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ON SOME EXISTENCE RESULTS OF MILD SOLUTIONS FOR NONLOCAL INTEGRODIFFERENTIAL CAUCHY PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study a class of integrodifferential evolution equations with nonlocal initial conditions in Banach spaces. Existence results of mild solutions are proved for a class of integrodifferential evolution equations with nonlocal initial conditions in Banach spaces. The main results are obtained by using the Schaefer fixed point theorem and semigroup theory. Finally, an example is given for demonstration.

Keywords: integrodifferential equations, nonlocal initial condition, completely continuous operator, Schaefer fixed point theorem, mild solutions.

Mathematics Subject Classification: 34G20.

1. INTRODUCTION

In this paper we discuss the nonlocal initial value integrodifferential problem (IVIDP for short)

$$u'(t) = Au(t) + f\left(t, u(t), \int_0^t k(t, s, u(s))ds\right), \quad t \in (0, b), \tag{1.1}$$

$$u(0) = g(u) + u_0, \tag{1.2}$$

where A is the infinitesimal generator of a C_0-semigroup $T(t)$ in a Banach space X and $f : [0, b] \times X \times X \to X$, $k : [0, b] \times [0, b] \times X \to X$, $g : C([0, b]; X) \to X$ are given X-valued functions.

The study of initial value problems with nonlocal conditions arises to deal specially with some situations in physics. For the importance of nonlocal conditions in different fields, we refer the reader to [1–3], [4, 10, 12–15] and the references contained therein. The integrodifferential equation with nonlocal conditions has been investigated extensively. We infer to some of the papers below. Balachandran and Chandrasekaran [5] studied the nonlocal Cauchy problem for semilinear integrodifferential equations with deviating argument. Balachandran and Kumar [6] studied nonlinear integrodifferential equations with time varying delays by using the resolvent operator and the Schafer fixed point theorem. Lin and Liu [7] studied the nonlocal Cauchy problem for semilinear integrodifferential equations by using resolvent operators. Liu and Ezzinbi [8] investigated non-autonomous integrodifferential equations with nonlocal conditions. Liang, Liu and Xiao [9] studied the integrodifferential equations without Lipschitz conditions on nonlinear and nonlocal terms.

In some references [8,9], authors assume g as f are Lipschitz continuous, and give the compactness conditions of g as $T(t)$ be compact and g be uniformly bounded. Particulary, Xue [16] studied the existence of semilinear differential equations with nonlocal initial conditions

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad t \in (0, b), \\ u(0) &= g(u) + u_0, \end{aligned}$$

under the following conditions of g, f and $T(\cdot)$:

Case 1: g and f are compact, $T(t)$ is aC_0-semigroup.

Case 2: g is Lipschitz continuous, f is compact and $T(t)$ is a C_0-semigroup.

Case 3: g is Lipschitz continuous and $T(t)$ is compact.

The purpose of this paper is to prove the new existence of mild solutions for a nonlocal IVIDP (1.1) and (1.2) with the help of the Schaefer fixed point theorem and the above assumptions on g, f and $T(\cdot)$.

Let $(X, \|\cdot\|)$ be a real Banach space. Denoted by $C([0, b]; X)$ the space of X-valued continuous functions on $[0, b]$ with the norm $|u| = \sup\{\|u(t)\|, t \in [0, b]\}$ denote by $L(0, b; X)$ the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|u\|_1 = \int_0^b \|u(t)\|dt$.

By a *mild solution* of the nonlocal IVIDP (1.1) and (1.2) we mean the function $u \in C([0, b]; X)$ which satisfies

$$u(t) = T(t)u_0 + T(t)g(u) + \int_0^t T(t-s)f\left(s, u(s), \int_0^s k(s, \theta, u(\theta))d\theta\right)ds \tag{1.3}$$

for all $t \in [0, b]$.

A C_0-semigroup $T(t)$ is said to be *compact* if $T(t)$ is compact for any $t > 0$. If the semigroup $T(t)$ is compact then $t \mapsto T(t)x$ are equicontinuous at all $t > 0$ with respect to x in all bounded subsets of X; i.e., the semigroup $T(t)$ is *equicontinuous*.

