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EXISTENCE AND SOLUTION SETS OF IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MULTIPLE DELAY

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Abstract. In this paper, we present some existence results of solutions and study the topological structure of solution sets for the following first-order impulsive neutral functional differential inclusions with initial condition:

$$\begin{cases} \dfrac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t) + \displaystyle\sum_{i=1}^{n_*} y(t - T_i), & \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(t) = \phi(t), & t \in [-r, 0], \end{cases}$$

where $J := [0, b]$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$ ($m \in \mathbb{N}^*$), F is a set-valued map and g is single map. The functions I_k characterize the jump of the solutions at impulse points t_k ($k = 1, \ldots, m$). Our existence result relies on a nonlinear alternative for compact u.s.c. maps. Then, we present some existence results and investigate the compactness of solution sets, some regularity of operator solutions and absolute retract (in short AR). The continuous dependence of solutions on parameters in the convex case is also examined. Applications to a problem from control theory are provided.

Keywords: impulsive functional differential inclusions, decomposable set, parameter differential inclusions, AR-set, control theory.

Mathematics Subject Classification: 34K45, 34A60, 54C55, 54C60.

1. INTRODUCTION

The dynamics of many processes in physics, population dynamics, biology, medicine may be subject to abrupt changes such that shocks, perturbations (see for instance [1, 37] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models may be described by impulsive differential equations. The mathematical study of boundary value problems for differential equations with impulses were considered in 1960 by Milman and Myshkis [42] and then followed by a period of active research which culminated in 1968 with the monograph by Halanay and Wexler [29].

Moreover, it is well known that time delay is an important factor of mathematical models in ecology. Usually, time delays in those models have two cases: discrete delay and distributed time delay (continuous delay)[48].

For the impulsive model with distributed time delay, papers [27, 34, 41, 52] have investigated some ecological models with distributed time delay and impulsive control strategy. Impulsive functional differential equations with multiple delay arise in the study of pulse vaccination strategies. In [24] the authors consider the following model:

$$\begin{cases} S'(t) = b - bS(t) - \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} + \gamma I(t-\tau)e^{-b\tau}, \\ E'(t) = \int\limits_{t-\omega}^{t} \dfrac{\beta S(u)I(u)}{N(u)} e^{-b(t-u)}du, \\ I'(t) = \dfrac{\beta e^{-b\omega} S(t-\omega)I(t-\omega)}{1+\alpha S(t-\omega)} - (b+\omega)I(t), \\ R'(t) = \int\limits_{t-\omega}^{t} \gamma I(u)e^{-b(t-u)}du, \\ S(t_k^+) = (1-\theta)S(t_k^-), & t = kT, k \in \mathbb{N}, \\ E(t_k^+) = E(t_k^-), & t = kT, k \in \mathbb{N}, \\ I(t_k^+) = I(t_k^-), & t = kT, k \in \mathbb{N}, \\ R(t_k^+) = R(t_k^-) + \theta S(t_k^-), & t = kT, k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, \dots, \}$, $S(t) + N(t) + I(t) = 1$ for all $t \geq 0$, and

(S) denotes the susceptible,
(I) the infectives,
(R) the removed group,
(E) the exposed but not yet infectious.

Important contributions to the study of the mathematical aspects of such equations have been undertaken in [10, 39, 46, 50] among others. Functional differential equations and inclusions with impulsive effects with fixed moments have been recently addressed by Djebali *et al.* [17], Yujun [56] and Yujun and Erxin [57]. Some existence results on impulsive functional differential equations with finite or infinite delay may be found in [44, 45] as well. During the last couple of years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied (see the book by Aubin [4], as well as the paper [30] and the references therein).

In this paper, we consider first order impulsive functional differential inclusions with multiple delays of the form:

$$\begin{cases} \dfrac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t) + \displaystyle\sum_{i=1}^{n_*} y(t - T_i), \ a.e. \quad t \in J \backslash \{t_1, \ldots, t_m\}, \\ \quad y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \qquad\qquad\qquad k = 1, \ldots, m, \\ \qquad\qquad y(t) = \phi(t), \qquad\qquad\qquad\qquad\qquad t \in [-r, 0], \end{cases} \tag{1.2}$$

where $n_* \in \{1, 2, \ldots\}$, $r = \max_{1 \le i \le n_*} T_i$, $J := [0, b]$, $F : J \times D \to \mathbb{R}^n$ is a given function, $D = C([-r, 0], \mathbb{R}^n)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$ and $I_k \in C(\mathbb{R}^n, \mathbb{R}^n), k = 1, 2, \ldots, m$, are given functions satisfying some assumptions that will be specified later.

For any function y defined on $[-r, b]$ and any $t \in J \backslash \{t_1, \ldots, t_m\}$ we denote by y_t the element of D defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$.

For the single case, some existence results of solutions for the problem (1.2) have been obtained in [44, 45]. Our goal in this work is to complement and extend some of these results to the case of differential inclusions; moreover the right-hand side multi-valued nonlinearity may be either convex or nonconvex.

Some auxiliary results from multi-valued analysis are gathered together in Section 2. In the first part of this work, we prove some existence results based on the nonlinear alternative of the Leary Schauder type (in the convex case), on the Bressan-Colombo selection theorem and on the Covitz and Nadler fixed point theorem for contraction multi-valued maps in a generalized metric space (in the nonconvex case). The compactness of the solution set and some geometric properties are also provided. This is the content of Section 4. We will also discuss the question of dependance on parameters in Section 5. The applicability of the obtained results, to a problem from control theory is presented in Section 6. We end the paper with a rich bibliography.

2. PRELIMINARIES

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $(E, |\cdot|)$ be a Banach space, denote by $\mathcal{P}(E) = \{Y \subset E \colon Y \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ is closed}\}$, $\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ is bounded}\}$, $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ is convex}\}$, $\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ is compact}\}$, and $\mathcal{P}_{wkcp}(E) = \{Y \in \mathcal{P}(E) \colon Y \text{ is weakly compact}\}$.

Let (X, d) and (Y, ρ) be two metric spaces and $G : X \to \mathcal{P}_{cl}(Y)$ be a multi-valued map. A single-valued map $g : X \to Y$ is said to be a selection of G and we write $g \subset G$ whenever $g(x) \in G(x)$ for every $x \in X$.

G is called *upper semi-continuous* (*u.s.c. for short*) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $G^{-1}(V) = \{x \in X : G(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Y. Equivalently, G is *u.s.c.* if the set $G^{+1}(V) = \{x \in X : G(x) \subset V\}$ is open for any open set V in Y.

The following two results are easily deduced from the limit properties.

Lemma 2.1 (see e.g. [6, Theorem 1.4.13]). *If $G : X \longrightarrow \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x \to x_0} G(x) = G(x_0).$$

Lemma 2.2 (see e.g. [6, Lemma 1.1.9]). *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}\,(\limsup_{n\to\infty} K_n) = \bigcap_{N>0} \overline{co}\,(\bigcup_{n\geq N} K_n),$$

where $\overline{co}\, A$ refers to the closure of the convex hull of A.

The second one is due to Mazur (1933).

Lemma 2.3 (Mazur's Lemma [43, Theorem 21.4]). *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, \ldots, m$ and $\sum_{k=1}^{m} \alpha_{mk} = 1$, which converges strongly to x.*

G is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $G(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that $G(A) = \bigcup\{G(x) : x \in A\} \subset K$. G is compact if $G(X)$ is relatively compact. It is called locally compact if, for each $x \in X$, there exists an open neighborhood U of x such that $G(U)$ is relatively compact. G is quasicompact if, for each subset $A \subset X$, $G(A)$ is relatively compact.

Definition 2.4. A multi-valued map $F : J \to \mathcal{P}_{cl}(Y)$ is said to be measurable provided for every open $U \subset Y$, the set $F^{+1}(U)$ is Lebesgue measurable.

Lemma 2.5 ([14, 26]). *The mapping F is measurable if and only if for each $x \in Y$, the function $\zeta : J \to [0, +\infty)$ defined by*

$$\zeta(t) = \operatorname{dist}(x, F(t)) = \inf\{|x - y| \, : \, y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following two lemmas are needed in this paper. The first one is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.6 ([26, Theorem 19.7]). *Let Y be a separable metric space and $F : [a, b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

Lemma 2.7 ([60, Lemma 3.2]). *Let $F : [0, b] \to \mathcal{P}(Y)$ be a measurable multi-valued map and $u : [a, b] \to Y$ a measurable function. Then for any measurable $v : [a, b] \to (0, +\infty)$, there exists a measurable selection f_v of F such that for a.e. $t \in [a, b]$,*

$$|u(t) - f_v(t)| \leq d(u(t), F(t)) + v(t).$$

Corollary 2.8. *Let $F : [0, b] \to \mathcal{P}_{cp}(Y)$ be a measurable multi-valued map and $u : [0, b] \to Y$ a measurable function. Then there exists a measurable selection f of F such that for a.e. $t \in [0, b]$,*

$$|u(t) - f(t)| \leq d(u(t), F(t)).$$

Proof. Taking $v(t) = v_n(t) = \frac{1}{n}$ in Lemma 2.7, we get a measurable selection f_n of F such that

$$|u(t) - f_n(t)| \leq d(u(t), F(t)) + 1/n.$$

Using the fact that F has compact values, we may pass to a subsequence if necessary to get that $\{f_n(\cdot)\}$ converges to a measurable function f, yielding our claim. □

We denote the graph of G to be the set $\mathcal{G}r(G) = \{(x, y) \in X \times Y : y \in G(x)\}$.

