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Dedicated to Professor Ion Colojoară on his 80th birthday

CONVOLUTION ALGEBRAS FOR TOPOLOGICAL GROUPOIDS WITH LOCALLY COMPACT FIBRES

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Abstract. The aim of this paper is to introduce various convolution algebras associated with a topological groupoid with locally compact fibres. Instead of working with continuous functions on G, we consider functions having a uniformly continuity property on fibres. We assume that the groupoid is endowed with a system of measures (supported on its fibres) subject to the "left invariance" condition in the groupoid sense.

Keywords: convolution algebra, groupoid, uniform continuity, Haar system.

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1. INTRODUCTION

A groupoid is the natural extension of a group. The shortest way to define a groupoid is to say it means a small category where each morphism is invertible. If a groupoid has only one object, then it is a group. Much of the rich theory associated with locally compact second countable groups can be expressed in terms of algebra and measure theory. In fact, if G is an analytic Borel group and μ a quasi-invariant Borel measure on G, then there is a topology on G relative to which G is a locally compact group. This topology generates the given Borel structure and is uniquely determined by this fact. Moreover, μ is equivalent to Haar measure on G [3].

Several generalizations of the Haar measure to the setting of groupoids were taken into considerations in the literature (see [2, 6, 8, 10, 14]). For instance, starting from the case of a locally compact second countable group G acting on a standard Borel space X, Mackey [4] took into consideration the analytic Borel groupoids that can be endowed with a measure class C (a generalization of the class obtained in the group action case by a product of quasi-invariant measures μ on X with measures in the Haar measure class on G) such that each measure in the class C has a decomposition satisfying a kind of quasi-invariance condition. Hahn proved in [6] that the measure class C of a general measure groupoid contains a σ-finite measure ν which is translate-invariant in the groupoid sense. The measure v, or more precisely, the measures in the decomposition of ν with respect to the range map, share many of the properties of the Haar measure for groups, to which it reduces if the groupoid is a group. According to a result of Ramsay, Mackey's groupoids may be assumed to have locally compact topologies. More precisely, a Mackey's groupoid G has an inessential reduction G_0 which has a locally compact metric topology in which it is a topological groupoid [7]. On the other hand, there are situations in which it is not natural to fix a certain measure class C on a groupoid. In this case, in the setting of locally compact groupoids, the analogue of the Haar measure associated with a locally compact group is a system of measures, called a Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". The construction of the C^*-algebra of a locally compact Hausdorff groupoid G endowed with a Haar system (due to Renault [8]) extends the case of a group. The space $C_c(G)$ of continuous functions with compact support is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. The multiplication in $C_c(G)$ is a convolution and the convolution of two functions is defined using the Haar system. The C^*-algebra of G is the completion of $C_c(G)$.

However unlike the case for locally compact groups:

1. Haar systems on groupoids need not exist. Also, when a Haar system does exist, it need not be unique. The continuity assumption has topological consequences for G (it entails that the range map is open [13]). In the same time, according to a result of Seda [11], the continuity assumption is essential for construction of the C^*-algebra of G in the sense of Renault [8].
2. The result of Ramsay [7] is only a partial generalization to groupoids of the Mackey result [3].
3. If G is a topological group such that the singleton sets are closed, then G is Hausdorff. However for general topological groupoids G this is no longer true. If G is a locally compact groupoid, not necessarily Hausdorff, then as pointed out by A. Connes [2], one has to modify the choice of $C_c(G)$ (because $C_c(G)$ it is too small to capture the topological or differential structure of G). In this case $C_c(G)$ is replaced with $\mathcal{C}_c(G)$, the space of complex valued functions on G spanned by functions f which vanishes outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. Since in a non-Hausdorff space a compact set may not be closed, the functions in $\mathcal{C}_c(G)$ are not necessarily continuous on G. According [5], if $(U_i)_i$ is a covering of G by open Hausdorff subsets, then the functions in $\mathcal{C}_c(G)$ are finite sums $\sum_i f_i$, where f_i is a continuous compactly supported function on U_i. In the Hausdorff case $C_c(G)$ and $\mathcal{C}_c(G)$ coincide.

Starting from the fact that in many cases the topological properties of the space of a topological groupoid are less important than the properties of its fibres, the groupoids considered in this paper will be topological, not necessarily locally compact, groupoids G such that the topology induced on fibres is locally compact Hausdorff. Instead of continuous functions on G, we will work with functions having a uniformly continuity property on fibres and instead of the family of compact subsets, we will work with a family of subsets having similar properties as conditionally compact subsets in the sense of [9]. We will endow these groupoids with pre-Haar systems meaning systems of measures subject to the "left invariance" condition (which do not necessarily satisfy the "continuity" property). The aim of the paper is to introduce various convolution algebras in this framework. In the case when G is locally compact and G can be endowed with a Haar system, the usual convolution algebra $C_c(G)$ (respectively, $\mathcal{C}_c(G)$ in the non-Hausdorff case) can be recovered as a particular $*$-subalgebra of the $*$-algebras constructed here.