In order to derive an important estimate, we need the following generalized Gronwall lemma.

Lemma 1.1. *Let $b \geq t \geq s \geq 0$ the following inequality holds*

$$\|u(t)\| \leq \bar{a} + \bar{b} \int_0^t \|u(s)\| ds + \int_0^t \left(\bar{c} \int_0^s \|u(\theta)\| d\theta \right) ds, \tag{1.4}$$

where $u \in C([0,b], X)$, $\bar{a}, \bar{b}, \bar{c} > 0$. Then, for $b \geq t \geq 0$, the following inequality is valid:

$$\|u(t)\| \leq \bar{a} \exp\left(b\bar{b} + b^2\bar{c}\right).$$

Proof. Denote

$$v(t) = \bar{a} + \bar{b} \int_0^t \|u(s)\| ds + \int_0^t \left(\bar{c} \int_0^s \|u(\theta)\| d\theta \right) ds.$$

Then $\|u(t)\| \leq v(t)$ and $v(t)$ is nondecreasing for $t > 0$. Since

$$v(t) \leq \bar{a} + \int_0^t \left[\bar{b} + \int_0^s \bar{c} ds \right] v(s) ds,$$

by the classical Gronwall Lemma,

$$v(t) \leq \bar{a} \exp\left(\int_0^t \left[\bar{b} + \int_0^s \bar{c} ds \right] ds \right) \leq \bar{a} \exp\left(b\bar{b} + b^2\bar{c}\right).$$

This completes the proof. $\square$

To prove the existence results in this paper we need the following fixed point theorem by Schaefer.

Lemma 1.2 ([11])**.** *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a continuous and compact map, and let the set $\{x \in S : x = \lambda F x$ for some $\lambda \in (0,1)\}$ be bounded. Then F has at least one fixed point in S.*

In this paper we suppose that A generates a C_0-semigroup $T(t)$ on X. And, without loss of generality, we always suppose that $u_0 = 0$.

2. MAIN RESULTS

In this section we give some existence results of the nonlocal IVIDP (1.1) and (1.2). Here we list the following main results.

(Hg) (1) $g : C([0,b]; X) \to X$ is continuous and compact.
(2) There exist $M > 0$ such that $\|g(u)\| \leq M$ for $u \in C([0,b]; X)$.

(Hk) There exists a continuous function a_k: $[0,b]\times[0,b]\to[0,\infty)$ and an increasing continuous function $\Omega_k : R^+ \to R^+$ such that $\|k(t,s,x)\| \le a_k(t,s)\Omega_k(\|x\|)$ for all $x \in X$ and a.e. $t, s \in [0,b]$.

(Hf) (1) $f(\cdot,x,y)$ is measurable for $x,y\in X$, $f(t,\cdot,\cdot)$ is continuous for a.e. $t\in[0,b]$.
(2) There exist a function $a_f(\cdot)\in L^1(0,b,R^+)$ and an increasing continuous function $\Omega_f : R^+\to R^+$ such that $\|f(t,x,y)\| \le a_f(t)\Omega_f(\|x\|+\|y\|)$ for all $x,y\in X$ and a.e. $t\in[0,b]$.
(3) $f : [0,b]\times X\times X\to X$ is compact.

Theorem 2.1. *If* (Hg) *and* (Hk) *and* (Hf) *are satisfied, then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$\int_0^b \Omega(s)ds < \frac{1}{N}\int_{NM}^{\infty} \frac{ds}{2\Omega_k(s)+\Omega_f(s)}, \tag{2.1}$$

where

$$\Omega(t) = \max\left\{ a_f(t), a_k(t,t), \int_0^t \frac{\partial a_k(t,\theta)}{\partial t} d\theta \right\} \text{ and } N = \sup\{\|T(t)\|, t\in[0,b]\}.$$

Next, we give an existence result when g is Lipschitz:

(Hg′) There exists a constant $l_g < 1/N$ such that $\|g(u)-g(v)\| \le l_g|u-v|$ for $u,v\in C([0,b];X)$.