Definition 2.9. G is closed if $\mathcal{G}r(G)$ is a closed subset of $X \times Y$, i.e. for every sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ as $n \to \infty$ with $y_n \in F(x_n)$, then $y_* \in G(x_*)$.

We recall the following two results. The first one is classical.

Lemma 2.10 ([16, Proposition 1.2])**.** *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathcal{G}r(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.11. *If $G : X \to \mathcal{P}_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

Proof. Assume that G is not u.s.c. at some point x. Then there exists an open neighborhood U of $G(x)$ in Y, a sequence $\{x_n\}$ which converges to x, and for every $l \in \mathbb{N}$ there exists $n_l \in \mathbb{N}$ such that $G(x_{n_l}) \not\subset U$. Then for each $l = 1, 2, \ldots$, there are y_{n_l} such that $y_{n_l} \in G(x_{n_l})$ and $y_{n_l} \notin U$; this implies that $y_{n_l} \in Y \backslash U$. Moreover, $\{y_{n_l} : l \in \mathbb{N}\} \subset G(\overline{\{x_n : n \geq 1\}})$. Since G is quasicompact, there exists a subsequence of $\{y_{n_l} : l \in \mathbb{N}\}$ which converges to y. G closed implies that $y \in G(x) \subset U$; but this is a contradiction to the assumption that $y_{n_l} \notin U$ for each n_l. □

Given a separable Banach space $(E, |\cdot|)$, for a multi-valued map $F : J \times E \to \mathcal{P}(E)$, denote

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t, x)\}.$$

Definition 2.12. F is said:

(a) integrable if it has a summable selection $f \in L^1(J, E)$,
(b) integrably bounded, if there exists $q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, z)\|_{\mathcal{P}} \leq q(t) \text{ for a.e. } t \in J \text{ and every } z \in E.$$

Definition 2.13. A multi-valued map F is called a Carathéodory function if:

(a) the function $t \mapsto F(t, x)$ is measurable for each $x \in E$,
(b) for a.e. $t \in J$, the map $x \mapsto F(t, x)$ is upper semi-continuous.

Furthermore, F is L^1-Carathéodory if it is locally integrably bounded, i.e., for each positive r, there exists $h_r \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h_r(t) \text{ for a.e. } t \in J \text{ and all } |x| \leq r.$$

Lemma 2.14 ([40]). *Given a Banach space E, let $F : [a,b] \times E \longrightarrow \mathcal{P}_{cp,cv}(E)$ be an L^1-Carathédory multi-valued map such that for each $y \in C([a,b],E)$, $S_{F,y} \neq \emptyset$ and let Γ be a linear continuous mapping from $L^1([a,b],E)$ into $C([a,b],E)$. Then the operator*

$$\begin{array}{rl}\Gamma \circ S_F : C([a,b],E) & \longrightarrow \mathcal{P}_{cp,cv}(C([a,b],E)),\\ y & \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})\end{array}$$

has a closed graph in $C([a,b],E) \times C([a,b],E)$.

For each $x \in C(J,E)$, the set

$$S_{F,x} = \left\{ f \in L^1(J,E) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,b] \right\}$$

is known as the set of selection functions.

Remark 2.15. (a) For each $x \in C(J,E)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t,x(t))$ is convex for a.e. $t \in J$.
(b) From [58, Theorem 5.10] (see also [40] when E is finite-dimensional), we know that $S_{F,x}$ is nonempty if and only if the mapping $t \mapsto \inf\{|v| : v \in F(t,x(t))\}$ belongs to $L^1(J)$. It is bounded if and only if the mapping $t \mapsto \|F(t,x(t))\|_{\mathcal{P}} = \sup\{|v| : v \in F(t,x(t))\}$ belongs to $L^1(J)$; this particularly holds true when F is L^1-Carathéodory. For the sake of completeness, we refer also to Theorem 1.3.5 in [35] which states that $S_{F,x}$ contains a measurable selection whenever x is measurable and F is a Carathéodory function.

For further readings and details on multi-valued analysis, we refer to the books by Andres and Górniewicz [3], Aubin and Celina [5], Aubin and Frankowska [6], Deimling [16], Górniewicz [26], Hu and Papageorgiou [31, 32], Kamenski *et al.* [35], and Tolstonogov [53].

3. EXISTENCE RESULTS

Let $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1,\dots,m$, and let y_k be the restriction of a function y to J_k. In order to define solutions for the problem (1.2), consider the space of piece-wise continuous functions

$$\begin{aligned} PC = \{y\colon [0,b] \to \mathbb{R}^n \mid y_k \in C(J_k,\mathbb{R}^n),\ k = 0,\dots,m, \text{ such that}\\ y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k) = y(t_k^-) \text{ for } k = 1,\dots,m\}.\end{aligned}$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty : k = 0,\dots,m\}, \quad \|y_k\|_\infty = \sup_{t\in[t_k,t_{k+1}]} |y(t)|$$

it is a Banach space. Moreover, if

$$\Omega = \{y : [-r, b] \to \mathbb{R}^n \mid y \in PC([0, b], \mathbb{R}^n) \cap D\},$$

then Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup\{|y(t)| : \ t \in [-r, b]\}.$$

Definition 3.1. A function $y \in \Omega \cap \bigcup_{k=1}^{k=m} AC(J_k, \mathbb{R})$, is said to be a solution of (1.2) if y satisfies the equation $\frac{d}{dt}(y(t) - g(t, y_t)) = v(t) + \sum_{i=1}^{n_*} y(t - T_i)$ a.e. on J, $t \neq t_k$, $k = 1, \ldots, m$ and the conditions $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, $k = 1, \ldots, m$, $v \in S_{F,y}$ and $y(t) = \phi(t)$ on $[-r, 0]$.

Lemma 3.2. *Let $f : D \to \mathbb{R}^n$ be a continuous function and assume that the function $t \to g(t, y_t)$ belongs to PC. Then y is the unique solution of the initial value problem*

$$\begin{cases} \frac{d}{dt}(y(t) - g(t, y_t)) = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i) & a.e.\ t \in J \backslash \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k) = I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{3.1}$$

where $r = \max_{1 \le i \le n_*} T_i$ if and only if y is a solution of the impulsive integral functional differential equation

$$y(t) = \begin{cases} \phi(t), \ \ if\ t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) - \sum_{0<t_k<t} \Delta_k(g(t_k^-, y_{t_k^-})) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s) ds + \\ + \int_0^t f(y_s) ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s) ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \ \ if\ \ t \in [0, b], \end{cases} \tag{3.2}$$

where $\Delta_k(g(t_k^-, y_{t_k^-})) = g(t_k^+, y_{t_k^+}) - g(t_k, y_{t_k})$.

Proof. Let y be a possible solution of the problem (3.1). Then $y|_{[-r,t_1]}$ is a solution to $\dfrac{d}{dt}(y(t) - g(t, y_t)) = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i)$ for $t \in [0, b]$. Assume that $t_k < t \le t_{k+1}$,

$k = 1, \ldots, m$. Integration of the above inequality yields

$$y(t_1^-) - y(0) - (g(t_1^-, y_{t_1^-}) - g(0,\phi)) = \int_0^{t_1} f(y_s)ds + \sum_{i=1}^{n_*}\int_0^{t_1} y(s-T_i)ds,$$

$$y(t_1^-) - y(0) - (g(t_1^-, y_{t_1^-}) - g(0,\phi)) = \int_0^{t_1} f(y_s)ds + \sum_{i=1}^{n_*}\int_{-T_i}^{t_1-T_i} y(s)ds,$$

$$y(t_2^-) - y(t_1^+) - (g(t_2^-, y_{t_2^-}) - g(t_1^+, y_{t_1^+})) = \int_{t_1}^{t_2} f(y_s)ds + \sum_{i=1}^{n_*}\int_{t_1}^{t_2} y(s-T_i)ds,$$

$$y(t_2^-) - y(t_1^-) - (g(t_2^-, y_{t_2^-}) - g(t_1^+, y_{t_1^+})) = I_1(y(t_1^-)) + \int_{t_1}^{t_2} f(y_s)ds + + \sum_{i=1}^{n_*}\int_{t_1-T_i}^{t_2-T_i} y(s)ds,$$

$\vdots$

$\vdots$

$\vdots$

$$y(t_k^-) - y(t_{k-1}^+) - (g(t_k^-, y_{t_k^-}) - g(t_{k-1}^+, y_{t_{k-1}^+})) = \int_{t_{k-1}}^{t_k} f(y_s)ds + + \sum_{i=1}^{n_*}\int_{t_{k-1}}^{t_k} y(s-T_i)ds,$$

$$y(t_k^-) - y(t_{k-1}^-) - (g(t_k^-, y_{t_k^-}) - g(t_{k-1}^+, y_{t_{k-1}^+})) = I_k(y(t_k^-)) + \int_{t_{k-1}}^{t_k} f(y_s)ds + + \sum_{i=1}^{n_*}\int_{t_{k-1}-T_i}^{t_k-T_i} y(s-T_i)ds,$$

$$y(t) - y(t_k^-) - (g(t, y_t) - g(t_k^+, y_{t_k^+})) = I_k(y(t_k^-)) + \int_{t_k}^{t} f(y_s)ds + \sum_{i=1}^{n_*}\int_{t_k-T_i}^{t-T_i} y(s)ds.$$

