EXISTENCE AND UNIQUENESS RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE CONDITIONS

LinLi Lv, JinRong Wang, Wei Wei

Abstract. In this paper, we study the existence and uniqueness of fractional differential equations with boundary value conditions. A new generalized singular type Gronwall inequality is given to obtain important a priori bounds. Existence and uniqueness results of solutions are established by virtue of fractional calculus and fixed point method under some weak conditions. An example is given to illustrate the results.

Keywords: fractional differential equations, boundary value conditions, singular Gronwall inequality, existence, uniqueness.

Mathematics Subject Classification: 26A33, 34B05.

1. INTRODUCTION

In this paper, we study the following boundary value problems (BVP for short) for fractional differential equations involving the Caputo derivative

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), & 0 < \alpha < 1, \ t \in J = [0, T], \\ ay(0) + by(T) = c, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $f : J \times R \to R$ will be specified latter and a, b, c are real constants with $a + b \neq 0$.

Very recently, fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. We can find numerous applications in viscoelasticity, electrochemistry, control and electromagnetic. There has been a significant development in fractional differential equations. One can see the monographs of Kilbas *et al.* [7], Miller and Ross [8], Lakshmikantham *et al.* [10], Podlubny [11]. Particulary, Agarwal *et al.* [1] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions

involving the Caputo fractional derivative in finite dimensional spaces. Particularly, fractional differential equations and optimal controls in Banach spaces are studied by Balachandran *et al.* [2, 3], Benchohra *et al.* [4], N'Guérékata [5, 6], Mophou and N'Guérékata [9], Wang *et al.* [12–19], Zhou *et al.* [20–22] and etc.

The existence of solutions for this kind of BVP has been studied by Benchohra *et al.* [4]. Let us mention, however, the assumptions on f are strong (f is continuous and satisfies uniformly Lipschitz condition or uniformly bounded). We will present the new existence and uniqueness results for the fractional BVP (1.1) by virtue of fractional calculus and fixed point method under some weak conditions. Compared with the results appeared in [4], there are at least two differences: (i) the assumptions on f are more general and easy to check; (ii) a priori bounds is established by a new singular type Gronwall inequality (Lemma 3.1) given by us.

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Section 3, we give a generalized singular type Gronwall inequality which can be used to establish the estimate of fixed point set $\{y = \lambda Fy, \lambda \in (0, 1)\}$. In Section 4, we give two main results (Theorems 4.1–4.2), the first result based on Banach contraction principle, the second result based on Schaefer's fixed point theorem. Finally, an example is given to demonstrate the application of our main results.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. We denote by $C(J, R)$ the Banach space of all continuous functions from J into R with the norm $\|y\|_\infty := \sup\{|y(t)| : t \in J\}$. For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^p(J,R)} := \begin{cases} \left(\int\limits_J |m(t)|^p dt \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \{ \sup\limits_{t \in J - \bar{J}} |m(t)| \}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure of $\bar{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^p(J,R)} < \infty$.

We need some basic definitions and properties of fractional calculus theory which are used in this paper. For more details, see [7].

Definition 2.1. *The fractional order integral of the function $h \in L^1([a, b], R_+)$ of order $\alpha \in R_+$ is defined by*

$$I_a^\alpha h(t) = \int\limits_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the Gamma function.

Definition 2.2. *For a function h given on the interval $[a, b]$, the α-th Riemann-Liouville fractional order derivative of h, is defined by*

$$(D_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. *For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by*

$$(^cD_{a+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.4. *Let $\alpha > 0$, then the differential equation $^cD^{\alpha}h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in R$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5. *Let $\alpha > 0$, then*

$$I^{\alpha}(^cD^{\alpha}h)(t) = h(t) + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in R$, $i = 0, 1, 2, \cdots, n-1$, $n = [\alpha] + 1$.

Now, let us recall the definition of a solution of the fractional BVP (1.1).

Definition 2.6 ([1, Definition 3.1]). *A function $y \in C^1(J, R)$ is said to be a solution of the fractional BVP (1.1) if y satisfies the equation $^cD^{\alpha}y(t) = f(t, y(t))$ a.e. on J, and the condition $ay(0) + by(T) = c$.*

For the existence of solutions for the fractional BVP (1.1), we need the following auxiliary lemma.

