T is in Figure 2.

In what follows, we step by step show that how one can find the reordering of the operations $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5\}$. The new ordering will be $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_4$.

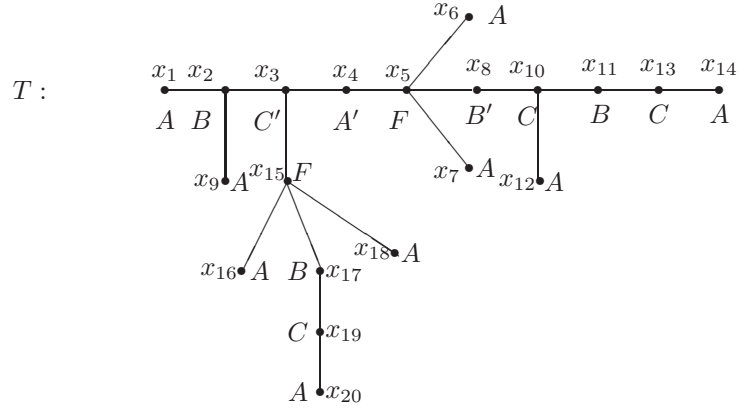


Fig. 2

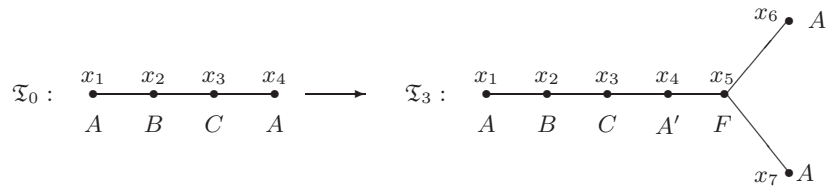


Fig. 3

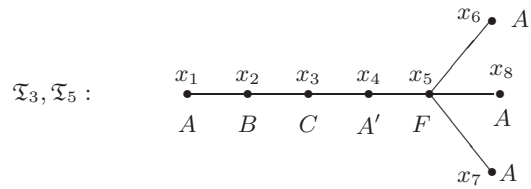


Fig. 4

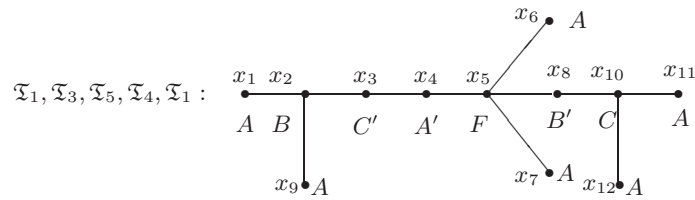
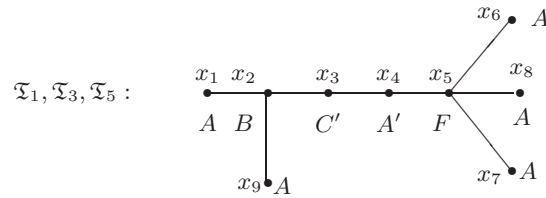