Then

$$y(t_1) - y(0) - (g(t_1, y_{t_1}) - g(0,\phi)) = \int_0^{t_1} f(y_s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{t_1 - T_i} y(s)ds,$$

$$y(t_2) - y(t_1^-) - (g(t_2, y_{t_2}) - g(t_1^+, y_{t_1^+})) = I_1(y(t_1^-)) + \int_{t_1}^{t_2} f(y_s)ds + \sum_{i=1}^{n_*} \int_{t_1 - T_i}^{t_2 - T_i} y(s)ds,$$

$$\vdots$$

$$y(t_k^-) - y(t_{k-1}) - (g(t_k, y_{t_k}) - g(t_{k-1}^+, y_{t_{k-1}^+})) = I_k(y(t_k^-)) + \int_{t_{k-1}}^{t_k} f(y_s)ds + \\ + \sum_{i=1}^{n_*} \int_{t_{k-1} - T_i}^{t_k - T_i} y(s - T_i)ds,$$

$$y(t) - y(t_k^-) - (g(t, y_t) - g(t_k^+, y_{t_k^+})) = I_k(y(t_k^-)) + \int_{t_k}^{t} f(y_s)ds + \sum_{i=1}^{n_*} \int_{t_k - T_i}^{t - T_i} y(s)ds.$$

Adding these together, we get

$$y(t) = y(0) + g(t, y_t) - g(0,\phi) + \sum_{0<t_k<t} (g(t_k, y_{t_k}) - g(t_k^+, y_{t_k^+})) + \\ + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(y_s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{t - T_i} y(s)ds,$$

$$y(t) = \phi(0) + g(t, y_t) - g(0,\phi) - \sum_{0<t_k<t} \Delta_k(g(t_k^-, y_{t_k^-})) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \\ + \int_0^t f(y_s)ds + \sum_{i=1}^{n_*} \int_0^{t - T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds.$$

□

Remark 3.3. If g is a continuous function, then the solution of the problem (3.1) is of the form

$$y(t) = \phi(0) + g(t, y_t) - g(0,\phi) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(y_s)ds + \\ + \sum_{i=1}^{n_*} \int_0^{t - T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds, \quad t \in [0, b].$$

3.1. CONVEX CASE

Let us introduce the following hypotheses:

($\mathcal{H}_1$) The function $F : J \times D \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a Carathéodory map.

($\mathcal{H}_2$) There exists a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\|F(t, x)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|_D) \textit{ for a.e. } t \in J \textit{ and each } x \in D$$

with

$$\int_c^\infty \frac{ds}{s + \psi(s)} = \infty,$$

where

$$c = \frac{1}{1 - d_1}\left[\|\phi\|_D + d_2 + \|g(0, \phi)\|_D + \sum_{i=1}^{n_*} T_i\|\phi\|_D\right].$$

($\mathcal{H}_3$) For every bounded set $B \in \Omega$, the set $\{t : t \mapsto g(t, y_t),\ y \in B\}$ is equicontinuous in Ω, g is continuous and there exist constants $d_1 \in [0, 1)$ and $d_2 > 0$ such that

$$\|g(t, x)\|_D \leq d_1\|x\|_D + d_2 \text{ for all } x \in D.$$

Theorem 3.4. *Assume that the hypotheses* ($\mathcal{H}_1$)–($\mathcal{H}_3$) *hold. Then the IVP* (1.2) *has at least one solution.*

Proof. Transform the problem (1.2) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by:

$$N(y) =$$
$$= \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum\limits_{0<t_k<t} I_k(y(t_k^-)) + \\ \int\limits_0^t f(s)ds + \sum_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^{0} \phi(s)ds, & \text{if } t \in J \end{cases} \right\},$$

where $f \in S_{F,y}$. Clearly, the fixed points of the operator N are solutions of the problem (1.2). We shall show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [23]. The proof is given in several steps.

Step 1. $N(y)$ is convex for each $y \in \Omega$.

Indeed, if h_1, h_2 belong to $N(y)$ then there exist f_1, $f_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_i(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f_i(s)ds +$$
$$+ \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} y(s)ds, \quad i = 1,\ 2.$$

Let $0 \leq d \leq 1$. Then for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \\ + \int_0^t [df_1(s) + (1-d)f_2(s)]ds + \\ + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds.$$

Since $S_{F,y}$ is convex (because F has a convex value), then

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2. N maps bounded sets into bounded sets in Ω.
Indeed, it is enough to show that there exists a positive constant l such that for each $y \in \mathcal{B}_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$ one has $\|N(y)\|_{\mathcal{P}(\Omega)} \leq l$. Let $y \in \mathcal{B}_q$ and $h \in N(y)$. Then there exist $f \in S_{F,y}$ such that, for each $t \in J$, we have

$$h(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{0<t_k<t} I_k(y(t_k^-)) + \int_0^t f(s)ds + \\ + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds.$$

By $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ we have, for each $t \in J$,

$$|h(t)| \leq |\phi(0)| + \|g(t, y_t)\|_D + \|g(0, \phi)\|_D + \sum_{0<t_k<t} |I_k(y(t_k^-))| + \\ + \int_0^t |f(s)|ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} |y(s)|ds + \sum_{i=1}^{n_*} \int_{-T_i}^0 |\phi(s)|ds \leq \\ \leq \|\phi\|_D + d_1 q + d_2 + \|g(0, \phi)\|_D + \sum_{k=1}^m \sup_{u \in B(0,q)} |I_k(u)| + \\ + \int_0^t p(s)\psi(\|y_s\|_D)ds + bqn_* + r\|\phi\|n_* \leq \\ \leq \|\phi\| + d_1 q + d_2 + \|g(0, \phi)\|_D + m \sup_{u \in B(0,q)} |I_k(u)| + \\ + b\|p\|_{L^1}\psi(q) + bqn_* + r\|\phi\|n_* := l.$$

Step 3. N maps bounded sets into equicontinuous sets of Ω.

Using $(\mathcal{H}_3)$ it suffices to show that the operator $N_* : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N_*(y) = \\ = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \phi(0) + \int\limits_0^t f(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \\ + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^0 \phi(s)ds + \sum\limits_{0<t_k<t} I_k(y(t_k^-)), & \text{if } t \in J \end{cases} \right\},$$

where $f \in S_{F,y}$.

As in [12, Theorem 3.2] we can prove that $N_*(\mathcal{B}_q)$ is equicontinuous.

Step 4. N has closed graph.

Let $y^n \to y^*$, $h_n \in N(y^n)$ and $h_n \to h^*$. We shall prove that $h^* \in N(y^*)$. $h_n \in N(y^n)$ means that there exists $f_n \in S_{F,y^n}$ such that, for each $t \in J$,

$$h_n(t) = \phi(0) + g(t, y_t^n) + g(0,\phi) + \sum_{0<t_k<t} I_k(y^n(t_k^-)) + \\ + \int_0^t f_n(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y^n(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds.$$

We have to prove that there exists $v^* \in S_{F,y^*}$ such that, for each $t \in J$,

$$h^*(t) = \phi(0) + g(t, y_t^*) - g(0,\phi) + \sum_{0<t_k<t} I_k(y^*(t_k^-)) + \\ + \int_0^t f^*(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y^*(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds.$$

Clearly, since I_k, $k = 1, \ldots, m$, are continuous, we obtain that

$$\Bigg\| \Big(h_n(t) - \phi(0) - g(t, y_t^n) - g(0,\phi) - \\ - \sum_{0<t_k<t} I_k(y^n(t_k^-)) - \sum_{i=1}^{n_*} \int_0^{t-T_i} y^n(s)ds - \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds \Big) - \\ - \Big(h^*(t) - \phi(0) - g(t, y_t^*) - g(0,\phi) - \\ - \sum_{0<t_k<t} I_k(y^*(t_k^-)) - \sum_{i=1}^{n_*} \int_0^{t-T_i} y^*(s)ds - \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)ds \Big) \Bigg\|_\Omega$$

tends to 0 as $n \to \infty$. Consider the operator

$$\Gamma : L^1 \to \Omega,$$

$$f \mapsto \Gamma(f)(t) = \int_0^t f(s)ds.$$

We can see that the operator Γ is linear and continuous. Indeed, one has

$$\|\Gamma(f)\|_\Omega \le \|p\|_{L^1}\psi(q).$$

From Lemma 2.14, it follows that $\Gamma \circ S_F$ is a closed graph operator. Since

$$h_n(t) - \phi(0) - g(t, y_t^n) - g(0, y_0) - \sum_{0<t_k<t} I_k(y^n(t_k^-)) - \\ - \sum_{i=1}^{n_*} \int_0^{t-T_i} y^n(s)ds - \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds \in \Gamma(S_{F,y_n}),$$

it follows from Lemma 2.14 that for some $f^* \in S_{F,y^*}$

$$h^*(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0) + g(t, y_t^*) - g(0, \phi) + \sum_{0<t_k<t} I_k(y^*(t_k^-)) + \\ + \int_0^t f^*(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y^*(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds, & t \in J. \end{cases}$$

Step 5. A priori bounds on solutions.