Lemma 2.7 ([1, Lemma 3.2]). *A function $y \in C(J, R)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}f(s)ds - c\right],$$

if and only if y is a solution of the following fractional BVP

$$\begin{cases} ^cD^{\alpha}y(t) = f(t), & 0 < \alpha < 1, \ t \in J, \\ ay(0) + by(T) = c. \end{cases} \tag{2.1}$$

As a consequence of Lemmas 2.7, we have the following result which is useful in what follows.

Lemma 2.8. *A function $y \in C(J, R)$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds - c \right],$$

if and only if y is a solution of the fractional BVP (1.1).

Lemma 2.9 (Bochner theorem)**.** *A measurable function $f \colon J \to R$ is Bochner integrable if $|f|$ is Lebesgue integrable.*

Lemma 2.10 (Ascoli-Arzela theorem)**.** *Let $\mathcal{S} = \{s(t)\}$ is a real-valued (n-dimensional) vector function family of continuous mappings $s : [a, b] \to R$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, then there exists a uniformly convergent function sequence $\{s_n(t)\}(n = 1, 2, \ldots, t \in [a, b])$ in $\mathcal{S}$.*

Theorem 2.11. *(Schaefer's fixed point theorem) Let $F : C(J, R) \to C(J, R)$ completely continuous operator. If the set*

$$E(F) = \{x \in C(J, R) : x = \lambda F x \quad \text{for some} \quad \lambda \in [0, 1]\}$$

is bounded, then F has at least a fixed point.

3. GRONWALL'S INEQUALITY
WITH MIXED TYPE SINGULAR INTEGRAL OPERATOR

In order to apply the Schaefer fixed point theorem to show the existence of solutions, we need a new generalized singular type Gronwall inequality with mixed type singular integral operator. It will play an essential role in the study of fractional BVP.

Lemma 3.1. *Let $y \in C(J, R)$ satisfy the following inequality:*

$$|y(t)| \leq a + b \int_0^t (t-s)^{\alpha-1} |y(s)|^\lambda ds + c \int_0^T (T-s)^{\alpha-1} |y(s)|^\lambda ds, \tag{3.1}$$

where $\alpha \in (0, 1)$, $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{1-\alpha}$, $a, b, c \geq 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$|y(t)| \leq M^*.$$

Proof. Let $M := (b + c) \left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}} > 0$, and

$$x(t) = \begin{cases} 1, & |y(t)| \leq 1, \\ y(t), & |y(t)| > 1. \end{cases}$$

$$|y(t)| \le |x(t)| \le a + 1 + b \int_0^t (t-s)^{\alpha-1} |x(s)|^\lambda ds + c \int_0^T (T-s)^{\alpha-1} |x(s)|^\lambda ds \le$$

$$\le (a+1) + b \Big(\int_0^t (t-s)^{p(\alpha-1)} ds \Big)^{\frac{1}{p}} \Big(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \Big)^{\frac{p-1}{p}} +$$

$$+ c \Big(\int_0^T (T-s)^{p(\alpha-1)} ds \Big)^{\frac{1}{p}} \Big(\int_0^T |x(s)|^{\frac{\lambda p}{p-1}} ds \Big)^{\frac{p-1}{p}} \le$$

$$\le (a+1) + b \left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}} \int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds +$$

$$+ c \left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}} \int_0^T |x(s)|^{\frac{\lambda p}{p-1}} ds \le$$

$$\le (a+1) + (b+c) \left[\frac{T^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right]^{\frac{1}{p}} \int_0^T |x(s)|^{\frac{\lambda p}{p-1}} ds =$$

$$= (a+1) + M \int_0^T |x(s)|^{\frac{\lambda p}{p-1}} ds \le$$

$$\le (a+1) + M \int_0^T |x(s)| ds.$$

By the standard Gronwall inequality, we have

$$|y(t)| \le |x(t)| \le (a+1)e^{MT} := M^*. \qquad \square$$

4. MAIN RESULTS

Before stating and proving the main results, we introduce the following hypotheses.