Fig. 5

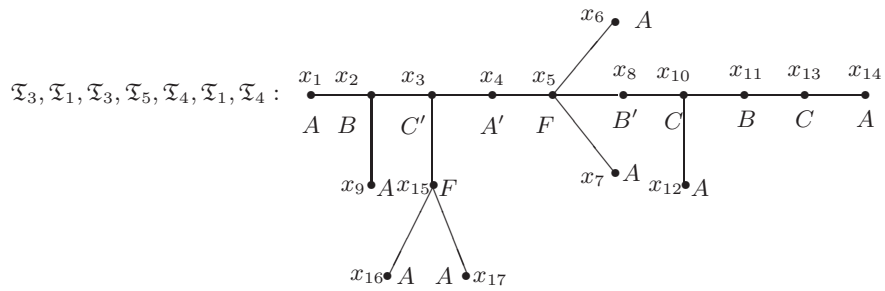


Fig. 6

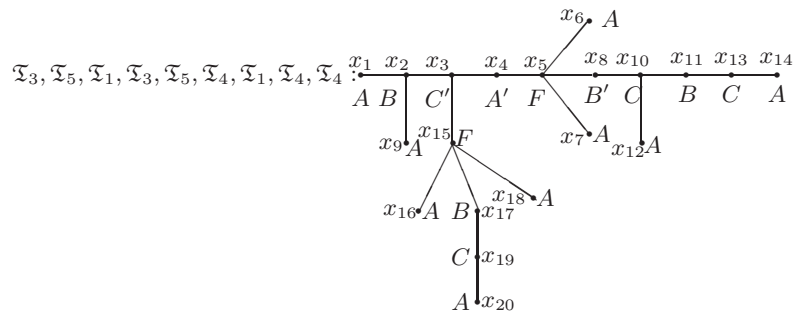


Fig. 7

In order to show that each tree in the family \mathcal{F} is in Class 2, we first present three lemmas.

Lemma 2.5. *Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of T and $\gamma_2(T) = m + k + 3$, where k is the number of operations of type \mathfrak{T}_3 .*

Proof. By Part (4) of Observation 2.1, the set $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of T . This implies that $\gamma_2(T) \leq m + k + 3$. The proof of $\gamma_2(T) = m + k + 3$ is by induction on m . If $m = 0$, then clearly the statement is true. Let $m \geq 1$ and that the statement holds for all trees which are obtained from P_4 by applying $m - 1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \dots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let T_{m-1} be the tree obtained from P_4 by the first $m - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$. If $\mathfrak{T}^m = \mathfrak{T}_3$, then T has been obtained from T_{m-1} by adding a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy to T . By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + (k - 1) + 3 = m + k + 1$ and the result follows by Proposition 2.3. If $\mathfrak{T}^m = \mathfrak{T}_5$, then T has been obtained from T_{m-1} by adding a vertex x and an edge xy to the tree T_{m-1} where $sta_{T_{m-1}}(y) = F$. Then, by the choice of reordering, y is a strong support vertex in T_{m-1} . By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + k + 3 = m + k + 2$ and the result follows by Proposition 2.2 (Part 2). If $\mathfrak{T}^m = \mathfrak{T}_4$, then the result follows by the inductive hypothesis and Proposition 2.2 (Part 1). Now consider the two remaining cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T has been obtained from T_{m-1} by adding a vertex x and an edge xy , where y is a support vertex of T_{m-1} . Suppose that w is a leaf adjacent to y and z is a vertex of status B, C, C' or B' adjacent to y by Observation 2.1, Parts (2) and (3). First assume y is in the original P_4 . Then, by the choice of reordering, $\mathfrak{T}^1 = \mathfrak{T}^2 = \dots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at y . Therefore $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$ -set S' , $S' \cup \{x\}$ is a 2-dominating set of T and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if S is a $\gamma_2(T)$ -set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of T_{m-1} . This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

Now assume y is not in the original P_4 . By the choice of reordering, we may assume for some positive integer s , $\mathfrak{T}^m = \dots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ and each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path zyw . Therefore, z is a leaf in T_{s-1} and so $sta_{T_{s-1}}(z) = A$ and $\deg_T(z) = 2$. By Proposition 2.3, $\gamma_2(T_{s-1}) + (m - s) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T has been obtained from T_{m-1} by adding a vertex x and an edge xy , where y is adjacent to a support vertex z of T_{m-1} with $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$ -set S' , $S' \cup \{x\}$ is a 2-dominating set of T and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if S is a $\gamma_2(T)$ -set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of T_{m-1} . This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis. \square

Lemma 2.6. *Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then:*

1. *for every $v \in V(T)$, there exists a $\gamma_2(T)$ -set containing v ,*
 2. *if $v \in A(T)$, then there is a $\gamma_2(T)$ -set S containing v and its support vertex.*
- Therefore, $S - \{v\}$ is a 2-dominating set of $T - \{v\}$.*

Proof. The proof is by induction on m . If $m = 0$, then clearly the statements are true. Let $m \geq 1$ and the statements hold for all trees which are obtained from a labeled P_4 by applying at most $m - 1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Let T_{m-1} be the tree obtained from P_4 by the first $m - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \dots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_3\}$.