Let y be a possible solution of the problem (1.2). Let y be a possible solution of the equation $y \in \lambda N(y)$, for some $\lambda \in (0, 1)$. Then there exists $f \in S_{F,y}$ such that

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{0<t_k<t} I_k(y(t_k^-)), \\ + \int_0^t f(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds, & \text{if } t \in J. \end{cases}$$

Thus

$$y(t) = \lambda \left[\phi(0) + g(t, y_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + \int_0^t f(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds \right]$$

for all $t \in [0, t_1]$. Hence

$$\begin{aligned} |y(t)| \le & |\phi(0)| + \|g(t, y_t)\|_D + \|g(0,\phi)\|_D + \\ & + \sum_{i=1}^{n_*} \int_{-T_i}^{0} |\phi(s)|ds + \int_0^t |f(s)|ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} |y(s)|ds \le \\ \le & \|\phi\|_D + d_1\|y_t\|_D + d_2 + \|g(0,\phi)\|_D + \\ & + \sum_{i=1}^{n_*} T_i\|\phi\|_\infty + n_* \int_0^t |y(s)|ds + \int_0^t p(s)\psi(\|y_s\|_D)ds. \end{aligned} \tag{3.3}$$

We consider the function

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\},\ t \in [0, t_1]. \tag{3.4}$$

Therefore,

$$\mu(t) \le \frac{1}{1-d_1}\left[L_* + \int_0^t p_*(s)(\mu(s) + \psi(\mu(s)))ds\right], \tag{3.5}$$

where

$$L_* = \|\phi\|_D + d_2 + \|g(0,\phi)\|_D + \sum_{i=1}^{n_*} T_i\|\phi\|_D$$

and

$$p_*(t) = n_* + p(t), \quad t \in [0, t_1].$$

Denoting by $\beta(t)$ the right hand side of the last inequality we have

$$\mu(t) \le \beta(t), \quad t \in [0, t_1],$$

and

$$\beta(0) = \frac{1}{1-d_1}\left[\|\phi\|_D + d_2 + \|g(0,\phi)\|_D + \sum_{i=1}^{n_*} T_i\|\phi\|_D\right],$$

and

$$\begin{aligned} \beta'(t) = & \frac{1}{1-d_1} p_*(t)[\psi(\mu(t)) + \mu(t)] \le \\ \le & \frac{1}{1-d_1} p_*(t)[\psi(\beta(t)) + \beta(t)]. \end{aligned}$$

This implies that for each $t \in [0, t_1]$

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)+s} \le \frac{1}{1-d_1}\int_0^{t_1} p_*(s)ds < \frac{1}{1-d_1}\int_c^{\infty} \frac{ds}{\psi(s)+s}.$$

Thus from $(\mathcal{H}_2)$ there exists a constant K_1 such that $\beta(t) \leq K_1,\ t \in [-r, t_1]$, and hence

$$\sup\{|y(t)| :\ t \in [-r, t_1]\} \leq K_1.$$

Let $t \in (t_1, t_2]$. Then

$$y(t) = \lambda \left[y(t_1^+) + g(t, y_t) - g(t_1, y_{t_1}) + \int\limits_{t_1}^{t} f(s)ds + \sum_{i=1}^{n_*} \int\limits_{t_1 - T_i}^{t - T_i} y(s)ds \right]$$

and

$$y(t_1^+) = y(t_1) + I_1(y(t_1)).$$

Thus

$$|y(t_1^+)| \leq |y(t_1)| + |I_1(y(t_1))| \leq K_1 + \sup\{|I_1(u)| : |u| \leq K_1\}.$$

Thus analogous to the above proof we can show that there exists $K_2 > 0$ such that

$$\sup\{|y(t)| : t \in [t_1, t_2]\} \leq K_2.$$

We continue this process and also take into account that

$$y(t) = \lambda \left[y(t_m^+) + g(t, y_t) - g(t_m, y_{t_m}) + \int\limits_{t_m}^{t} f(s)ds + \sum_{i=1}^{n_*} \int\limits_{t_m - T_i}^{t - T_i} y(s)ds \right],\ t \in (t_m, b],$$

and

$$y(t_m^+) = y(t_m) + I_1(y(t_m)).$$

We obtain that there exists a constant K_m such that

$$\sup\{|y(t)| : t \in [t_m, b]\} \leq K_m.$$

Consequently, for each possible solution y to $z = \lambda P(z)$ for some $\lambda \in (0,1)$ we have

$$\|y\|_\Omega \leq \max\{K_i : i = 1, \ldots, m\} := \overline{K}.$$

Set

$$U = \{y \in \Omega : \|y\|_\Omega < \overline{K} + 1\}.$$

and consider the operator $N : \overline{U} \to \mathcal{P}_{cv,cp}(\Omega)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \gamma N(y)$ for some $\gamma \in (0,1)$. As a consequence of the Leray-Schauder nonlinear alternative [23], we deduce that N has a fixed point y in U, which is a solution of the problem (1.2). □

3.2. THE NONCONVEX CASE

In this section we present a result for the problem (1.2) in the spirit of the linear alternative of Laray-Schauder type [23] for single-valued maps, combined with a selection theorem due to Bressan and Colombo [13] for lower semi-continuous multivalued maps with decomposable values.

Let $\mathcal{A}$ be a subset of $J \times \mathcal{D}$. $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$ where $\mathcal{J}$ is Lebesgue measurable in J and $\mathcal{D}$ is Borel measurable in $\mathcal{D}$. A subset $\mathcal{A}$ of $L^1(J, E)$ is decomposable if for all $w, v \in \mathcal{A}$ and $\mathcal{J} \subset J$ measurable, $w\mathcal{X}_{\mathcal{J}} + v\mathcal{X}_{J-\mathcal{J}} \in A$, where $\mathcal{X}$ stands for the characteristic function.

Let $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c) if the set $\{x \in X : F(x) \cap B \neq \emptyset\}$ is open for any open set $B \in \mathbb{R}^n$.

Definition 3.5. Let Y be a separable metric space and let $N : Y \to P(L^1(J, \mathbb{R}^n))$ be a multivalued operator. We say that N has property (BC) if:

1. N is lower semi-continuous (l.s.c.),
2. N has nonempty closed and decomposable values.

Let $F : J \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$F : \Omega \to \mathcal{P}(L^1(J, \mathbb{R}^n))$$

by letting

$$\mathcal{F}(y) = \{g \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y_t) \ for \ a.e. \ t \in J\}.$$

The operator F is called the Niemytzki operator associated to F.

Definition 3.6. Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. We say F is of lower semi continuous type (l.s.c. type) if its associated Niemytzky operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo [13].

Theorem 3.7. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection. i.e. there exists a continuous function (single-valued) $\tilde{g} : Y \to L^1(J, \mathbb{R}^n)$ such that $\tilde{g}(y) \in N(y)$ for every $y \in Y$.*

Let us introduce the following hypotheses which are used in the sequel:

$(\mathcal{A}_1)$ $F : J \times D \to \mathcal{P}(\mathbb{R}^n)$ is nonempty compact valued multivalued map such that:
- (a) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
- (b) $x \mapsto F(t, x)$ is lower semi-continuous for a.e. $t \in J$.

($\mathcal{A}_2$) For each $q > 0$, there exists a function $h_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq h_q(t) \text{ for a.e. } t \in J \text{ and for } x \in D \text{ with } \|x\|_D \leq q.$$

The following lemma is crucial in the proof of our main theorem.

Lemma 3.8 ([22])**.** *Let $F : J \times D \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with noneempty, compact values. Assume that ($\mathcal{A}_1$)-($\mathcal{A}_2$) hold. Then f is of l.s.c. type.*

Theorem 3.9. *Suppose that ($\mathcal{H}_2$)-($\mathcal{H}_3$) and ($\mathcal{A}_1$)-($\mathcal{A}_2$) hold. Then the problem* (1.2) *has at least one solution.*

Proof. ($\mathcal{A}_1$) and ($\mathcal{A}_2$) imply by Lemma 3.8 that F is of lower semi-continuous type. Then from Theorem 3.7 there exists a continuous function $f : \Omega \to L^1(J, \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the following problem:

$$\begin{cases} \frac{d}{dt}[y(t) - g(t, y_t)] = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i), & a.e.\ t \in J \backslash \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, \ldots, m, \\ y(t) = \phi(t), & t \in [-r, 0]. \end{cases} \tag{3.6}$$

Remark 3.10. If $y \in \Omega$ is a solution of the problem (3.6), then y is solution to the problem (1.2).

Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(y) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum\limits_{0<t_k<t} I_k(y(t_k^-)) + & \\ + \int\limits_0^t f(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^{0} \phi(s)ds, & \text{if } t \in J. \end{cases}$$

As in Theorem 3.4, we can prove that the single-valued operator G is compact and there exists $M_* > 0$ such that for all possible solutions y, we have $\|y\|_\Omega < M_*$. Now, we only check that N_1 is continuous. Let $\{y^n : n \in \mathbb{N}\}$ converges to some limit y_* in Ω. Then

$$\begin{aligned} \|N_1(y^n) - N_1(y)\|_\Omega \quad \leq \quad & \|g(\cdot, y_\cdot^n) - g(\cdot, y_\cdot)\|_D + \int\limits_0^b |f(y_s^n) - f(y_s)|ds + \\ & + \sum_{k=1}^m |I_k(y^n(t_k^-)) - I_k(y(t_k^-))|. \end{aligned}$$

Since the functions f and I_k, $k = 1, \ldots, m$, are continuous, we have

$$\begin{aligned} \|N_1(y_n) - N_1(y)\|_\Omega \quad \leq \quad & \|g(\cdot, y_\cdot^n) - g(\cdot, y_\cdot)\|_D + \int\limits_0^b |f(y_s^n) - f(y_s)|ds + \\ & + \sum_{k=1}^m |I_k(y_n(t_k^-)) - I_k(y(t_k^-))|, \end{aligned}$$

which, by continuity of f and I_k $(k = 1, \ldots, m)$, tends to 0 as $n \to \infty$. Let

$$U = \{y \in \Omega : \|y\|_\Omega < M_*\}.$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N_1 y$ for in $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of the Leray-Schauder type [23], we deduce that N_1 has a fixed point $y \in U$ which is a solution of the problem (3.6), hence a solution to the problem (1.2). □

In this part, we present a second existence result to the problem (1.2) with a non-convex valued right-hand side. First, consider the Hausdorff pseudo-metric distance

$$H_d\colon\ \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}$$

where $d(A,b) = \inf\limits_{a\in A} d(a,b)$ and $d(a,B) = \inf\limits_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(E), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [36]). In particular, H_d satisfies the triangle inequality.

Definition 3.11. A multi-valued operator $N\colon\ E \to \mathcal{P}_{cl}(E)$ is called:

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for all } x, y \in E,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is γ-Lipschitz, then for every $\gamma' > \gamma$,

$$N(x) \subset N(y) + \gamma' d(x,y) B(0,1) \text{ for all } x, y \in E.$$

Our proofs are based on the following classical fixed point theorem for contraction multi-valued operators proved by Covitz and Nadler [15] in 1970 (see also Deimling, [16, Theorem 11.1]).

Lemma 3.12. *Let (X,d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let us introduce the following hypotheses:

($\overline{\mathcal{A}}_1$) $F : J \times D \longrightarrow \mathcal{P}_{cp}(\mathbb{R}^n)$; $t \longmapsto F(t,x)$ is measurable for each $x \in D$.

($\overline{\mathcal{A}}_2$) There exists constants c_k, such that

$$|I_k(x) - I_k(y)| \le c_k|x - y| \text{ for each } k = 1, \ldots, m, \text{ and for all } x, y \in \mathbb{R}^n.$$

($\overline{\mathcal{A}}_3$) There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d(F(t,x), F(t,y)) \le l(t)|x - y| \text{ for a.e. } t \in J \text{ and all } x, y \in D,$$

with

$$H_d(0, F(t,0)) \le l(t) \text{ for a.e. } t \in J.$$

($\overline{\mathcal{A}}_4$) There exist $c_* > 0$ such that

$$\|g(t,u) - g(t,u_*)\|_D \le c_* \|u - u_*\|_D \quad \text{for all } u, u_* \in D,\ t \in J.$$

Theorem 3.13. *Let assumptions* ($\overline{\mathcal{A}}_1$)–($\overline{\mathcal{A}}_4$) *be satisfied. If* $c_* + \sum_{k=1}^{k=m} c_k < 1$, *then the problem* (1.2) *has at least one solution.*

Proof. In order to transform the problem (1.2) into a fixed point problem, let the multi-valued operator $N : \Omega \to \mathcal{P}(\Omega)$ be as defined in Theorem 3.4. We shall show that N satisfies the assumptions of Lemma 3.12.

(a) $N(y) \in \mathcal{P}_{cl}(\Omega)$ for each $y \in \Omega$. Indeed, let $\{h_n : n \in \mathbb{N}\} \subset N(y)$ be a sequence converge to h. Then there exists a sequence $f_n \in S_{F,y}$ such that

$$h_n(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \phi(0) + g(t,y_t) - g(0,\phi) + \sum\limits_{0<t_k<t} I_k(y(t_k^-)) + \\ + \int\limits_0^t f_n(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^0 \phi(s)ds, & \text{if } t \in J. \end{cases}$$

Since $F(\cdot,\cdot)$ has compact values, let $w(\cdot) \in F(\cdot,0)$ be a measurable function such that

$$|f(t) - w(t)| = d(g(t), F(t,0)).$$

From ($\overline{\mathcal{A}}_1$) and ($\overline{\mathcal{A}}_2$), we infer that for a.e. $t \in [0,b]$

$$\begin{aligned} |f_n(t)| &\le |f_n(t) - w(t)| + |w(t)| \le \\ &\le l(t)\|y\|_\Omega + l(t) := \widehat{M}(t), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Then the Lebesgue dominated convergence theorem implies that, as $n \to \infty$,

$$\|f_n - f\|_{L^1} \to 0 \ \text{ and thus } \ h_n(t) \to h(t)$$

with

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \phi(0) + g(t,y_t) - g(0,\phi) + \sum\limits_{0<t_k<t} I_k(y(t_k^-)) + \\ + \int\limits_0^t f(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^0 \phi(s)ds, & \text{if } t \in J, \end{cases}$$

proving that $h \in N(y)$.

(b) There exists $\gamma < 1$ such that

$$H_d(N(y), N(\overline{y})) \le \gamma \|y - \overline{y}\|_\Omega \text{ for all } y, \overline{y} \in \Omega.$$

Let $y, \overline{y} \in \Omega$ and $h \in N(y)$. Then there exists $v(t) \in F(t, y_t)$ such that

$$h(t) = \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + \int_0^t v(s)\, ds + \\ + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

From $(\overline{\mathcal{A}}_3)$ it follows that

$$H_d(F(t, y_t), F(t, \overline{y}_t)) \leq l(t)\|y_t - \overline{y}_t\|_D.$$

Hence, there is $w \in F(t, \overline{y}_t)$ such that

$$|v(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_D, \quad t \in J.$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$ given by

$$U(t) = \{w \in \mathbb{R}^n : |v(t) - w| \leq l(t)\|y_t - \overline{y}_t\|_D\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}_t)$ is measurable (see [6,14,26]), by Lemma 2.6, there exists a function $\overline{v}(t)$, which is a measurable selection for V. Thus $\overline{v}(t) \in F(t, \overline{y}_t)$ and

$$|v(t) - \overline{v}(t)| \leq l(t)\|y_t - \overline{y}_t\|_D \text{ for a.e. } t \in J.$$

Let us define for a.e. $t \in J$

$$\begin{aligned} \overline{h}(t) &= \phi(0) + g(t, \overline{y}_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + \\ &\quad + \int_0^t \overline{v}(s)\, ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} \overline{y}(s)ds + \sum_{0<t_k<t} I_k(\overline{y}(t_k^-)). \end{aligned}$$

Then we have

$$\begin{aligned}|h(t)-\overline{h}(t)| &\le \int_0^t |v(s)-\overline{v}(s)|\,ds + \sum_{k=1}^{n_*}\int_0^{t-T_i}|y(s)-\overline{y}(s)|ds+\\ &+\sum_{0<t_k<t}|I_k(y(t_k^-))-I_k(\overline{y}(t_k^-))| + \|g(t,y_t)-g(t,\overline{y}_t)\|_D \le\\ &\le \int_0^t l(s)\|y_s-\overline{y}_s\|_{\mathcal{D}}ds + n_*\int_0^t |y(s)-\overline{y}(s)|ds+\\ &+\sum_{0<t_k<t} c_k|y(t_k)-\overline{y}(t_k)| + c_*\|y_t-\overline{y}_t\|_D \le\\ &\le \int_0^t l(s)e^{\tau L(s)}ds\|y-\overline{y}\|_* + \int_0^t n_*e^{\tau L(s)}ds\|y-\overline{y}\|_*+\\ &+\sum_{0<t_k<t} c_k e^{\tau L(t)}\|y-\overline{y}\|_* + e^{\tau L(t)}c_*\|y-\overline{y}\|_* \le\\ &\le \int_0^t \frac{1}{\tau}(e^{\tau L(s)})'ds\|y-\overline{y}\|_* + \left(c_*+\sum_{k=1}^m c_k\right)e^{\tau L(t)}\|y-\overline{y}\|_* \le\\ &\le e^{\tau L(t)}\left(c_*+\frac{1}{\tau}+\sum_{k=1}^m c_k\right)\|y-\overline{y}\|_*.\end{aligned}$$

Thus

$$e^{-\tau L(t)}|h(t)-\overline{h}(t)| \quad\le\quad \left(c_*+\frac{1}{\tau}+\sum_{k=1}^m c_k\right)\|y-\overline{y}\|_*,$$

where $L(t)=\int_0^t l^*(s)ds$ and

$$l^*(t)=\begin{cases}0, & t\in[-r,0],\\ l(t)+n_*, & t\in[0,b],\end{cases}$$

and τ is sufficiently large and $\|\cdot\|_*$ is the Bielecki-type norm on Ω defined by

$$\|y\|_* = \sup\{e^{-\tau L(t)}|y(t)| : -r\le t\le b\}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y),N(\overline{y})) \le \left(c_*+\frac{1}{\tau}+\sum_{k=1}^m c_k\right)\|y-\overline{y}\|_* \text{ for all } y,\overline{y}\in\Omega.$$

So, N is a contraction. Thus, by Lemma 3.12, N has a fixed point y, which is a solution to (1.2). □

4. TOPOLOGICAL STRUCTURE OF SOLUTIONS SET

In this section we prove that the solutions set of the problem (1.2) is compact and the operator solution is u.s.c.