(H1) The function $f : J \times R \to R$ is Lebesgue measurable with respect to t on J.

(H2) There exists a constant $\alpha_1 \in [0, \alpha)$ and real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(J, R_+)$ such that

$$|f(t, u_1) - f(t, u_2)| \le m(t)|u_1 - u_2| \quad \text{for each} \quad t \in J \quad \text{and all} \quad u_1, u_2 \in R.$$

(H3) There exists a constant $\alpha_2 \in [0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J, R_+)$ such that

$$|f(t, y)| \le h(t) \quad \text{for each} \quad t \in J \quad \text{and all} \quad y \in R.$$

For brevity, let $M = \|m\|_{L^{\frac{1}{\alpha_1}}(J,R_+)}$, $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,R_+)}$.

Our first result is based on the Banach contraction principle.

Theorem 4.1. *Assume that* (H1)–(H3) *hold. If*

$$\Omega_{\alpha,\alpha_1,T}(t) = \frac{M}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(t^{\alpha-\alpha_1} + \frac{|b|T^{\alpha-\alpha_1}}{|a+b|} \right) \leq \omega < 1 \quad \text{for all} \quad t \in J, \quad (4.1)$$

then the system (1.1) *has a unique solution.*

Proof. For each $t \in J$, we have

$$\int_0^t \left|(t-s)^{\alpha-1}f(s,y(s))\right| ds \leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \leq$$

$$\leq \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}} ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}} ds \right)^{\alpha_2} \leq$$

$$\leq \frac{T^{\alpha-\alpha_2}H}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.$$

Thus, $|(t-s)^{\alpha-1}f(s,y(s))|$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$ and $y \in C(J,R)$. Then $(t-s)^{\alpha-1}f(s,y(s))$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in J$ due to Lemma 2.9.

Hence, the fractional BVP (1.1) is equivalent to the following fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s,y(s))ds -$$

$$- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}f(s,y(s))ds - c \right], \quad t \in J.$$

Let

$$r \geq \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|}.$$

Now we define the operator F on $B_r := \{y \in C(J,R) : \|y\|_\infty \leq r\}$ as follows

$$(Fy)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s,y(s))ds -$$

$$- \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}f(s,y(s))ds - c \right], \quad t \in J. \qquad (4.2)$$

Therefore, the existence of a solution of the fractional BVP (1.1) is equivalent to that the operator F has a fixed point in B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fy \in B_r$ for every $y \in B_r$.

In fact, for $y \in B_r$ and all $t \in J$, one can verify that F is continuous on J, i.e., $Fy \in C(J, R)$, and

$$|(Fy)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s, y(s))|ds+$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|f(s, y(s))|ds + \frac{|c|}{|a+b|} \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds+$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{|c|}{|a+b|} \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}ds \right)^{1-\alpha_2} \left(\int_0^t (h(s))^{\frac{1}{\alpha_2}}ds \right)^{\alpha_2} +$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_2}}ds \right)^{1-\alpha_2} \left(\int_0^T (h(s))^{\frac{1}{\alpha_2}}ds \right)^{\alpha_2} +$$

$$+ \frac{|c|}{|a+b|} \leq$$

$$\leq \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|b|}{|a+b|} \times \frac{T^{\alpha-\alpha_2}H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{|c|}{|a+b|} \leq r,$$

which implies that $\|Fy\|_\infty \leq r$. Thus, we can conclude that for all $y \in B_r$, $Fy \in B_r$. i.e., $F : B_r \to B_r$ is well defined.

Step 2. F is a contraction mapping on B_r.