(1) Since by Lemma 2.5, $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a $\gamma_2(T)$ -set, we assume that $v \in C(T) \cup C'(T) \cup F(T)$. We consider three cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or \mathfrak{T}_5 . Then T is obtained from T_{m-1} by adding a vertex x and an edge xy , where $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. Since $C(T) \cup C'(T) \cup F(T) = C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})$, by the inductive hypothesis v is contained in some $\gamma_2(T_{m-1})$ -set S . Now $S \cup \{x\}$ is a $\gamma_2(T)$ -set containing v by Lemma 2.5.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by adding a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy , where $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. We have $C(T) \cup C'(T) \cup F(T) = (C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})) \cup \{x\}$. If $v \in V(T_{m-1})$, then by the inductive hypothesis there is a $\gamma_2(T_{m-1})$ -set S containing v and $S \cup \{x_1, x_2\}$ is a $\gamma_2(T)$ -set by Lemma 2.5. Let $v = x$. By the choice of reordering, for some integer $0 \leq s \leq m - 1$, each of $\mathfrak{T}^m, \mathfrak{T}^{m-1}, \dots, \mathfrak{T}^{s+1}$ adds a star $K_{1,2}$ and joins its center to y but \mathfrak{T}^s does not add a star $K_{1,2}$ to y . If $s < m - 1$, then we may assume \mathfrak{T}^{m-1} adds a star $K_{1,2}$ with center x' and leaves x'_1, x'_2 . Obviously, we can rearrange the order of the operations to have $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-2}, \mathfrak{T}^m, \mathfrak{T}^{m-1}$. By the inductive hypothesis, the tree T' obtained from P_4 by the operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-2}, \mathfrak{T}^m$ has a $\gamma_2(T')$ -set S containing v . Then $S \cup \{x'_1, x'_2\}$ is a $\gamma_2(T)$ -set containing v by Lemma 2.5. Now we assume $s = m - 1$. Let first $\text{sta}(y) = B'$ or C' . Then, by the choice of reordering, we may assume $\mathfrak{T}^s \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$. We consider two subcases.

Subcase 2.1. $\mathfrak{T}^{m-1} = \mathfrak{T}_1$. This forces that y is adjacent to a strong support vertex z with status B or C and $\text{deg}(z) = 3$. By Lemma 2.5 and the inductive hypothesis, z is contained in a $\gamma_2(T_{m-1})$ -set S . Now obviously $(S \setminus \{z\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$ -set containing v .

Subcase 2.2. $\mathfrak{T}^{m-1} = \mathfrak{T}_2$. Then T_{m-1} is obtained from T_{m-2} by adding a vertex u and an edge uz , where z is a vertex of status B or C adjacent to the support vertex y of status C or B and degree 2 in T_{m-2} . Thus we have $\text{deg}_{T_{m-1}}(z) \geq 3$, $\text{sta}_{T_{m-1}}(z) = B$ or C and $\text{sta}_{T_{m-1}}(y) = C'$ or B' . Let z' be a vertex adjacent to z other than y and u . By the inductive hypothesis, z' is contained in a $\gamma_2(T_{m-1})$ -set say S . Then we have $z \in S$ or $y \in S$. By Lemma 2.5, $(S \setminus \{z, y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$ -set containing v .

Now let $sta(y) = A$. Then y is a leaf in T_{m-1} , and by the inductive hypotheses there is a $\gamma_2(T_{m-1})$ -set S containing y and its support vertex and so $(S \setminus \{y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$ -set containing v .

Finally, let $sta(y) = A'$. Then \mathfrak{T}^{m-1} adds a star $K_{1,2}$ with center x' and leaves x'_1, x'_2 and changes the status of y from A to A' . Thus y is a leaf in T_{m-2} , and by the inductive hypothesis there is a $\gamma_2(T_{m-2})$ -set S containing y and its support vertex w . Now obviously $(S \setminus \{y\}) \cup \{x'_1, x'_2, x, x_1, x_2\}$ is a $\gamma_2(T)$ -set containing v .