Theorem 4.1. *Under the assumptions of Theorem 3.4, the solution set for the problem* (1.2) *is compact, and the operator solution* $S(\cdot) : D \to \mathcal{P}(\Omega)$ *defined by*

$$S(\phi) = \{y \in \Omega : y \textit{ is a solution of } (1.2)\}$$

is u.s.c.

Proof. Compactness of the solution set. Let $\phi \in D$. Then

$$S(\phi) = \{y \in \Omega : y \text{ is a solution of the problem } (1.2)\}.$$

From Step 5 of Theorem 3.4, there exists $\widetilde{M}$ such that for every $y \in S(\phi)$, $\|y\|_\Omega \leq \widetilde{M}$. Since N is completely continuous, $N(S(\phi))$ is relatively compact in Ω. Let $y \in S(\phi)$. Then $y \in N(y)$. Hence $S(\phi) \subset \overline{N(S(\phi))}$, where N is defined in the proof of Theorem 3.4. It remains to prove that $S_F(a)$ is a closed subset in Ω. Let $\{y_n : n \in \mathbb{N}\} \subset S(\phi)$ be such that $(y^n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists v_n such that $v_n(t) \in F(t, y^n_t)$, a.e. $t \in J$ and

$$y^n(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y^n_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + & \\ + \int_0^t v^n(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y^n(s)ds + \sum_{0<t_k<t} I_k(y^n(t_k^-)), & \text{if } t \in J. \end{cases}$$

As in Step 3 of Theorem 3.4, we can prove that there exists v such that $v(t) \in F(t, y_t)$ and

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + & \\ + \int_0^t v(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & \text{if } t \in J. \end{cases}$$

Therefore, $y \in S(\phi)$, which yields that $S(\phi)$ is closed, hence a compact subset in Ω.

We will show that $S(\cdot)$ is u.s.c. by proving that the graph

$$\Gamma(\varphi) := \{(y, \varphi) \in \Omega \times D : y \in S(\varphi)\}$$

of $S(\varphi)$ is closed. Let$(y^n,\varphi^n)\in\Gamma(\varphi)$, i.e., $y^n\in S(\varphi^n)$, and let $(y^n,\varphi^n)\to(y,\varphi)$ as $n\to\infty$. Since $y^n\in S(\varphi^n)$, there exists $v^n\in L^1(J,\mathbb{R}^n)$ such that

$$y^n(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ \phi(0)+g(t,y^n_t)-g(0,\phi)+\sum\limits_{i=1}^{n_*}\int\limits_{-T_i}^{0}\phi(s)ds+ & \\ +\int\limits_0^t v^n(s)ds+\sum\limits_{i=1}^{n_*}\int\limits_0^{t-T_i}y^n(s)ds+\sum\limits_{0<t_k<t}I_k(y^n(t_k^-)), & \text{if } t\in J.\end{cases}$$

Using the fact that (y^n,φ^n) converge to (y,φ), there exists $M>0$ such that

$$\|\varphi^n\|_D\le M \text{ for all } n\in\mathbb{N}.$$

As in Theorem 1.2, we can prove that there exists $\overline{M}>0$ such that

$$\|y^n\|_\Omega\le\overline{M} \text{ for all } n\in\mathbb{N}.$$

By $(\mathcal{H}_2)$, we have

$$|v^n(t)|\le p(t)\psi(M),\ t\in J.$$

Thus, $v^n(t)\in p(t)\psi(M)\bar{B}(0,1):=\chi(t)$ a.e. $t\in J$. It is clear that $\chi:J\to\mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is a multivalued map that is integrable bounded. Since $\{v^n(\cdot):n\ge1\}\in\chi(\cdot)$, we may pass to a subsequence if necessary to obtain that v^n converges to v in $L^1(J,\mathbb{R}^n)$.

It remains to prove that $v\in F(t,y_t)$, for a.e. $t\in J$. Lemma 2.3 yields the existence of $\alpha_i^n\ge0$, $i=n,\dots,k(n)$ such that $\sum_{i=1}^{k(n)}\alpha_i^n=1$ and the sequence of convex combinations $g_n(\cdot)=\sum_{i=1}^{k(n)}\alpha_i^n v_i(\cdot)$ converges strongly to v in L^1. Since F takes convex values, using Lemma 2.2, we obtain that

$$\begin{aligned} v(t)\in\bigcap_{n\ge1}&\overline{\{g_n(t)\}},\quad \text{a.e. } t\in J\ \subset\\ \subset\bigcap_{n\ge1}&\overline{co}\{v_k(t),\ k\ge n\}\subset\\ \subset\bigcap_{n\ge1}&\overline{co}\{\bigcup_{k\ge n}F(t,y_t^k)\}=\\ =\overline{co}(&\limsup_{k\to\infty}F(t,y_t^k)).\end{aligned}\tag{4.1}$$

Since F is u.s.c. with compact values, then by Lemma 2.1, we have

$$\limsup_{n\to\infty}F(t,y_t^n)=F(t,y_t) \text{ for a.e. } t\in J.$$

This with (4.1) imply that $v(t)\in\overline{co}\,F(t,y_t)$. Since $F(\cdot,\cdot)$ has closed, convex values, we deduce that $v(t)\in F(t,y_t)$ for a.e. $t\in J$.

Let

$$z(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ \phi(0)+g(t,y_t)-g(0,\phi)+\sum_{i=1}^{n_*}\int\limits_{-T_i}^{0}\phi(s)ds+ & \\ +\int\limits_0^t v(s)ds+\sum_{i=1}^{n_*}\int\limits_0^{t-T_i}y(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)), & \text{if } t\in J.\end{cases}$$

Since the functions I_k, $k=1,\dots,m$ are continuous, we obtain the estimates

$$\|y^n-z\|_\Omega\le\|g(t,y_t^n)-g(t,y_t)\|_D+\int\limits_0^b|\bar{v}^n(s)-v(s)|ds+$$
$$+\sum_{k=1}^{m}|I_k(y^n(t_k))-I_k(y(t_k))|+\sum_{i=1}^{n_*}\int\limits_0^{t-T_i}|y^n(s)-y(s)|ds.$$

The right-hand side of the above expression tends to 0 as $n\to+\infty$. Hence,

$$y(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ \phi(0)+g(t,y_t)-g(0,\phi)+\sum_{i=1}^{n_*}\int\limits_{-T_i}^{0}\phi(s)ds+ & \\ +\int\limits_0^t v(s)ds+\sum_{i=1}^{n_*}\int\limits_0^{t-T_i}y(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)), & \text{if } t\in J.\end{cases}$$

Thus, $y\in S(\varphi)$, Now show that $S(\varphi)$ maps bounded sets into relatively compact sets of Ω. Let B be a compact set in $\mathbb{R}^n$ and let $\{y^n\}\subset S(B)$. Then there exist $\{\varphi^n\}\subset B$ such that $y^n\in S(\varphi^n)$. Since $\{\varphi^n\}$ is a compact sequence, there exists a subsequence of $\{\varphi^n\}$ converging to φ, so from $(\mathcal{H}_2)$, there exists $M_*>0$ such that

$$\|y^n\|_\Omega\le M_*,\ n\in\mathbb{N}.$$

We can show that $\{y^n : n\in\mathbb{N}\}$ is equicontinous in Ω. As a consequence of the Arzelá-Ascoli Theorem, we conclude that there exists a subsequence of $\{y^n\}$ converging to y in Ω. By a similar argument to the one above, we can prove that

$$y(t)=\begin{cases}\phi(t), & \text{if } t\in[-r,0],\\ \phi(0)+g(t,y_t)-g(0,\phi)+\sum_{i=1}^{n_*}\int\limits_{-T_i}^{0}\phi(s)ds+ & \\ +\int\limits_0^t v(s)ds+\sum_{i=1}^{n_*}\int\limits_0^{t-T_i}y(s)ds+\sum_{0<t_k<t}I_k(y(t_k^-)), & \text{if } t\in J,\end{cases}$$

where $v\in S_{F,y}$. Thus $y\in S(\varphi)$. This implies that $S(\varphi)$ is u.s.c. □

In this part we show that the solution set of the problem (1.2) is AR.

Definition 4.2. A space X is called an absolute retract (in short $X \in AR$) provided that for every space Y, every closed subset $B \subseteq Y$ and any continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$. In other words, for every space Y and for any embedding $f : X \longrightarrow Y$, the set $f(X)$ is a retract of Y.

Proposition 4.3 ([49])**.** *Let C be a closed, convex subset of a Banach space E and let $N : C \to \mathcal{P}_{cp,cv}(C)$ be a contraction multivalued map. Then $Fix(N)$ is a nonempty, compact AR-space.*

Our contribution is the following.