Theorem 2.2. *If* (Hg′), (Hg)(2), (Hk) *and* (Hf) *are satisfied, then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *when* (2.1) *holds.*

Above we suppose that g is uniformly bounded. Next, we give existence results without the hypothesis (Hg)(2).

Theorem 2.3. *If* (Hg)(1), (Hk) *and* (Hf) *are satisfied, then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$\int_0^b \Omega(s)ds < \liminf_{T\to\infty} \frac{T - N\alpha(T)}{N(2\Omega_k(T)+\Omega_f(T))}, \tag{2.2}$$

where $\alpha(T) = \sup\{\|g(u)\|; |u|\le T\}$.

Theorem 2.4. *If* (Hg′), (Hk) *and* (Hf) *are satisfied, then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$\int_0^b \Omega(s)ds < \liminf_{T\to\infty} \frac{T - Nl_gT}{N(2\Omega_k(T)+\Omega_f(T))}. \tag{2.3}$$

Next, we give an existence result when g is Lipschitz and the semigroup $T(t)$ is compact.

Theorem 2.5. *Assume that* (Hg′), (Hk), (Hf)(1), (Hf)(2) *are satisfied, and assume that* $T(t)$ *is compact. Then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$\int_0^b \Omega(s)ds < \liminf_{T\to\infty} \frac{T - Nl_gT}{N(2\Omega_k(T) + \Omega_f(T))}. \tag{2.4}$$

At last we would like to discuss the IVIDP (1.1) and (1.2) under the following growth conditions of f, k and g.

(Hf)(2′) There exists $m_f > 0$ such that

$$\|f(t,x,y)\| \le m_f(1 + \|x\| + \|y\|),$$

for a.e. $t \in [0,b]$ and $x, y \in X$.

(Hk′) There exists $m_k > 0$ such that

$$\|k(t,s,x)\| \le m_k(1 + \|x\|),$$

for a.e. $t, s \in [0,b]$ and $x \in X$.

(Hg)(2′) There exist constants c, d such that for $u \in C([0,b];X)$, $\|g(u)\| \le c|u| + d$.

Theorem 2.6. *Assume* (Hg)(1), (Hg)(2′), (Hk′), (Hf)(1), (Hf)(2′), *and assume* (Hf)(3) *is true, or* $T(t)$ *is compact. Then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$Nc\exp(Nm_fb + b^2Nm_k) < 1. \tag{2.5}$$

Theorem 2.7. *Assume* (Hg′), (Hk′), (Hf)(1), (Hf)(2′), *and assume* (Hf)(3) *is true or* $T(t)$ *is compact. Then there is at least one mild solution for the IVIDP* (1.1) *and* (1.2) *provided that*

$$Nl_g\exp(Nm_fb + b^2Nm_k) < 1. \tag{2.6}$$

3. PROOFS OF MAIN RESULTS

We define $K : C([0,b];X) \to C([0,b];X)$ by

$$(Ku)(t) = \int_0^t T(t-s)f\left(s, u(s), \int_0^s k(s,\theta,u(\theta))d\theta\right)ds \tag{3.1}$$

for $t \in [0,b]$. To prove the existence results we need following lemmas.

Lemma 3.1. *If* (Hk) *and* (Hf) *hold, then* K *is continuous and compact; i.e.* K *is completely continuous.*

Proof. The continuity of K is proved as follows. For this purpose, we assume that $u_n \to u$ in $B_r = \{u \in C([0,b];X); |u| \le r\}$. It comes from the continuity of k that $k(s,\theta,u_n(\theta)) \to k(s,\theta,u(\theta))$ and $\|k(s,\theta,u_n(\theta))\| \le \int_0^s a_k(s,\theta)\Omega_k(r)d\theta$. By Lebesgue's convergence theorem, $\int_0^s k(s,\theta,u_n(\theta))d\theta \to \int_0^s k(s,\theta,u(\theta))d\theta$. Similarly, we have

$$f\Big(s,u_n(s),\int_0^s k(s,\theta,u_n(\theta))d\theta\Big) \to f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big) \text{ a.e. } t \in [0,b].$$

and

$$\Big\|f\Big(s,u_n(s),\int_0^s k(s,\theta,u_n(\theta))d\theta\Big)\Big\| \le a_f(s)\Omega_f\Big(r+\int_0^s a_k(s,\theta)\Omega_k(r)d\theta\Big).$$

