Opuscula Mathematica • Vol. 32 • No. 1 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.1.21

COMPACTLY SUPPORTED MULTI-WAVELETS

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Abstract. In this paper we show some construction of compactly supported multi-wavelets in $L^2(\mathbb{R}^d)$, $d \geq 2$ which is based on the one-dimensional case, when $d = 1$. We also demonstrate that some methods, which are useful in the construction of wavelets with a compact support at $d = 1$, can be adapted to higher-dimensional cases if $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ is an expansive matrix of a special form.

Keywords: compactly supported multi-wavelet, compactly supported scaling function, multiresolution analysis, expansive matrix.

Mathematics Subject Classification: 42C40.

1. INTRODUCTION AND PRELIMINARIES

It is well-known that if we consider a one-dimensional case, that is, an $L^2(\mathbb{R})$ space, then we are able to construct a compactly supported wavelet ψ associated with some multiresolution analysis. Such wavelets have been considered by Daubechies in [7] and [8]. A generalization of this problem to higher dimensions was given by Ayache [1, 2] and also Belogay and Wang [3]. Moreover, the general existence problem for Meyer type wavelets in two dimensions was solved by Bownik and Speegle in [4].

The purpose of this paper is to show that we are able to give a generalization of the Daubechies construction for $L^2(\mathbb{R}^d)$, where $d = 2, 3$. Note that for every fixed dimension d we consider only some special expansive matrices $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$, which have a different form than dilations from [1].

We assume in this paper that the basic notation and definitions, which are characteristic of *Wavelet Theory*, are known to the reader. Nevertheless, some well-known facts will be given. Specific information can be found in [6, 8, 12].

At the beginning we will give the three basic definitions directly associated with this work.

Definition 1.1. A multi-wavelet or equivalently a wavelet set for an expansive matrix A is a finite set of functions $\psi^r(x) \in L^2(\mathbb{R}^d)$, $r = 1, 2, \dots, s$ such that the system

$$\{|detA|^{\frac{j}{2}}\psi^r(A^j x - \gamma)\}$$

for $r = 1, 2, \dots, s$, $j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^d$ is an orthonormal basis in $L^2(\mathbb{R}^d)$.

Definition 1.2. We say, that a multi-wavelet $\Psi = \{\psi^r\}_{r=1,2,\dots,s}$ is compactly supported, when each function ψ^r, $r = 1, 2, \dots, s$ has a compact support.

Definition 1.3. A multiresolution analysis for an expansive matrix A is a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ which satisfy the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(d) $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1}$.
(e) $f(x) \in V_0$ if and only if $f(x-\gamma) \in V_0$ for all $\gamma \in \mathbb{Z}^d$.
(f) There is a function $\varphi \in V_0$, called a scaling function, such that the system $\{\varphi(x-\gamma)\}_{\gamma\in\mathbb{Z}^d}$ is an orthonormal basis in V_0.

Now, we will demonstrate a sketch of a well-known construction of a wavelet set associated with a multiresolution analysis for an expansive matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$.

Thus assume that we have given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ with a scaling function φ and an expansive matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$. Of course $|detA| = q \geq 2$, $q \in \mathbb{N}$.

Let $\mathcal{K} = \{k_0, k_1, \dots, k_{q-1}\}$ be a full collection of representatives of the quotient group $\mathbb{Z}^d/A\mathbb{Z}^d$. Then $E_r = A\mathbb{Z}^d + k_r$, $r = 0, 1, \dots, q-1$ generate the set $\mathbb{Z}^d/A\mathbb{Z}^d$. Consequently, if we have a function $G \in V_1$ then its Fourier transform takes the form

$$\widehat{G}(A^*\xi) = m_G(\xi)\widehat{\varphi}(\xi),$$

where $m_G(\xi) = \sum_{\gamma\in\mathbb{Z}^d} b(\gamma)e^{-i\langle\xi,\gamma\rangle}$. Observe that the polynomial m_G can be written in the following way:

$$m_G(\xi) = \sum_{r=0}^{q-1}\sum_{\gamma\in E_r} b(\gamma)e^{-i\langle\xi,\gamma\rangle} = \sum_{r=0}^{q-1} m_G^r(\xi),$$

where $m_G^r(\xi) = \sum_{\gamma\in E_r} b(\gamma)e^{-i\langle\xi,\gamma\rangle}$. What is more for $r = 0, 1, \dots, q-1$

$$m_G^r(\xi) = e^{-i\langle\xi,k_r\rangle}\mu_G^r(A^*\xi),$$

where $\mu_G^r(\xi) = \sum_{\gamma\in\mathbb{Z}^d} c^r(\gamma)e^{-i\langle\xi,\gamma\rangle}$, $c^r(\gamma) = b(A\gamma + k_r)$, $r = 0, 1, \dots, q-1$. Using this notation we obtain the following proposition.