Theorem 4.4. *Let $F\colon J \times D \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be multivalued. Assume that all conditions of Theorem 3.13 are satisfied. Then the solution set $S_{[-r,b]}(\phi) \in AR$.*

Proof. Let the multi-valued operator $N : \Omega \to \mathcal{P}(\Omega)$ be as defined in Theorem 3.4. Using the fact that $F(\cdot,\cdot)$ has convex and compact values by $(\overline{\mathcal{A}}_1)$–$(\overline{\mathcal{A}}_2)$, then for every $y \in \Omega$ we have $N(y) \in \mathcal{P}_{cv,cp}(\Omega)$. By some Bielecki-type norm on Ω we can prove that N is contraction. Hence, from proposition 4.3, the solution set $S_{[-r,b]}(\phi) = Fix(N)$ is a nonempty, compact AR-space. □

5. THE PARAMETER-DEPENDANT CASE

In this section, we consider the following parameter impulsive problem:

$$\begin{cases} \dfrac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t, \lambda) + \displaystyle\sum_{i=1}^{n_*} y(t - T_i) & \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-), \lambda), & k = 1, \ldots, m, \\ y(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{5.1}$$

where $n_* \in \{1, 2, \ldots\}$, $r = \max\limits_{1 \le i \le n_*} T_i$, $F : J \times D \times \Lambda \to \mathcal{P}_{cp}(\mathbb{R}^n)$ is a multi-valued map with compact values, $I_k(\cdot,\cdot) : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$, are continuous functions, (Λ, d_Λ) is a complete metric space.

In the case with no impulses, some existence results and properties of solutions for semilinear and evolutions of differential inclusions with parameters were studied by Hu *et al.* [33], Papageorgiou and Yannakakis [47] and Tolstonogov [54,55]; see also [7] for a parameter-dependant first-order Cauchy problem. Very recently the parameter problems of impulsive differential inclusions was studied by Djebali *et al.* [17], Graef and Ouahab [28].

5.1. THE CONVEX CASE

We will assume the following.

($\widetilde{\mathcal{B}}_1$) The multi-valued map $F(\cdot, x, \lambda) : [0, b] \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is measurable for all $x \in \mathbb{R}^n$ and $\lambda \in \Lambda$.

($\widetilde{\mathcal{B}}_2$) The multi-valued map $F(t, \cdot, \cdot) : D \times \Lambda \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is u.s.c. for a.e. $t \in [0, b]$.

($\widetilde{\mathcal{B}}_3$) There exists $\alpha \in [0, 1)$ and $p, q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, x, \lambda)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|_D) \text{ for a.e. } t \in J \text{ and for all } x \in E, \ \lambda \in \Lambda.$$

Theorem 5.1. *Assume that F satisfies ($\widetilde{\mathcal{B}}_1$)–($\widetilde{\mathcal{B}}_3$). Then for every fixed $\lambda \in \Lambda$, there exists $y(\cdot, \lambda) \in \Omega$ a solution of the problem* (5.1).

Proof. For fixed $\lambda \in \Lambda$, let $F_\lambda(t, y_t) = F(t, y_t, \lambda)$, $(t, y_t) \in [0, b] \times \mathbb{R}^n$ and let $I_k^\lambda(y) = I_k(y, \lambda)$, $k = 1, \ldots, m$. It is clear that $F_\lambda(\cdot, u)$ is a measurable multi-valued map for all $u \in \mathbb{R}^n$, $F_\lambda(t, \cdot)$ is u.s.c and

$$\|F_\lambda(t, x)\|_{\mathcal{P}} \leq p(t)\psi(\|x\|_D) \text{ for a.e. } t \in J \text{ and each } x \in D,$$

where $p \in L^1(J, \mathbb{R}^+)$ are as defined in ($\widetilde{\mathcal{B}}_3$). To transform the problem (5.1) into a fixed point problem, consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) = \\ = \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum\limits_{0<t_k<t} I_k(y(t_k^-), \lambda) + \\ + \int\limits_0^t v(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^0 \phi(s)ds, & \text{if } t \in J \end{cases} \right\},$$

where $v \in S_{F,y}$. Clearly, the fixed points of the operator N are solutions of the problem (5.1).

Define the mapping $S : \Lambda \to \mathcal{P}_{cp}(\mathbb{R}^n)$ by

$$S(\lambda) = \{y \in \Omega : y \text{ is a solution of the problem (5.1)}\}.$$

From Theorem 3.4, $S(\lambda) \neq \emptyset$ so that S is well defined. Next, we prove the upper semi-continuity of solutions in respect of the parameter λ. □

Proposition 5.2. *If hypotheses ($\widetilde{\mathcal{B}}_1$) − ($\widetilde{\mathcal{B}}_3$) hold, then S is u.s.c.*

Proof. Step 1. $S(\cdot) \in \mathcal{P}_{cp}(\mathbb{R}^n)$. Let $\lambda \in \Lambda$ and $y_n \in S(\lambda)$, $n \in \mathbb{N}$. Then there exists $v_n \in S_{F_\lambda, y_n}$ such that

$$y_n(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, (y_n)_t) - g(0, \phi) + \sum\limits_{i=1}^{n_*} \int\limits_{-T_i}^0 \phi(s)ds + \\ + \int\limits_0^t v_n(s)ds + \sum\limits_{i=1}^{n_*} \int\limits_0^{t-T_i} y_n(s)ds + \sum\limits_{0<t_k<t} I_k(y_n(t_k^-), \lambda), & \text{if } t \in J. \end{cases}$$

From $(\widetilde{\mathcal{B}}_3)$ and the continuity of $I_k,\ k = 1, \dots, m$, we can prove that there exists $M > 0$ such that $\|y_n\|_\Omega \leq M,\ n \in \mathbb{N}$. As in the proof of Theorem 3.4, Steps 2 to 3, we can easily prove that the set $\{y_n : n \geq 1\}$ is compact in Ω; hence there exists a subsequence of $\{y_n\}$ which converges to y in Ω. Since $\{v_n\}(t)$ is integrably bounded, then arguing as in the proof of Theorem 4.1, there exists a subsequence which converges weakly to v and then we obtain at the limit:

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, y_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + & \\ + \int_0^t v(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-), \lambda), & \text{if } t \in J. \end{cases}$$

Hence $S(\cdot) \in \mathcal{P}_{cp}(\mathbb{R}^n)$.

Step 2. $S(\cdot)$ is quasicompact. Let K be a compact set in Λ. To show that $\overline{S(K)}$ is compact, let $y_n \in S(\lambda_n),\ \lambda_n \in K$. Then there exists $v_n \in S_{F(\cdot,\cdot,\lambda_n,y_n)},\ n \in \mathbb{N}$, such that

$$y_n(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + g(t, (y_n)_t) - g(0, \phi) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)ds + & \\ + \int_0^t v_n(s)ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y_n(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-), \lambda_n), & \text{if } t \in J. \end{cases}$$

As mentioned in Step 1, $\{y_n : n \geq 1\}$ is compact in Ω. Then there exists a subsequence of $\{y_n\}$ which converges to y in Ω. Since K is compact, there exists a subsequence $\{\lambda_n : n \geq 1\}$ in K such that λ_n converges to $\lambda \in \Lambda$. As we did above, we can easily prove that there exists $v(\cdot) \in F(\cdot, y_\cdot), \lambda)$ such that y satisfies (5.1).

Step 3. $S(\cdot)$ is closed. For this, let $\lambda_n \in \Lambda$ be such that λ_n converge to λ and let $y_n \in S(\lambda_n),\ n \in \mathbb{N}$ be a sequence which converges to some limit y in Ω. Then y_n satisfies (5.1) and as we did above, we can use $(\widetilde{\mathcal{B}}_3)$ to show that the set $\{y_n : n \geq 1\}$ is equicontinuous in Ω. Hence, by the Arzelá-Ascoli Theorem, we conclude that there exists a subsequence of $\{y_n\}$ converging to some limit y in Ω and there exists a subsequence of $\{v_n\}$ which converges to $v(\cdot) \in F(\cdot, y_\cdot), \lambda)$ such that y satisfies (5.1). Therefore $S(\cdot)$ has a closed graph, hence u.s.c. by Lemma 2.10. □

6. APPLICATION TO CONTROL THEORY

Many problems in applied mathematics, such as those in control theory, mathematical economics, and mechanics, lead to the study of differential inclusions. In a differential inclusion the tangent at each state is prescribed by a multifunction instead of the usual single-value function in differential equations. For single-valued functions the controllability may be described by a nonlinear differential equations of the form

$$\begin{cases} y'(t) = f(t, y(t), u(t)), \quad t \in \mathbb{R}_+, \\ y(0) = a, \\ \quad u \in U, \end{cases} \tag{6.1}$$

with constrained control u. Here $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a single-valued function measurable in t and continuous in y, u. The time-varying set of constraints function $U : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is a measurable multi-valued function. By $u \in U$, we mean $u(t) \in U(t)$, for a.e. $t \in J$. The problem (6.1) is solved if there is a control function u for which the problem admits a solution. If we define the multi-function

$$F(t, x) = \{f(t, x, u) : u \in U\}, \tag{6.2}$$

then Filippov [19] and Ważewski [59] have shown that under some assumptions the control problem (6.1) coincides with the set of the Carathéodory solution of the following problem

$$\begin{cases} y'(t) \in F(t, y(t)), \quad t \in \mathbb{R}_+, \\ y(0) = a, \\ \quad u \in U, \end{cases} \tag{6.3}$$

with right-hand side given by (6.2).

The controllability of ordinary differential equations and inclusions were investigated by many authors (see [8, 9, 12, 20, 36] for instance and the references therein).

And impulsive differential equations and inclusions dealing with control theory were investigated by [2, 11, 25]. Indeed, the first motivation of the study of the concept of differential inclusions comes from the development of some studies in control theory. For more information about the relation between the differential inclusions and control theory, see for instance [6, 21, 38, 51, 53] and the references therein.