For $x, y \in B_r$ and any $t \in J$, using (H2) and Hölder inequality, we get

$$|(Fx)(t) - (Fy)(t)| \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| ds +$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s)|x(s) - y(s)| ds +$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m(s)|x(s) - y(s)| ds \leq$$

$$\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) ds + \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} m(s) ds \leq$$

$$\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^t (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} +$$

$$+ \frac{|b| \, \|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \left(\int_0^T (T-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_0^T (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \leq$$

$$\leq \frac{\|x-y\|_\infty}{\Gamma(\alpha)} \frac{t^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}(J, R_+)} +$$

$$+ \frac{|b|\|x-y\|_\infty}{|a+b|\Gamma(\alpha)} \frac{T^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|m\|_{L^{\frac{1}{\alpha_1}}(J, R_+)} \leq$$

$$\leq \left[\frac{M}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left(t^{\alpha-\alpha_1} + \frac{|b|T^{\alpha-\alpha_1}}{|a+b|} \right) \right] \|x-y\|_\infty.$$

So we obtain

$$\|Fx - Fy\|_\infty \leq \Omega_{\alpha, \alpha_1, T}(t) \|x-y\|_\infty.$$

Thus, F is a contraction due to the condition (4.1).

By Banach contraction principle, we can deduce that F has an unique fixed point which is just the unique solution of the fractional BVP (1.1). $\qquad\square$

Our second result is based on the well known Schaefer's fixed point theorem.
We make the following assumptions:
(H4) The function $f : J \times R \to R$ is continuous.

(H5) There exist constants $\lambda \in [0, 1 - \frac{1}{p}]$ for some $1 < p < \frac{1}{1-\alpha}$ and $N > 0$ such that

$$|f(t, u)| \leq N(1 + |u|^\lambda) \; for \; each \; t \in J \; and \; all \; u \in R.$$

Theorem 4.2. *Assume that* (H4)–(H5) *hold. Then the fractional BVP* (1.1) *has at least one solution on J.*

Proof. Transform the fractional BVP (1.1) into a fixed point problem. Consider the operator $F : C(J, R) \to C(J, R)$ defined as (4.2). It is obvious that F is well defined due to (H4).

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, R)$. Then for each $t \in J$, we have

$$|(Fy_n)(t) - (Fy)(t)| \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_n(s)) - f(s, y(s))| ds +$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, y_n(s)) - f(s, y(s))| ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{t \in J} |f(s, y_n(s)) - f(s, y(s))| ds +$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \sup_{t \in J} |f(s, y_n(s)) - f(s, y(s))| ds \leq$$

$$\leq \frac{\|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} ds + \frac{|b|}{|a+b|} \int_0^T (T-s)^{\alpha-1} ds \right] \leq$$

$$\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty.$$

Since f is continuous, we have

$$\|Fy_n - Fy\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) \|f(\cdot, y_n(\cdot)) - f(\cdot, y(\cdot))\|_\infty \to 0 \; as \; n \to \infty.$$

Step 2. F maps bounded sets into bounded sets in $C(J, R)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a $\ell > 0$ such that for each $y \in B_{\eta^*} = \{y \in C(J, R) : \|y\|_\infty \leq \eta^*\}$, we have $\|Fy\|_\infty \leq \ell$.

For each $t \in J$, we get

$$|(Fy)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s,y(s))|ds+$$

$$+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|f(s,y(s))|ds + \frac{|c|}{|a+b|}+$$

$$\leq \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(1+|y(s)|^\lambda)ds+$$

$$+ \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(1+|y(s)|^\lambda)ds + \frac{|c|}{|a+b|}+$$

$$\leq \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds + \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}ds + \frac{|c|}{|a+b|}+$$

$$+ \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|y(s)|^\lambda ds+$$

$$+ \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|y(s)|^\lambda ds \leq$$

$$\leq \frac{NT^\alpha}{\Gamma(\alpha+1)} + \frac{|b|NT^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}+$$

$$+ \frac{N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|y(s)|^\lambda ds+$$

$$+ \frac{|b|N}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|y(s)|^\lambda ds \leq$$

$$\leq \frac{NT^\alpha}{\Gamma(\alpha+1)} + \frac{|b|NT^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|c|}{|a+b|} + \frac{NT^\alpha(\eta^*)^\lambda}{\Gamma(\alpha+1)} + \frac{|b|NT^\alpha(\eta^*)^\lambda}{|a+b|\Gamma(\alpha+1)},$$