By Lebesgue's convergence theorem,

$$\begin{aligned}&|Ku_n - Ku| \le \\ &\le N\int_0^b \Big\|f\Big(s,u_n(s),\int_0^s k(s,\theta,u_n(\theta))d\theta\Big) - f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big)\Big\|ds \to 0\end{aligned}$$

as $n \to \infty$. So $Ku_n \to Ku$ in $C([0,b];X)$.

Form the Ascoli-Arzela theorem, to prove the compactness of K, we should prove that $KB_r \subset C([0,b];X)$ is equi-continuous and $KB_r(t) \subset X$ is pre-compact for $t \in [0,b]$ for any $r > 0$. For any $u \in B_r$, we have

$$\begin{aligned}&\|Ku(t+h) - Ku(t)\| \le \\ &\le N\int_t^{t+h} \Big\|f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big)\Big\|ds+ \\ &\quad + \int_0^t \Big\|[T(t+h-s)-T(t-s)]f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big)\Big\|ds \le \\ &\le N\int_t^{t+h} a_f(s)\Omega_f\Big(\|u(s)\| + \int_0^s a_k(s,\theta)\Omega_k(\|u(s)\|)d\theta\Big)ds+ \\ &\quad + N\int_0^t \Big\|[T(h)-I]f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big)\Big\|ds \le \\ &\le N\Omega_f\Big(r+\|a_k(b,b)\|_1\Omega_k(r)\Big)\int_t^{t+h} a_f(s)ds+ \\ &\quad + N\int_0^t \Big\|[T(h)-I]f\Big(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\Big)\Big\|ds.\end{aligned}$$

Since f is compact, $\left\|[T(h)-I]f\left(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\right)\right\| \to 0$ (as $h \to 0$) uniformly for $s \in [0,b]$ and $u \in B_r$. This implies that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left\|[T(h)-I]f\left(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\right)\right\| \le \epsilon$$

for $0 \le h < \delta$ and all $u \in B_r$. We know that

$$\|Ku(t+h)-Ku(t)\| \le N\Omega_f\Big(r+\|a_k(b,b)\|_1\Omega_k(r)\Big)\int_t^{t+h} a_f(s)ds + N\epsilon$$

for $0 \le h < \delta$ and all $u \in B_r$. So $KB_r \subset C([0,b];X)$ is equicontinuous.

The set $\left\{T(t-s)f\left(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\right); t,s \in [0,b], u \in B_r\right\}$ is pre-compact as f is compact and $T(\cdot)$ is a C_0 semigroup. So $KB_r(t) \subset X$ is pre-compact as

$$KB_r(t) \subset t\,\overline{\text{conv}}\left\{T(t-s)f\left(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\right); s \in [0,t], u \in B_r\right\}$$

for all $t \in [0,b]$. □

Define $J : C([0,b];X) \to C([0,b];X)$ by $(Ju)(t) = T(t)g(u)$. So u is the mild solution of IVIDP (1.1)and (1.2) if and only if u is the fixed point of $J+K$. We can prove the following lemma easily.

Lemma 3.2. *If* (Hg)(1) *and* (Hk) *are true then J is continuous and compact.*

Proof of Theorem 2.1. From the above we know that $J+K$ is continuous and compact. To prove existence, we need only prove that the set of fixed points of $\lambda(J+K)$ is uniformly bounded for $\lambda \in (0,1)$ by Schaefer's fixed point theorem (Lemma 1.2). Let $u = \lambda(J+K)u$, i.e., for $t \in [0,b]$

$$u(t) = \lambda T(t)g(u) + \lambda\int_0^t T(t-s)f\left(s,u(s),\int_0^s k(s,\theta,u(\theta))d\theta\right)ds.$$