Hereafter, we apply the existence results and structure topology and geometry obtained in Sections 3 and 4 to study the impulsive neutral problem, that is, the problem (1.2):

$$\begin{cases} \dfrac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t) + \displaystyle\sum_{i=1}^{n_*} y(t - T_i), & t \in J \backslash \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), & k = 1, \ldots, m, \\ y(t) = \phi(t) & t \in [-r, 0], \end{cases} \tag{6.4}$$

with F given by (6.2), $I_k : \mathbb{R} \to \mathbb{R}$, $x \to I_k(x) = b_k x$, $b_k \in \mathbb{R}$, $k = 1, \ldots, m$, $J =: [0,1]$ and $g : J \times D \to \mathbb{R}$ is a continuous function $0 < t_1 < t_2 < \ldots < t_m < 1$, $T_i \in \mathbb{R}_+$, $i = 1, \ldots, n_*$, $r = \max_{1 \le i \le n_*} T_i$.

We will need the following auxiliary result in order to prove our main controllability theorem.

Theorem 6.1 ([6]). *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measurable space, X a complete separable metric space and $F : \Omega \to \mathcal{P}(X)$ a measurable set value map with closed images. Consider a Carathéodory set-valued map G from $\Omega \times X$ to a complete separable metric space Y. Then the map*

$$\Omega \ni \omega \mapsto \overline{G(\omega, F(\omega))} \in \mathcal{P}(Y)$$

is measurable.

Next, we state our main existence result.

Theorem 6.2. *Assume that U and f satisfy the following hypotheses:*

($\overline{\mathcal{H}}1$) *$U : J \to \mathcal{P}_{cv,\,cp}(\mathbb{R}_+)$ is a measurable multi-function and has compact image.*

($\overline{\mathcal{H}}2$) *The function f is linear in the third argument, i.e. there exist Carathéodory functions $f_i : J \times D \to \mathbb{R}$ $(i = 1, 2)$ such that for a.e. $t \in J$,*

$$f(t, x, u) = f_1(t, x)u + f_2(t, x),\ \forall\, (x, u) \in D \times U.$$

($\overline{\mathcal{H}}3$) *There exist $k \in L^1(J, (0, +\infty))$ and a continuous nondecreasing function ψ such that*

$$|f(t, x, u)| \le k(t)\psi(\|x\|_D) \text{ for a.e. } t \in J, \forall\, x \in D \text{ and } \forall\, u \in U$$

with

$$\int_0^b k(s)ds < \int_0^\infty \frac{ds}{s + \psi(s)}.$$

($\overline{\mathcal{H}}4$) *For every $M > 0$, there exists $\epsilon > 0$ and a function $R : [0, \epsilon] \to \mathbb{R}_+$ with $\lim_{h\to 0} R(h) = 0$, such that for every $y \in \Omega$ satisfying $\|y\|_\Omega \le M$, we have*

$$|g(t, y_t) - g(s, y_s)| \le R(|t - s|) \text{ with } |t - s| < \epsilon.$$

and there exists $c_* > 0$ such that

$$|g(t, u)| \le c(\|u\|_D + 1) \text{ for every } u \in D.$$

Then the control boundary value problem (6.1) *has at least one solution.*

Proof. Claim 1. Since $U(\cdot)$ is measurable, we can find $u_n : [0, 1] \to \mathbb{R}$, $n \ge 1$, Lebesgue measurable functions such that

$$U(t) = \overline{\{u_n(t) :\ n \ge 1\}} \quad \text{for all } t \in [0, 1].$$

From ($\overline{\mathcal{H}}2$) and ($\overline{\mathcal{H}}3$) we have

$$F(t, y_t) = \overline{\{f_1(t, y_t)u_n(t) + f_2(t, y_t) :\ n \ge 1\}} \quad \text{for all } t \in [0, b].$$

This implies that the map $t \mapsto F(t, \cdot)$ is a measurable multifunction. By ($\overline{\mathcal{H}}3$) and ($\overline{\mathcal{H}}4$), we have that $F(\cdot, \cdot) \in \mathcal{P}_{cv}(\mathbb{R})$. Using the compactness of U and the continuity of f, we can easily show that $F(\cdot, \cdot) \in \mathcal{P}_{cp}(\mathbb{R})$; then $F(\cdot, \cdot) \in \mathcal{P}_{cp,\,cv}(\mathbb{R})$.

Claim 2. The selection set of F is not empty. Since U is a measurable multifunction and has compact image then $\overline{F(t,x)} = F(t,x)$. Let $x \in \mathbb{R}$ then from $(\overline{\mathcal{H}}1) - (\overline{\mathcal{H}}3)$ the map $(t,u) \to f(t,x,u)$ is L^1-Carathéodory. Hence from Theorem 6.1 $F(\cdot,x)$ is measurable.
Claim 3. Using the fact that U has a compact image and f is an L^1-Carathéodory function, hence we can easily show that $F(t,\cdot)$ is u.s.c. (see [18, Theorem 6.3, Claim 3]).
Claim 4. Let B be a bounded set in Ω, then there exists $M_* > 0$ such that

$$\|u\|_D \leq M_* \text{ for every } u \in B.$$

Then

$$|g(t,u)| \leq c(M_* + 1) \text{ for every } u \in B.$$

The first part of the condition $(\overline{\mathcal{H}}4)$ implies that

$$\{t \mapsto g(t,y_t) : \|y\|_\Omega \leq c(M_* + 1)\}$$

is equicontinouous. Therefore, all conditions of Theorems 3.4 and 4.1 are fulfilled, and then the problem (6.4) has at least one solution and solution set is compact. □

The following auxiliary lemma is concerned with measurability for two-variable multi-function.

Lemma 6.3 ([31])**.** *Let (Ω, A) be a measurable space, X, Y two separable metric spaces and let $F : \Omega \times X \to \mathcal{P}_{cl}(Y)$ be a multi-function such that:*

(i) *for every $x \in X$, $\omega \to F(\omega,x)$ is measurable,*
(ii) *for a.e. $\omega \in \Omega$, $x \to F(\omega,x)$ is continuous or H_d-continuous.*

Then the mapping $(\omega,x) \to F(\omega,x)$ is measurable.

Our contribution is the following.

Theorem 6.4. *Assume that U and f satisfy the following hypotheses:*

$(\overline{\mathcal{H}}5)$ *$U : J \to \mathcal{P}_{cp}(\mathbb{R})$ is a measurable multi-function.*
$(\overline{\mathcal{H}}6)$ *There exists $k \in L^1(J,(0,+\infty))$ such that*

$$|f(t,x,u) - f(t,y,u)| \leq k(t)\|x-y\|_D \text{ for a.e. } t \in J, \forall\, x \in D \text{ and } \forall\, u \in U.$$

$(\overline{\mathcal{H}}6)$ *There exists $p \in L^1(J,(0,+\infty))$ such that*

$$|f(t,x,u)| \leq p(t) \text{ for a.e. } t \in J, \forall\, x \in \mathbb{R} \text{ and } \forall\, u \in U.$$

$(\overline{\mathcal{H}}7)$ *there exists $c_* \in (0,1)$ such that*

$$|g(t,x) - g(t,z)| \leq c_*\|x-z\|_D \text{ for all } x,z \in D.$$

If $c_* + \sum_{i=1}^{m} |b_k| < 1$. Then the solution set of the problem (6.1) *is not empty.*

Proof. Clearly, $F(\cdot, x)$ is a measurable multi-function for any fixed x and $F(\cdot,\cdot) \in \mathcal{P}_{cp}(\mathbb{R})$. To prove that $F(t,\cdot)$ is k-Lipschitz, let $x, y \in D$ and $h \in F(t,x)$. Then there exists $u \in U$ such that $h(t) = f(t,x,u)$. From $(\overline{\mathcal{H}}7)$ we get successively the estimates

$$\begin{aligned} d(h, F(t,y)) &= \inf_{z \in F(t,y)} |h - z| = \\ &= \inf_{v \in U} |f(t,x,u) - f(t,y,v)| \leq \\ &\leq |f(t,x,u) - f(t,y,u)| \leq \\ &\leq k(t)\|x - y\|_D. \end{aligned}$$

By an analogous relation obtained by interchanging the roles of x and y, we find that for each $l \in F(t,y)$ it holds that

$$d(F(t,x), l) \leq k(t)\|x - y\|_D$$

and hence

$$H_d(F(t,x), F(t,y)) \leq k(t)\|x - y\|_D \quad \text{for each} \quad x, y \in \mathbb{R}.$$

So, $F(t,\cdot)$ is k-Lipschitz. Therefore $F(t,\cdot)$ is H_d-continuous and from Lemma 6.3, the two-variable multi-function $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable. Then Aumann's selection theorem (see Wagner [58, Theorem 5.10]) implies the existence of a measurable selection, hence $S_{F,y}$ has nonempty.

Then $F(t,\cdot)$ is in fact *u.s.c.* (see [16, Proposition 1.1]). Finally, notice that $F(t,0)$ is integrably bounded by $(\overline{\mathcal{H}}7)$. Consequently, all the conditions of Theorem 3.13 are met and the solution set of the problem (6.4) is not empty. □

Remark 6.5. If $F(\cdot,\cdot) \in \mathcal{P}_{cv}(\mathbb{R})$, then under the condition of Theorem 6.4 the solution set of the problem (6.4) is an *AR*-space (see Theorem 4.4).

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