Proposition 1.4 ([12])**.** *Let $G_0, G_1, \dots, G_{q-1}$ belong to the space V_1. Introduce the notation $m_k(\xi) = m_{G_k}(\xi)$, $\mu_k^r(\xi) = \mu_{G_k}^r(\xi)$, where $m_{G_k}(\xi)$, $\mu_{G_k}^r(\xi)$ are defined as above. Then we have:*

(a) *The system* $\{G_0(t-\gamma)\}_{\gamma\in\mathbb{Z}^d}$ *is orthonormal if and only if*

$$\sum_{r=0}^{q-1} |\mu_0^r(\xi)|^2 = \frac{1}{q} \qquad a.e.$$

(b) *The system* $\{G_j(t-\gamma)\}_{\gamma\in\mathbb{Z}^d, j=0,\ldots,s}$, $s \le q-1$ *is orthonormal if and only if vectors* $v_j(\xi) = (\sqrt{q}\mu_j^r(\xi))_{r=0}^{q-1}$, $j = 0,1,\ldots,s$ *are orthonormal in* $\mathbb{C}^q$ *for almost every* $\xi \in \mathbb{R}^d$.

(c) *The system* $\{G_j(t-\gamma)\}_{\gamma\in\mathbb{Z}^d, j=0,\ldots,q-1}$ *is an orthonormal basis in* V_1 *if and only if the matrix*

$$U(\xi) = [\sqrt{q}\mu_j^r(\xi)]_{r,j=0,1,\ldots,q-1}$$

is unitary for almost every $\xi \in \mathbb{R}^d$.

Thus, having a given multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ for an expansive matrix A, we can construct a wavelet set. Obviously we need to find a unitary matrix $U(\xi)$ defined in point (c) of Proposition 1.4, where the first row is given by point (a). Then, using the construction demonstrated before Proposition 1.4 we obtain the multi-wavelet $\Psi = \{\psi^s\}_{s=1,2,\ldots,q-1}$, which elements satisfy equations

$$\widehat{\psi}^s(A^*\xi) = \widehat{\varphi}(\xi)\sum_{r=0}^{q-1} e^{-i\langle\xi,k_r\rangle}\mu_s^r(A^*\xi),$$

where $\mu_s^r(\xi) = \sum_{\gamma\in\mathbb{Z}^d} c_s^r(\gamma)e^{-i\langle\xi,\gamma\rangle}$, $s = 1,2,\ldots,q-1$.

Next we will consider cases for an expansive matrix $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ with $q = 4,8$, where $q = |detA|$. The construction which was demonstrated above will be very useful.

2. THE CONSTRUCTION OF COMPACTLY SUPPORTED MULTI-WAVELETS

In this section we show that for some multiresolution analysis with a compactly supported scaling function φ and an expansive matrix A, where $q = |detA|$, $q = 4,8$ we are able to construct a compactly supported multi-wavelet Ψ. As we know it is possible if there exists a unitary matrix $U(\xi)$, $\xi \in \mathbb{R}^d$ as in Proposition 1.4, which elements are trigonometric polynomials and $\mu_0^r(\xi) = \mu_\varphi^r(\xi)$, for $r = 0,\ldots,q-1$.

The following fact is very useful for our further considerations.

Fact 2.1. *Let* $q = 4,8$ *and* $\mathbb{S}_q$ *be the unit sphere in* $\mathbb{C}^q$. *Then there exists the matrix* $F_q(x) \in \mathcal{M}_{q\times q}(\mathbb{C})$ *for* $x \in \mathbb{S}_q \subset \mathbb{C}^q$, *which rows* $u_i(x)$, $i = 1,\ldots,q$ *are orthogonal in* $\mathbb{C}^q$ *for almost every* $x \in \mathbb{S}_q$ *and take the following form:*

$$u_1(x) = (x_1, x_2, \ldots, x_q) = x,$$
$$u_i(x) = (w_1^i(x,\overline{x}), w_2^i(x,\overline{x}), \ldots, w_q^i(x,\overline{x})),$$

where $w_j^i(y,z)$ *for* $i = 2,3,\ldots,q$, $j = 1,2,\ldots,q$ *are polynomials with variables* $y = (y_1, y_2, \ldots, y_q)$, $z = (z_1, z_2, \ldots, z_q)$.