which implies that

$$\|Fy\|_\infty \leq \frac{NT^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|} + \frac{NT^\alpha(\eta^*)^\lambda}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) := \ell.$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J, R)$. Let $0 \le t_1 < t_2 \le T$, $y \in B_{\eta^*}$. Using (H5), we have

$$|(Fy)(t_2) - (Fy)(t_1)| =$$

$$= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s, y(s)) ds \right| \le$$

$$\le \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, y(s)) ds \right| +$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s)) ds \right| \le$$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] |f(s, y(s))| ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, y(s))| ds \le$$

$$\le \frac{N}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] (1 + |y(s)|^\lambda) ds +$$

$$+ \frac{N}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} (1 + |y(s)|^\lambda) ds \le$$

$$\le \frac{N(1 + (\eta^*)^\lambda)}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds + \frac{N(1 + (\eta^*)^\lambda)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \le$$

$$\le \frac{N(1 + (\eta^*)^\lambda)}{\Gamma(\alpha+1)} \left(|t_1^\alpha - t_2^\alpha| + 2(t_2 - t_1)^\alpha \right) \le \frac{3N(1 + (\eta^*)^\lambda)(t_2 - t_1)^\alpha}{\Gamma(\alpha+1)}.$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero, therefore F is equicontinuous. As a consequence of Step 1–3 together with the Arzela-Ascoli theorem (Lemma 2.10), we can conclude that $F : C(J, R) \to C(J, R)$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(F) = \{ y \in C(J, R) : y = \lambda Fy \text{ for some } \lambda \in (0, 1) \}$$

is bounded.

Let $y \in E(F)$, then $y = \lambda F y$ for some $\lambda \in (0,1)$. Thus, for each $t \in J$, we have

$$y(t) = \lambda \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds - \right.$$

$$\left. - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y(s)) ds - c \right] \right).$$

For each $t \in J$, we have

$$|y(t)| \leq \frac{\lambda N T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda |b| N T^\alpha}{|a+b| \Gamma(\alpha+1)} + \frac{\lambda |c|}{|a+b|} +$$

$$+ \frac{\lambda N}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s)|^\lambda ds + \frac{\lambda |b| N}{|a+b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |y(s)|^\lambda ds.$$

By Lemma 3.1, there exists a $M^* > 0$ such that

$$|y(t)| \leq M^*, \ t \in J.$$

Thus for every $t \in J$, we have

$$\|y\|_\infty \leq M^*.$$

This shows that the set $E(F)$ is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional BVP (1.1). $\qquad \square$

5. EXAMPLE

In this section we give an example to illustrate the usefulness of our main results.

Let us consider the following fractional boundary value problem,

$$\begin{cases} {}^c D^\alpha y(t) = \frac{e^{-t} |y(t)|}{(1+9e^t)(1+|y(t)|)}, \ \alpha \in (0,1), \ t \in J, \\ y(0) + y(T) = 0, \end{cases} \tag{5.1}$$

Set

$$f(t, y) = \frac{e^{-t} y}{(1+9e^t)(1+y)}, \ (t, y) \in J \times [0, \infty).$$

Let $y_1, y_2 \in [0, \infty)$ and $t \in J$. Then we have

$$|f(t, y_1) - f(t, y_2)| = \frac{e^{-t}}{(1+9e^t)} \left| \frac{y_1}{1+y_1} - \frac{y_2}{1+y_2} \right| =$$

$$= \frac{e^{-t} |y_1 - y_2|}{(1+9e^t)(1+y_1)(1+y_2)} \leq \frac{e^{-t}}{1+9e^t} |y_1 - y_2| \leq \frac{e^{-t}}{10} |y_1 - y_2|.$$

Obviously, for all $y \in [0, \infty)$ and each $t \in J$,

$$|f(t, y)| = \frac{e^{-t}}{1+9e^t} \left| \frac{y}{1+y} \right| \leq \frac{e^{-t}}{1+9e^t} \leq \frac{e^{-t}}{10}.$$

For $t \in J$, $\beta \in (0, \alpha)$, let $m(t) = h(t) = \frac{e^{-t}}{10} \in L^{\frac{1}{\beta}}(J, R_+)$, $M = \left\| \frac{e^{-t}}{10} \right\|_{L^{\frac{1}{\beta}}(J, R_+)}$.
Choosing suitable $\beta \in (0, \alpha)$, one can arrive at the following inequality

$$\Omega_{\alpha,\beta,T} = \frac{MT^{\alpha-\beta}}{\Gamma(\alpha)(\frac{\alpha-\beta}{1-\beta})^{1-\beta}} \times \frac{3}{2} < 1.$$

Thus all the assumptions in Theorem 4.1 are satisfied, our results can be applied to the problem (5.1).