Case 3. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding a path yxu to T_{m-1} , where $y \in A(T_{m-1})$. Thus y is a leaf in T_{m-1} . Suppose that z is the support vertex of y in T_{m-1} . If $v \in T_{m-1}$, then by the inductive hypothesis v is contained in some $\gamma_2(T_{m-2})$ -set S and $S \cup \{u\}$ is a $\gamma_2(T)$ -set by Lemma 2.5. Now let $v = x$. By the inductive hypothesis, there is a $\gamma_2(T_{m-1})$ -set S containing y and its support vertex and obviously $(S - \{y\}) \cup \{x, u\}$ is a $\gamma_2(T)$ -set containing v .

(2) Let u be the support vertex of v . Then by Part (2) of Observation 2.1, $sta(u) = B, C, B', C'$, or F . Now the result follows by Lemma 2.5, Part (1) of this theorem and the fact that each $\gamma_2(T)$ -set contains all leaves \square

Lemma 2.7. *Let $T \in \mathcal{F}$ and let T^* be a tree obtained from T by subdividing an edge of T . Then $\gamma_2(T^*) = \gamma_2(T)$.*

Proof. Let $T \in \mathcal{F}$. First note that $\gamma_2(T^*) \geq \gamma_2(T)$ and that any 2-dominating set of T^* of size $\gamma_2(T)$ is a $\gamma_2(T^*)$ -set. Let $e \in E(T)$ and let T^* be obtained from T by subdividing the edge e with vertex u . Let T be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on m . If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled P_4 by applying at most $m - 1$ operations. Suppose T_{m-1} is a tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$. When $e \in E(T_{m-1})$, let T_{m-1}^* be obtained from T_{m-1} by subdividing the edge e with vertex u . We consider three cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or \mathfrak{T}_5 . Then T is obtained from T_{m-1} by attaching the path yx to $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. If $e \in E(T_{m-1})$, then by the inductive hypothesis we have

$$\gamma_2(T^*) \leq \gamma_2(T_{m-1}^*) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T).$$

Let $e = xy$. By Lemmas 2.5 and 2.6, there exists a $\gamma_2(T_{m-1})$ -set S containing y . Now $S \cup \{x\}$ is a 2-dominating set of T^* of size $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Hence, $\gamma_2(T^*) = \gamma_2(T)$.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by attaching a star $K_{1,2}$ with center x and two leaves x_1, x_2 to the attacher $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup C'(T_{m-1}) \cup B'(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.3 and the inductive hypothesis we have

$$\gamma_2(T^*) = \gamma_2(T_{m-1}^*) + 2 = \gamma_2(T_{m-1}) + 2 = \gamma_2(T).$$

Let $e \in E(T) \setminus E(T_{m-1})$. By Lemma 2.6, there is a $\gamma_2(T)$ -set S containing x . Now S is a 2-dominating set of T^* of size $\gamma_2(T)$ if $e = xx_1$ or xx_2 and $(S - \{x\}) \cup \{u\}$ is a 2-dominating set for T^* of size $\gamma_2(T)$ if $e = xy$. Recall that u is the subdividing vertex.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by attaching the path yxw to the attacher $y \in A(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.2 and the inductive hypothesis $\gamma_2(T^*) = \gamma_2(T_{m-1}^*) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Let $e \notin E(T_{m-1})$. Without loss of generality, we may subdivide $e = yx$ with u . By Lemma 2.6, T_{m-1} has a $\gamma_2(T_{m-1})$ -set S containing y and its support vertex. Now $(S - \{y\}) \cup \{u, w\}$ is a $\gamma_2(T^*)$ -set of size $\gamma_2(T)$. This completes the proof. \square

An immediate consequence of Theorem 1.1 and Lemma 2.7 now follows.

Theorem 2.8. *Each tree in Family \mathcal{F} is in Class 2.*

In order to prove that any tree in Class 2 is indeed in \mathcal{F} we need the following lemma.