Proof. If $q = 4$, we take the matrix $F_4(x)$, which rows are of the following form:

$$\begin{aligned}
u_1(x) &= (x_1, x_2, x_3, x_4) = x,\\
u_2(x) &= (-\overline{x}_3, 0, \overline{x}_1, 0),\\
u_3(x) &= (0, -\overline{x}_4, 0, \overline{x}_2),\\
u_4(x) &= (x_1 p(x), x_2[p(x)-1], x_3 p(x), x_4[p(x)-1]),
\end{aligned}$$

where $p(x) = |x_2|^2 + |x_4|^2$ and $x \in \mathbb{S}_4 \subset \mathbb{C}^4$.

For $q = 8$ we define the rows of the matrix $F_8(x)$ as follows:

$$\begin{aligned}
u_1(x) &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = x,\\
u_2(x) &= (-\overline{x}_3, 0, \overline{x}_1, 0, 0, 0, 0, 0),\\
u_3(x) &= (0, -\overline{x}_4, 0, \overline{x}_2, 0, 0, 0, 0),\\
u_4(x) &= (0, 0, 0, 0, -\overline{x}_7, 0, \overline{x}_5, 0),\\
u_5(x) &= (0, 0, 0, 0, 0, -\overline{x}_8, 0, \overline{x}_6),\\
u_6(x) &= (x_1 p_1(x), -x_2 p_2(x), x_3 p_1(x), -x_4 p_2(x), 0, 0, 0, 0),\\
u_7(x) &= (0, 0, 0, 0, x_5 g_1(x), -x_6 g_2(x), x_7 g_1(x), -x_8 g_2(x)),\\
u_8(x) &= h(x)(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) - (0, 0, 0, 0, x_5, x_6, x_7, x_8),
\end{aligned}$$

where

$$\begin{aligned}
p_1(x) &= |x_2|^2 + |x_4|^2,\ p_2(x) = |x_1|^2 + |x_3|^2,\\
g_1(x) &= |x_6|^2 + |x_8|^2,\ g_2(x) = |x_5|^2 + |x_7|^2,\\
h(x) &= |x_5|^2 + |x_6|^2 + |x_7|^2 + |x_8|^2
\end{aligned}$$

and $x \in \mathbb{S}_8 \subset \mathbb{C}^8$.

An elementary computation show that matrices $F_4(x)$, $F_8(y)$ have orthogonal rows for almost every $x \in \mathbb{S}_4 \subset \mathbb{C}^4$, $y \in \mathbb{S}_8 \subset \mathbb{C}^8$. □

Assume now that we have a multiresolution analysis with a compactly supported scaling function $\varphi \in L^2(\mathbb{R}^d)$, $d \geq 1$ and an expansive matrix A, where $|detA| = q$, and $q = 4, 8$. Thus $\mu(\xi) := \sqrt{q}(\mu_0^0(\xi), \mu_0^1(\xi), \ldots, \mu_0^{q-1}(\xi))$ has coordinates which are trigonometric polynomials on $\mathbb{R}^d$, where functions $\mu_0^r(\xi) = \mu_\varphi^r(\xi)$, $\xi \in \mathbb{R}^d$ are constructed as in Section 1. If we use Fact 2.1 we obtain the matrix $U(\xi) := F(\mu(\xi))$, which rows are orthogonal in $\mathbb{C}^q$ for almost every $\xi \in \mathbb{R}^d$, and every element of $U(\xi)$ is a trigonometric polynomial on $\mathbb{R}^d$. Additionally, if we assume that $\mu_0^{2r}(\xi) = \mu_0^{2r+1}(\xi)$, $r = 0, \ldots, 2^{m-1} - 1$, where $m = 2$ or $m = 3$, then we get the matrix $U(\xi)$ defined as above, where rows have constant norm in $\mathbb{C}^q$ for almost every $\xi \in \mathbb{R}^d$.