Acknowledgements

REFERENCES

[1] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math. **109** (2010), 973–1033.

[2] K. Balachandran, J.Y. Park, *Nonlocal Cauchy problem for abstract fractional semilinear evolution equations*, Nonlinear Anal. **71** (2009), 4471–4475.

[3] K. Balachandran, S. Kiruthika, J.J. Trujillo, *Existence results for fractional impulsive integrodifferential equations in Banach spaces*, Commun. Nonlinear Sci. Numer. Simulat. **16** (2011), 1970–1977.

[4] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl. **3** (2008), 1–12.

[5] G.M. N'Guérékata, *A Cauchy problem for some fractional differential abstract differential equation with nonlocal conditions*, Nonlinear Anal. **70** (2009), 1873–1876.

[6] G.M. N'Guérékata, *Corrigendum: A Cauchy problem for some fractional differential equations*, Commun. Math. Anal. **7** (2009), 11–11.

[7] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, [in:] North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[8] K.S. Miller, B. Ross, *An introduction to the fractional calculus and differential equations*, John Wiley, New York, 1993.

[9] G.M. Mophou, G.M. N'Guérékata, *Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay*, Appl. Math. Comput. **216** (2010), 61–69.

[10] V. Lakshmikantham, S. Leela, J.V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[11] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[12] JinRong Wang, W. Wei, Y. Yang, *Fractional nonlocal integrodifferential equations of mixed type with time-varying generating operators and optimal control*, Opuscula Math. **30** (2010), 217–234.

[13] JinRong Wang, Y. Yang, W. Wei, *Nonlocal impulsive problems for fractional differential equations with time-varying generating operators in Banach spaces*, Opuscula Math. **30** (2010), 361–381.

[14] JinRong Wang, W. Wei, Y. Yang, *On some impulsive fractional differential equations in Banach spaces*, Opuscula Math. **30** (2010), 507–525.

[15] JinRong Wang, W. Wei, Y. Yang, *Fractional nonlocal integrodifferential equations and its optimal control in Banach spaces*, J. KSIAM **14** (2010), 79–91.

[16] JinRong Wang, Yong Zhou, *Time optimal control problem of a class of fractional distributed systems*, Int. J. Dyn. Diff. Eq. **3** (2011), 363–382.

[17] JinRong Wang, Yong Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. **12** (2011), 262–272.

[18] JinRong Wang et al., *A class of nonlocal integrodifferential equations via fractional derivative and its mild solutions*, Opuscula Math. **31** (2011), 119–135.

[19] JinRong Wang, W. Wei, Yong Zhou, *Fractional finite time delay evolution systems and optimal controls in infinite dimensional spaces*, J. Dyn. Contr. Syst. **17** (2011), in press.

[20] Yong Zhou, *Existence and uniqueness of fractional functional differential equations with unbounded delay*, Int. J. Dyn. Diff. Eq. **1** (2008), 239–244.

[21] Yong Zhou, Feng Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comp. Math. Appl. **59** (2010), 1063–1077.

[22] Yong Zhou, Feng Jiao, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. **11** (2010), 4465–4475.

LinLi Lv
lvlinli2008@126.com

Guizhou University
College of Technology
Guiyang, Guizhou 550004, P.R. China

JinRong Wang (corresponding author)
wjr9668@126.com

Guizhou University
College of Science
Guiyang, Guizhou 550025, P.R. China

Wei Wei
wwei@gzu.edu.cn

Guizhou University
College of Science
Guiyang, Guizhou 550025, P.R. China