Lemma 2.9. *Let $T \in \mathcal{F}$, $v \in B(T) \cup C(T) \cup F(T)$ and let T^* be obtained from T by adding a star $K_{1,2}$ and an edge joining the center of the star to v . Then $\text{sd}_{\gamma_2}(T^*) = 1$.*

Proof. Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \dots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on m . If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled P_4 by applying at most $m - 1$ operations. Suppose T_{m-1} is the tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$. When $v \in V(T_{m-1})$, let T_{m-1}^* be obtained from T_{m-1} by adding a star $K_{1,2}$ and an edge joining the center of the star to v . Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \dots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let z, z_1 and z_2 be the center and leaves of the added star to T , respectively. We consider five cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by adding a star $K_{1,2}$ and an edge joining the center x of the star to $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. If $v \in V(T_{m-1})$, then by the inductive hypothesis $\text{sd}_{\gamma_2}(T_{m-1}^*) = 1$. Since T^* is formed from T_{m-1}^* by adding a star $K_{1,2}$, by Proposition 2.3 we have $\text{sd}_{\gamma_2}(T^*) \leq \text{sd}_{\gamma_2}(T_{m-1}^*) = 1$. Thus by Theorem 1.1, $\text{sd}_{\gamma_2}(T^*) = 1$. If $v = x$, then let T' be obtained from T^* by subdividing the edge xz by inserting a vertex t . Since y and z are strong support vertices, for each $\gamma_2(T^*)$ -set S we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for T^* , a contradiction. Let D be a $\gamma_2(T')$ -set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for T^* . Therefore $\text{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding a path xw and an edge joining x to $y \in A(T_{m-1})$. First let $v \in V(T_{m-1}) - \{y\}$. Then by the inductive hypothesis $\text{sd}_{\gamma_2}(T_{m-1}^*) = 1$. Assume e is an edge of T_{m-1}^* such that subdividing e increases the 2-domination number. Let T'_{m-1} and T' be obtained from T_{m-1}^* and

T^* by subdividing the edge e , respectively. By Proposition 2.2 (Part (1)), $\gamma_2(T^*) = \gamma_2(T_{m-1}^*) + 1$ and $\gamma_2(T') = \gamma_2(T'_{m-1}) + 1$. Now

$$\gamma_2(T') = \gamma_2(T'_{m-1}) + 1 \geq \gamma_2(T_{m-1}^*) + 2 = \gamma_2(T^*) + 1.$$

Therefore, $\text{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Let $v = y$. Obviously, $\deg(x) = 2$. Let T' be obtained from T^* by subdividing the edge yz by inserting a vertex t . Suppose that S is a $\gamma_2(T')$ -set. Since $\deg(x) = 2$, $y \in S$ or $x \in S$. We may assume $y \in S$, otherwise $(S - \{x\}) \cup \{y\}$ is a $\gamma_2(T')$ -set. Since t is a subdividing vertex, $\deg(t) = 2$. To dominate t we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for T^* and so $\text{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Now let $v = x$. Then $\deg_{T^*}(y) = 2$. Suppose that $w \neq x$ is adjacent to y . Let T' be obtained from T^* by subdividing the edge xz by inserting a vertex t . Suppose that S is a $\gamma_2(T')$ -set. Since $\deg_{T'}(y) = 2$, $y \in S$ or $\{x, w\} \subseteq S$. To dominate t we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for T^* and so $\text{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_5$. Then T is obtained from T_{m-1} by adding a vertex x and an edge joining x to $y \in F(T_{m-1})$. By the choice of reordering, we may assume $\mathfrak{T}^m = \dots = \mathfrak{T}^{k+1} = \mathfrak{T}_5$ and $\mathfrak{T}^k = \mathfrak{T}_3$ which adds a star $K_{1,2}$ with center y . Suppose T_{k-1} is the tree obtained by applying the first $k - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{k-1}$. If $v \in V(T_{k-1})$, then Proposition 3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$, then let T' be obtained from T^* by subdividing the edge yz by inserting a vertex t . Since y and z are strong support vertices, for each $\gamma_2(T^*)$ -set S we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for T^* , a contradiction. Let D be a $\gamma_2(T')$ -set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for T^* . Therefore $\text{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

Case 4. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T is obtained from T_{m-1} by adding a vertex x and an edge joining x to a support vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$. If y belongs to the original P_4 , then obviously $\mathfrak{T}^1 = \dots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at y . This forces $v = y$ and as Case 3, it is easy to see that subdividing the edge yz increases the 2-domination number. Suppose y is not contained in the original P_4 . By the choice of reordering, we may assume $\mathfrak{T}^m = \dots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ where each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds a path yw and an edge joining y to some vertex in T_{s-1} . Suppose T_{s-1} is the tree obtained by applying the first $s - 1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then Proposition 2.3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$ then, as before, we can see that subdividing the edge yz increases the 2-domination number.