We have

$$\|u(t)\| \le NM + N\int_0^t a_f(s)\Omega_f\Big(\|u(s)\| + \int_0^s a_k(s,\theta)\Omega_k(\|u(\theta)\|)d\theta\Big)ds.$$

Denote

$$x(t) = NM + N\int_0^t a_f(s)\Omega_f\Big(\|u(s)\| + \int_0^s a_k(s,\theta)\Omega_k(\|u(\theta)\|)d\theta\Big)ds, \tag{3.2}$$

then we know that $x(0) = NM$ and $\|u(t)\| \leq x(t)$ for $t \in [0, b]$, and

$$x'(t) = a_f(t)\Omega_f\left(\|u(t)\| + \int_0^t a_k(t,\theta)\Omega_k(\|u(\theta)\|)d\theta\right) \leq$$
$$\leq Na_f(t)\Omega_f\left(x(t) + \int_0^t a_k(t,\theta)\Omega_k(x(\theta))d\theta\right)$$

for a.e. $t \in [0, b]$.

From (Hk), (Hf) and (3.2), we know that x is increasing. Let

$$y(t) = x(t) + \int_0^t a_k(t,\theta)\Omega_k(x(\theta))d\theta.$$

Then $y(0) = x(0)$ and $x(t) \leq y(t)$. Furthermore,

$$y'(t) = x'(t) + a_k(t,t)\Omega_k(x(t)) + \int_0^t \frac{\partial a_k(t,\theta)}{\partial t}\Omega_k(x(\theta))d\theta \leq$$
$$\leq Na_f(t)\Omega_f(y(t)) + a_k(t,t)\Omega_k(y(t)) + \int_0^t \frac{\partial a_k(t,\theta)}{\partial t}\Omega_k(y(\theta))d\theta \leq$$
$$\leq N\Omega(t)\{2\Omega_k(y(t)) + \Omega_f(y(t))\},$$

where $\Omega(t) = \max\{a_f(s), a_k(t,t), \int_0^t \frac{\partial a_k(t,\theta)}{\partial t}d\theta\}$.

This implies

$$\int_{y(0)=NM}^{y(t)} \frac{ds}{N(2\Omega_k(s) + \Omega_f(s))} \leq \int_0^t \Omega(s)ds \leq \int_0^b \Omega(s)ds < \int_{NM}^{\infty} \frac{ds}{N(2\Omega_k(s) + \Omega_f(s))},$$

for $t \in [0, b]$. This implies that there is a constant $r > 0$ such that $x(t) \leq r$, where r depends only on b and on the functions Ω, a_f, a_k and is independent of λ. We complete the proof as $\|u(t)\| \leq r$ for $u \in \{u; u = \lambda(J + K)u$ for some $\lambda \in (0, 1)\}$. □

For the next lemma, let $L : C([0, b]; X) \to C([0, b]; X)$ be defined by $(Lu)(t) = u(t) - T(t)g(u)$.

Lemma 3.3. *If* (Hg$'$) *holds then L is bijective and L^{-1} is Lipschitz continuous with constant* $1/(1 - Nl_g)$.

Proof of Theorem 2.2. Clearly u is a mild solution of the IVIDP and (1.2) if and only if u is the fixed point of $L^{-1}K$. Similarly with Theorem 2.1 we need only prove that the set $\{u; \lambda u = (L^{-1}K)u$ for some $\lambda > 1\}$ is bounded as $L^{-1}K$ be continuous and compact due to the fixed point theorem of Schaefer. If $\lambda u = L^{-1}Ku$. Then for any $t \in [0, b]$

$$\lambda u(t) = T(t)g(\lambda u) + \int_0^t T(t-s) f\left(s, u(s), \int_0^s k(s, \theta, u(\theta)) d\theta\right) ds.$$

We have

$$\|u(t)\| \leq \frac{1}{\lambda} NM + \frac{1}{\lambda} N \int_0^t a_f(s) \Omega_f\left(\|u(s)\| + \int_0^s a_k(s,\theta)\Omega_k(\|u(s)\|) d\theta\right) ds \leq$$

$$\leq NM + N \int_0^t \Omega_f\left(\|u(s)\| + \int_0^s a_k(s,\theta)\Omega_k(\|u(s)\|) d\theta\right) ds.$$