Thus by Proposition 1.4 we obtain a set $\Psi = \{\psi^s\}_{s=1,\ldots,q-1}$ such that the system $\{|detA|^{\frac{j}{2}} \psi^s(A^j x - \gamma)\}$ for $s = 1, 2, \ldots, q-1$, $j \in \mathbb{Z}$, $\gamma \in \mathbb{Z}^d$ is an orthonormal basis in $L^2(\mathbb{R}^d)$, and each function ψ^s is compactly supported.

Note that the rows $u_i(x)$, $i = 2, \ldots, 7$ of matrix $F_8(x)$, $x \in \mathbb{S}_8$ in Fact 2.1 was constructed by the form of matrix $F_4(x)$, $x \in \mathbb{S}_4$. The last vector $u_8(x)$ can be constructed by the Schmidt orthogonalization.

Remark 2.2. It is easy to notice that applying the notation as in Fact 2.1 we can construct the rows $u_i(x)$, $i = 2, \dots, 2^m - 1$ of matrix $F_{2^m}(x)$ using the rows of matrix $F_{2^{m-1}}(x)$, where $m \geq 2$. As in the previous example the last vector $u_{2^m}(x)$ can be obtained by the Schmidt orthogonalization. Of course for the matrix $F_2(x)$ we have $u_1(x) = (x_1, x_2) = x$, $u_2(x) = (-\overline{x}_2, \overline{x}_1)$, $x \in \mathbb{S}_2$. This gives us the method to construct a compactly supported multi-wavelet Ψ whenever $\varphi \in L^2(\mathbb{R}^d)$ is a compactly supported scaling function, $|detA| = 2^m$, $\mathbb{N} \ni m \geq 1$, and $\mu_0^{2r}(\xi) = \mu_0^{2r+1}(\xi)$, $r = 0, \dots, 2^{m-1} - 1$.

Note that assuming in Fact 2.1 that coordinates of the vector $x \in \mathbb{S}_q$ satisfy equalities $x_i = x_{i+2}$, $i = 1, \dots, q-2$, where $x = (x_1, \dots, x_q)$ it is possible to get matrices $F_4(x), F_8(x)$, which are unitary. Indeed, rows of the matrix $F_4(x)$ are of the form:

$$
\begin{aligned}
u_1(x) &= (x_1, x_2, x_1, x_2) = x, \\
u_2(x) &= \frac{1}{\sqrt{2}}(1, 0, -1, 0), \\
u_3(x) &= \frac{1}{\sqrt{2}}(0, 1, 0, -1), \\
u_4(x) &= (\overline{x}_2, -\overline{x}_1, \overline{x}_2, -\overline{x}_1),
\end{aligned}
$$

where $x \in \mathbb{S}_4 \subset \mathbb{C}^4$.

For matrix $F_8(x)$ we have:

$$
\begin{aligned}
u_1(x) &= (x_1, x_2, x_1, x_2, x_1, x_2, x_1, x_2) = x, \\
u_2(x) &= \frac{1}{\sqrt{2}}(1, 0, 0, 0, -1, 0, 0, 0), \\
u_3(x) &= \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1, 0, 0), \\
u_4(x) &= \frac{1}{\sqrt{2}}(0, 0, 1, 0, 0, 0, -1, 0), \\
u_5(x) &= \frac{1}{\sqrt{2}}(0, 0, 0, 1, 0, 0, 0, -1), \\
u_6(x) &= \sqrt{2}(\overline{x}_2, -\overline{x}_1, 0, 0, \overline{x}_2, -\overline{x}_1, 0, 0), \\
u_7(x) &= \sqrt{2}(0, 0, \overline{x}_2, -\overline{x}_1, 0, 0, \overline{x}_2, -\overline{x}_1), \\
u_8(x) &= (x_1, x_2, -x_1, -x_2, x_1, x_2, -x_1, -x_2),
\end{aligned}
$$

where $x \in \mathbb{S}_8 \subset \mathbb{C}^8$.

Thus we see that using the same argumentation as in Remark 2.2 we may construct unitary matrices $F_q(x)$, whenever $q = 2^m$, $\mathbb{N} \ni m \geq 1$ and $(x_1, \dots, x_q) = x \in \mathbb{S}_q$, where $x_i = x_{i+2}$, $i = 1, \dots, q-2$.