Case 5. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T is obtained from T_{m-1} by adding a vertex x and an edge that joins x to a vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$, where y is adjacent to a support vertex z of degree 2 in T_{m-1} . If y belongs to the original P_4 , then the result follows as Case 4. Suppose y is not contained in the original P_4 . By the choice of reordering, we may assume $\mathfrak{T}^m = \dots = \mathfrak{T}^{s+1} = \mathfrak{T}_2$ where each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path zw and an edge joining z to y in T_{s-1} .

Suppose T_{s-1} is the tree obtained by applying the first $s-1$ operations $\mathfrak{T}^1, \dots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then the result follows by Proposition 2.2 (Part (3)) and the inductive hypothesis. Let $v = y$. We show that subdividing the edge zz_1 , where z_1 is a leaf at z , increases the 2-domination number. Let T' be obtained from T^* by subdividing the edge zz_1 by inserting a vertex u . Let S be a $\gamma_2(T)$ -set. Since $\deg_T(z) = 2$, we may assume $y \in S$. Now to dominate u we must have $u \in S$ or $z \in S$. Then clearly $S - \{u, z\}$ is a 2-dominating set for T^* . It follows that $\text{sd}_{\gamma_2}(T) = 1$. This completes the proof. \square

Theorem 2.10. *A tree T of order $n \geq 3$ is in Class 2 if and only if $T \in \mathcal{F}$.*

Proof. By Theorem 2.8, we only need to prove that every tree in Class 2 is in \mathcal{F} . We prove this by induction on n . Since $\text{sd}_{\gamma_2}(T) = 2$, we have $n \geq 4$. If $n = 4$, then the only tree T of order 4 and $\text{sd}_{\gamma_2}(T) = 2$ is $P_4 \in \mathcal{F}$. Let $n \geq 5$ and assume the statement holds for every tree in Class 2 of order less than n . Let T be a tree of order n and $\text{sd}_{\gamma_2}(T) = 2$. Assume $P = v_1v_2 \dots v_r$ is the longest path in T . Obviously, $\deg(v_1) = \deg(v_r) = 1$ and $r \geq 4$. Suppose T is rooted at v_r .

First let $\deg(v_2) \geq 3$. Then v_2 is a strong support vertex. Let $v_1 = u_1, u_2, \dots, u_{\deg(v_2)-1}$ be the leaves adjacent to v_2 and $T_1 = T - T_{v_2}$. By Proposition 2.3, $\text{sd}_{\gamma_2}(T_1) = 2$ and by the inductive hypothesis, $T_1 \in \mathcal{F}$. Since $\text{sd}_{\gamma_2}(T) = 2$, by Lemma 2.9, $\text{sta}_{T_1}(v_3) = A, A', B'$, or C' , and hence T can be obtained from T_1 by applying operation \mathfrak{T}_3 once and operation $\mathfrak{T}_5, \deg(v_2) - 3$ times.