Just as proved in Theorem 2.1 we know there is a constant r which is independent of λ, such that $|u| \leq r$ for all $u \in \{u; \lambda u = (L^{-1}K)u$ for some $\lambda > 1\}$. So we have proved this theorem. □

Proof of Theorem 2.3. By Lemma 3.1 and Lemma 3.2 we know that $J + K$ is continuous and compact. From (2.2) there exists a constant $r > 0$ such that

$$\int_0^b \Omega(s) ds \leq \frac{r - N\alpha(r)}{N(2\Omega_k(r) + \Omega_f(r))}. \tag{3.3}$$

For any $u \in B_r$ and $v = Ju + Ku$, we get

$$\|v(t)\| \leq N\alpha(r) + N \int_0^t \Omega_f\left(\|r\| + \int_0^s a_k(s,\theta)\Omega_k(r) d\theta\right) ds \leq r,$$

for $t \in [0, b]$. It implies that$(J + K)B_r \subset B_r$. By Schauder's fixed point theorem, we know that there is at least one fixed point $u \in B_r$ of the completely continuous map $J + K$, and u is a mild solution. □

Proof of Theorem 2.4. By Lemma 3.1 and Lemma 3.3 we know that $L^{-1}K$ is continuous and compact. From (2.3) there exists a constant number $r > 0$ such that

$$\int_0^b \Omega(s) ds \leq \frac{r - Nl_g r - N\|g(0)\|}{N(2\Omega_k(r) + \Omega_f(r))}. \tag{3.4}$$

For any $u \in B_r$ and $v = L^{-1}Ku$, we get

$$\|v(t)\| \leq Nl_g|v| + N\|g(0)\| + N \int_0^t \Omega_f\left(\|r\| + \int_0^s a_k(s,\theta)\Omega_k(r) d\theta\right) ds,$$

for $t \in [0, b]$. It implies that $|v| \leq r$, i.e., $L^{-1}KB_r \subset B_r$. By Schauder's fixed point theorem, there is at least one fixed point $u \in B_r$ of the completely continuous map $L^{-1}K$, and u is a mild solution. □

Proof of Theorem 2.5. By the standard argument, it is not difficult to verify that K is completely continuous under (Hk), (Hf)(1), (Hf)(2) and the condition of compactness of the semigroup $T(t)$. So $L^{-1}K$ is completely continuous. Similarly with the proof of Theorem 2.4, we complete the the proof of this theorem. □

Proof of Theorem 2.6. Since the map $J+K$ is completely continuous, by Lemma 1.2, we need only prove that the set $\{u; u = \lambda(J+K)u$ for some $\lambda \in (0,1)\}$ is bounded. For any $u \in \{u; u = \lambda(J+K)u$ for some $\lambda \in (0,1)\}$, we have

$$\begin{aligned}\|u(t)\| &\leq \lambda N[c|u| + d + (m_f b + m_k)b] + \lambda N m_f \int_0^t \|u(s)\| ds + \\ &+ \lambda N \int_0^t \left(m_k \int_0^s \|u(\theta)\| d\theta \right) ds \leq \\ &\leq N[c|u| + d + (m_f b + m_k)b] + N m_f \int_0^t \|u(s)\| ds + \\ &+ \int_0^t \left(N m_k \int_0^s \|u(\theta)\| d\theta \right) ds.\end{aligned}$$

This implies that for $t \in [0, b]$

$$|u| \leq \frac{N(d + m_f b^2 + m_k b)\exp(N m_f b + b^2 N m_k)}{1 - Nc\exp(N m_f b + b^2 N m_k)},$$

by Lemma 1.2. □

Proof of Theorem 2.7. Since the map $L^{-1}K$ is completely continuous, by Schaefer's fixed point theorem (Lemma 1.2), we need only prove that the set $\{u; u = \lambda(L^{-1}K)u$ for some $\lambda \in (0,1)\}$ is bounded. For any $u \in \{u; u = \lambda(L^{-1}K)u$ for some $\lambda \in (0,1)\}$, similarly with the estimation above, we know that

$$\begin{aligned}\|u\| = \|\lambda L^{-1}Ku\| &\leq N[l_g|u| + \|g(0)\| + (m_f b + m_k)b] + N m_f \int_0^t \|u(s)\| ds + \\ &+ \int_0^t \left(N m_k \int_0^s \|u(\theta)\| d\theta \right) ds.\end{aligned}$$