Nevertheless, we will consider only cases when $q = |detA|$, $q = 4, 8$.

3. EXAMPLES OF TRIGONOMETRIC POLYNOMIALS

In this part we show examples of trigonometric polynomials $m(\xi)$ which satisfy conditions (W1), (W2), (W3) of Theorem 3.1, where an expansive matrix $A = [a_{i,j}] \in \mathcal{M}_{d\times d}(\mathbb{Z})$ is a lower or upper triangular matrix and $a_{i,j} \in 2\mathbb{Z}$. We consider cases when $d = 2$, and $d = 3$, but some examples can be generalized to any dimension.

First, we give a well-known theorem, which is based on Cohen's orthonormality criterion [5].

Theorem 3.1 ([5]). *Let $m(\xi) = \sum_{\gamma\in E} a_\gamma e^{-i\langle\xi,\gamma\rangle}$, $E \subset \mathbb{Z}^d$, $\sharp E < \infty$. Moreover, let $A \in \mathcal{M}_{d\times d}(\mathbb{Z})$ be an expansive matrix, and $B = A^*$. Assume also that the following conditions are satisfied:*

(W1) $\quad \sum_{k\in\mathcal{K}} |m(\xi - 2\pi B^{-1}k)|^2 = 1, \qquad \xi \in \mathbb{R}^d,$

where $\mathcal{K}$ is a full collection of representatives of $\mathbb{Z}^d/B\mathbb{Z}^d$.

(W2) $\quad m(0) = 1,$ *where* $0 = (0, 0, \dots, 0) \in \mathbb{Z}^d,$

(W3) $\quad m(\xi)$ *satisfies Cohen's criterion.*

Then the infinite product

$$\Theta(\xi) = \prod_{j=1}^{\infty} m(B^{-j}\xi)$$

is convergent uniformly on compact sets in $\mathbb{R}^d$, that is, $\Theta(\xi)$ is a continuous function. Besides, Θ is in $L^2(\mathbb{R}^d)$ and the function φ which is given by the equation $\widehat{\varphi}(\xi) = (2\pi)^{-\frac{d}{2}}\Theta(\xi)$ has a support contained in some compact set Q. Moreover, φ is a scaling function of some multiresolution analysis for an expansive matrix A.

Example 3.2. Let A be the expansive matrix given by $A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$ and $B = A^*$. We define a trigonometric polynomial $m(\xi)$, $(\xi_1, \xi_2) = \xi \in \mathbb{R}^2$ as follows:

$$m(\xi) = m_1(\xi_1)m_2(\xi_2),$$

where $m_1(\xi_1) = \frac{1}{2}(1 + e^{i\xi_1})$, $\xi_1 \in \mathbb{R}$, $m_2(\xi_2)$, $\xi_2 \in \mathbb{R}$ is a mono-dimensional trigonometric polynomial satisfying conditions (W1), (W2) in one-dimensional version of Theorem 3.1, and $m_2(\xi_2) \neq 0$, for $\xi_2 \in (-\pi, \pi)$.

It is easy to see that taking $\widetilde{\mathcal{K}} = \{(i,j),\ i,j \in \{0,-1\}\}$ as the full collection of representatives of $\mathbb{Z}^2/B\mathbb{Z}^2$, polynomial $m(\xi)$ satisfies conditions (W1), (W2) of Theorem 3.1 with the matrix B, and $m(\xi) \neq 0$, for $\xi \in (-\pi, \pi)^2$. Here we show that $m(\xi)$ satisfies Cohen's criterion.

Let K be the parallelogram with vertices $(-3\pi, -\pi)$, $(-\pi, -\pi)$, (π, π), $(3\pi, \pi)$. It is easy to check that the polynomial $m(\xi)$ satisfies Cohen's criterion with set K. Indeed, observe that we have below the equality

$$B^{-1}K = \Big[-\frac{\pi}{2}, \frac{\pi}{2} \Big]^2.$$

Since $m(\xi) \neq 0$, $\xi \in (-\pi, \pi)^2$, we may write $m(\xi) \neq 0$, $\xi \in B^{-j}K \subset D$, for some compact set $D \subset (-\pi, \pi)^2$, and every $j \geq 1$. Thus, in view of compactness of D we obtain