Now let $\deg(v_2) = 2$. First let $\deg(v_3) = 2$. Then by Proposition 2.2 (Part (1)), $\gamma_2(T) = \gamma_2(T - T_{v_2}) + 1$ and $\text{sd}_{\gamma_2}(T) \leq \text{sd}_{\gamma_2}(T - T_{v_2})$. Therefore $\text{sd}_{\gamma_2}(T - T_{v_2}) = 2$ and by the inductive hypothesis, $T - T_{v_2} \in \mathcal{F}$. Now T can be obtained from $T - T_{v_2}$ by operation \mathfrak{T}_4 . Now let $\deg(v_3) \geq 3$. First assume that v_3 is adjacent to a support vertex u such that $u \neq v_2$. Let w be a leaf adjacent to u . As before, we may assume that $\deg(u) = 2$. Let T' be obtained from T by subdividing the edge v_3u by inserting a vertex s . For any $\gamma_2(T)$ -set S of T , $|S \cap \{v_1, v_2, v_3\}| \geq 2$ and $|S \cap \{s, u, w\}| \geq 2$. Obviously, $(S - \{v_1, v_2, v_3, s, u, w\}) \cup \{v_1, v_3, w\}$ is a 2-dominating set for T with cardinality less than $|S|$. Therefore, $\text{sd}_{\gamma_2}(T) = 1$, a contradiction. Thus v_3 is adjacent to $\deg(v_3) - 2$ leaves. Let $u_1, \dots, u_{\deg(v_3)-2}$ be the leaves adjacent to v_3 . Assume $T' = T - \{u_1, \dots, u_{\deg(v_3)-2}, v_1, v_2\}$. By Proposition 2.2 (Part 3) $\gamma_2(T) = \gamma_2(T') + \deg(v_3) - 1$ and $\text{sd}_{\gamma_2}(T) \leq \text{sd}_{\gamma_2}(T')$. Since $\text{sd}_{\gamma_2}(T) = 2$, by Theorem 1.1, $\text{sd}_{\gamma_2}(T') = 2$. Hence, by the inductive hypothesis, $T' \in \mathcal{F}$. Since v_3 is a leaf in T' , $\text{sta}_{T'}(v_3) = A$ and T can be obtained from T' by applying operation \mathfrak{T}_4 once and operations \mathfrak{T}_1 or $\mathfrak{T}_2, \deg(v_3) - 2$ times. Thus $T \in \mathcal{F}$ and the proof is complete. \square

REFERENCES

- [1] H. Aram, S.M. Sheikholeslami, O. Favaron, *Domination subdivision numbers of trees*, Discrete Math. **309** (2009), 622–628.
- [2] M. Atapour, A. Khodkar, S.M. Sheikholeslami, *Characterization of double domination subdivision number of trees*, Discrete Appl. Math. **155** (2007), 1700–1707.

- [3] M. Atapour, S.M. Sheikholeslami, A. Khodkar, *Trees whose Roman domination subdivision number is 2*, Util. Math. **82** (2010), 227–240.
- [4] M. Atapour, S.M. Sheikholeslami, A. Hansberg, L. Volkmann, A. Khodkar, *2-domination subdivision number of graphs*, AKCE J. Graphs Combin. **5** (2008), 169–177.
- [5] M. Blidia, M. Chellali, T.W. Haynes, *Characterizations of trees with equal paired and double domination numbers*, Discrete Math. **306** (2006), 1840–1845.
- [6] M. Blidia, M. Chellali, L. Volkmann, *Bounds on the 2-domination number of graphs*, Util. Math. **71** (2006), 209–216.
- [7] M. Chellali, *Bounds on the 2-domination number in cactus graphs*, Opuscula Math. **26** (2006), 5–12.
- [8] P. Dankelmann, J.H. Hattingh, M.A. Henning, H.C. Swart, *Trees with equal domination and restrained domination numbers*, J. Global Optim. **34** (2006), 597–607.
- [9] M. Dorflinga, W. Goddard, M.A. Henning, C.M. Mynhardt, *Construction of trees and graphs with equal domination parameters*, Discrete Math. **306** (2006), 2647–2654.
- [10] T.W. Haynes, M.A. Henning, *A characterization of i -excellent trees*, Discrete Math. **248** (2002), 69–77.
- [11] T.W. Haynes, M.A. Henning, L. Hopkins, *Total domination subdivision numbers of trees*, Discrete Math. **286** (2004), 195–202.
- [12] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
- [13] A. Hansberg, L. Volkmann, *On graphs with equal domination and 2-domination numbers*, Discrete Math. **308** (2008), 2277–2281.
- [14] H. Karami, A. Khodkar, S.M. Sheikholeslami, *Trees whose double domination number is twice their domination number*, Congr. Numer. **186** (2007), 49–56.
- [15] R.S. Shaheen, *Bounds for the 2-domination number of toroidal grid graphs*, Int. J. Comput. Math. **86** (2009), 584–588.
- [16] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc., 2000.

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