By Lemma 1.2 again,

$$|u| \leq \frac{N(\|g(0)\| + m_f b^2 + m_k b)\exp(Nm_f b + b^2 N m_k)}{1 - Nl_g \exp(Nm_f b + b^2 N m_k)}.$$

The proof is complete. □

4. AN EXAMPLE

Let Ω be a bounded domain in R^n with smooth boundary, and $X = L^2(\Omega)$. Consider the following nonlinear integrodifferential equation in X

$$\frac{\partial u(t,y)}{\partial t} = \triangle u(t,y) + \frac{\gamma_1 u(t,y)}{(1+t)(1+t^2)}\sin(u(t,y)) + \int\limits_0^t \frac{\gamma_2 u(s,y)}{(1+t)(1+t^2)^2(1+s)^2}ds, \tag{4.1}$$

with nonlocal conditions

$$u(0) = u_0(y) + \int\limits_\Omega \int\limits_0^b h(t,y)\log(1+|u(t,s)|^{\frac{1}{2}})dtds, \ y \in \Omega, \tag{4.2}$$

or

$$u(0) = u_0(y) + \gamma_3 u(t,y), \ y \in \Omega, \tag{4.3}$$

where $\gamma_1, \gamma_2, \gamma_3 \in R$, $h(t,y) \in C([0,b] \times \bar{\Omega})$. Set $A = \triangle$, $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,

$$f\left(t, u(t), \int\limits_0^t k(t,s,u(s))ds\right)(y) = \frac{\gamma_1 u(t,y)}{(1+t)(1+t^2)}\sin(u(t,y)) + \int\limits_0^t \frac{\gamma_2 u(s,y)}{(1+t)(1+t^2)^2(1+s)^2}ds.$$

Define nonlocal conditions

$$g(u)(y) = u_0(y) + \int\limits_\Omega \int\limits_0^b h(t,y)\log(1+|u(t,s)|^{\frac{1}{2}})dtds, \tag{4.4}$$

or

$$g(u)(y) = u_0(y) + \gamma_3 u(t,y). \tag{4.5}$$

It is easy to see that A generates a compact C_0-compact semigroup in X, and

$$\left\| f\left(t, u(t), \int\limits_0^t k(t,s,u(s))ds\right)\right\| \leq |\gamma|(1+\|u\|+\|j\|), |\gamma| = \max\{|\gamma_1|, |\gamma_2|\}$$

where

$$j = \int\limits_0^t \frac{\gamma_2 u(s,y)}{(1+t)(1+t^2)^2(1+s)^2} ds, \quad k(t,s,u(s)(y)) = \frac{\gamma_2 u(s,y)}{(1+t)(1+t^2)^2(1+s)^2}$$

and

$$\|k(t,s,u(s))\| \leq |\gamma_2|(1+\|u\|).$$

For nonlocal conditions (4.4),

$$\|g(u)\| \leq b(mes(\Omega)) \max_{t\in[0,b], y\in\bar{\Omega}} |h(t,y)|[\|u\| + (mes(\Omega))^{\frac{1}{2}}], u \in C([0,b];X),$$

and g is compact (see Example of [9]). Furthermore, γ_1 γ_2, b and $h(t,y)$ can be chosen such that (2.5) is also satisfied. For nonlocal conditions (4.5),

$$\|g(u_1) - g(u_2)\| \leq |\gamma_3|\|u_1 - u_2\|.$$

Hence, g is Lipschitz. Furthermore, γ_1, γ_2, γ_3 and b can be chosen such that (2.6) is also satisfied. Obviously, it satisfies all the assumptions given in our Theorem 2.6 (or Theorem 2.7), the problem has at least one mild solution in $C([0,b];L^2(\Omega))$.

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