$$\exists_{c>0} : |m(B^{-j}\xi)| > c, \qquad \xi \in K,\ j \geq 1.$$

Therefore Cohen's criterion is satisfied. What is more, in this case we have $\mu_0^r(\xi) = \mu_0^{r+2}(\xi)$, $r = 0, \dots, q-3$ with $q = 4$. Thus we may construct a compactly supported multi-wavelet $\Psi = \{\psi^1, \psi^2, \psi^3\}$ using Theorem 3.1 and the unitary matrix $U(\xi) := F_4(\mu(\xi))$, where $F_4(x)$, $x = (x_1, x_2, x_3, x_4)$ is the unitary matrix given in Section 2 with $x_i = x_{i+2}$, $i = 1, 2$.

Example 3.3. Now take the expansive matrix $A = \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix}$ and $B = A^*$. In this case a trigonometric polynomial $m(\xi)$, $(\xi_1, \xi_2) = \xi \in \mathbb{R}^2$ is defined as follows:

$$m(\xi) = m_1(\xi_1)m_2(2\xi_2)m_3(\xi_2),$$

where polynomials $m_j(\xi_j) = \frac{1}{2}(1 + e^{i\xi_j})$, $j = 2, 3$, $m_1(\xi_1)$, $\xi_1 \in \mathbb{R}$ is a mono-dimensional trigonometric polynomial defined in the same way as in previous example. As the full collection of representatives of $\mathbb{Z}^2/B\mathbb{Z}^2$ we choose $\mathcal{K} = \{(-i, -j),\ i \in \{0, 1\},\ j \in \{0, 1, 2, 3\}\}$. Of course $m(\xi)$ satisfies conditions (W1), (W2) of Theorem 3.1 with the matrix B, and $m(\xi) \neq 0$, for $\xi \in (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

To show that $m(\xi)$ satisfy Cohen's criterion we may define the set K as the parallelogram with vertices $(-2\pi, -\pi)$, $(0, -\pi)$, $(0, \pi)$, $(2\pi, \pi)$.

Just as before we can show that the polynomial $m(\xi)$ satisfies Cohen's criterion with set K. Moreover, $\mu_0^r(\xi) = \mu_0^{r+2}(\xi)$, $r = 0, \dots, q-3$ with $q = 8$. Like in the previous example the unitary matrix $U(\xi) := F_8(\mu(\xi))$ gives us a compactly supported multi-wavelet $\Psi = \{\psi^r,\ r \in \{1, \dots, 7\}\}$.

By analogy, we can generalize our examples to the dimension $d = 3$. Just notice that for an expansive matrix A of the form

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2l_1 & 2 & 0 \\ 2l_2 & 2l_3 & 2 \end{bmatrix},$$

where $l_i \in \mathbb{Z}$, $i = 1, 2, 3$ we may always construct a trigonometric polynomial $m(\xi)$, where $(\xi_1, \xi_2, \xi_3) = \xi \in \mathbb{R}^3$, satisfying conditions (W1), (W2) of Theorem 3.1 and the inequality $m(\xi) \neq 0$, $\xi \in (-\pi, \pi)^3$. We can just define a polynomial

$$m(\xi) = m_1(\xi_1)m_2(\xi_2)m_3(\xi_3),$$

where $m_i(\xi_i)$, $\xi_i \in \mathbb{R}$, $i = 1, 2, 3$ are mono-dimensional trigonometric polynomials defined in the same way as before. All we need to construct a compactly supported wavelet set is to find l_i, $i = 1, 2, 3$, such that $m(\xi)$ satisfies Cohen's criterion and matrix $U(\xi) := F_8(\mu(\xi))$ is unitary. As we know, it can be done when $\mu_0^r(\xi) = \mu_0^{r+2}(\xi)$, $r = 0, \ldots, 5$, or $\mu_0^{2r}(\xi) = \mu_0^{2r+1}(\xi)$, $r = 0, 1, 2, 3$.

Thus we see that our observations from Example 3.2 can be generalized in the same way to any dimension $d \geq 3$.

A similar extension to $d = 3$ can be done for Example 3.3.

It is clear that previous conclusions are also true for upper triangular matrices. Obviously, mono-dimensional trigonometric polynomials which were used above can be easily constructed. The full parametrization of such polynomials was presented in [11].

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Received: May 6, 2010.
Revised: March 17, 2011.
Accepted: May 6, 2011.