2. SPACES OF FUNCTIONS ON GROUPOIDS WITH LOCALLY COMPACT FIBRES

A *groupoid* is a set G endowed with a *product map*

$$(x, y) \mapsto xy \left[: G^{(2)} \to G\right],$$

where $G^{(2)}$ is a subset of $G \times G$ called the *set of composable pairs*, and an *inverse map*

$$x \mapsto x^{-1} \ [: G \to G]$$

such that the following conditions hold:

(1) If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z) \in G^{(2)}$, $(x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.
(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.
(3) For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z, x) \in G^{(2)}$, then $(zx)x^{-1} = z$.
(4) For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x, y) \in G^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on G, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range map*, respectively the *source* (*domain*) *map*. It follows easily from the definition that they have a common image called the *unit space* of G, which is denoted $G^{(0)}$. Its elements are *units* in the sense that $xd(x) = r(x)x = x$.

The fibres of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. Also for $u, v \in G^{(0)}$, $G^u_v = G^u \cap G_v$.

If A and B are subsets of G, one may form the following subsets of G:

$$A^{-1} = \left\{x \in G : \ x^{-1} \in A\right\}$$
$$AB = \left\{xy : (x, y) \in G^{(2)} \cap (A \times B)\right\}.$$

A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure. This means that the inverse map $x \mapsto x^{-1} \ [: G \to G]$ is continuous, as well as the product map $(x, y) \mapsto xy \ \left[: G^{(2)} \to G\right]$ is continuous, where $G^{(2)}$ has the induced topology from $G \times G$.

In the following a topological space X is said to be *locally compact* if every point $x \in X$ has a compact Hausdorff neighborhood. If X, Y and Z are sets and $p : X \to Z$ and $q : Y \to Z$ are maps, then the *fibred product* they determine, denoted $X_p \times_q Y$, is defined to be

$$X_p \times_q Y = \{(x, y) \in X \times Y : \ p(x) = q(y)\}.$$

It is well known that any continuous function with compact support defined on a topological group is uniformly continuous. Let us show that in a suitable sense (introduced in [1]) the property of topological groups generalizes to topological groupoids.

Definition 2.1 ([1, Definition 3.1, p. 39])**.** Let G be a topological groupoid and E be a Banach space. The function $h : G \to E$ is said to be *left "uniformly continuous on fibres"* if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space $G^{(0)}$ such that:

$$\|h(x) - h(y)\| < \varepsilon \text{ for all } (x, y) \in G_r \times_r G, \ x^{-1}y \in W.$$

The function $h : G \to E$ is said to be *right "uniformly continuous on fibres"* if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space such that:

$$\|h(x) - h(y)\| < \varepsilon \text{ for all } (x, y) \in G_d \times_d G, \ yx^{-1} \in W.$$

It is easy to see that h is left "uniformly continuous on fibres" if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space such that:

$$\|h(x) - h(xs)\| < \varepsilon \text{ for all } s \in W \text{ and } x \in G_{r(s)}$$

and that h is right "uniformly continuous on fibres" if and only if for each $\varepsilon > 0$ there is a neighborhood W of the unit space such that:

$$\|h(x) - h(sx)\| < \varepsilon \text{ for all } s \in W \text{ and } x \in G^{d(s)}.$$

Lemma 2.2. *Let X, Y, Z and S be four topological spaces and let $p : X \to Z$ and $q : Y \to Z$ be two continuous maps. Let K be a subset of Y with the property that $K \cap q^{-1}(\{z\})$ is a compact set for all $z \in Z$ and $p^{-1}(q(K \setminus V))$ is a closed set for every open neighborhood V of $q^{-1}(\{z\}) \cap K$ and for every $z \in Z$. If W is an open subset of S and $f : X_p \times_q K \to S$ is a continuous function, where $X_p \times_q K$ has the induced topology from $X \times K$, then*

$$L = \left\{x \in X : f(x, y) \in W \textit{ for all } y \in K \cap q^{-1}(\{p(x)\})\right\}$$

is an open subset of X.

Proof. Let $x_0 \in L$ and $y \in K$ with $q(y) = p(x_0)$. Since $f(x_0, y) \in W$ and f is continuous in (x_0, y), it follows that there are two open sets U_y and V_y such that $(x_0, y) \in U_y \times V_y$ and $f((U_y \times V_y) \cap (X_p \times_q K)) \subset W$.

Since $\{V_y\}_{y \in K \cap q^{-1}(\{p(x_0)\})}$ is an open covering of the compact set $K \cap q^{-1}(\{p(x_0)\})$, it follows that there are $y_1, y_2, \dots, y_n \in K$ such that $\bigcup_{i=1,n} V_{y_i} \supset K \cap q^{-1}(\{p(x_0)\})$.

Let us denote $U = \bigcap_{i=1,n} U_{y_i}$ and $V = \bigcup_{i=1,n} V_{y_i}$. Then U is an open neighborhood of x_0 and V is an open neighborhood of $q^{-1}(\{p(x_0)\}) \cap K$. Furthermore $U_0 = U \cap \left(X \setminus p^{-1}(q(K \setminus V))\right)$ is an open neighborhood of x_0 and $f((U_0 \times K) \cap (X_p \times_q (K))) \subset W$. Consequently, $x_0 \in U_0 \subset L$. □

Proposition 2.3. *Let G be a topological groupoid and E be a Banach space. Let W and W_0 be two symmetric open neighborhoods of $G^{(0)}$ and let K be a subset of G with the property that there is a K_0 such that:*

1. *$KW \subset K_0$, respectively $WK \subset K_0$.*
2. *For all $u \in G^{(0)}$, $K_0 \cap G_u$, respectively $K_0 \cap G^u$, are compact sets for all $u \in G^{(0)}$.*
3. *For all $u \in G^{(0)}$, $r(K_0 \setminus V)$, respectively $d(K_0 \setminus V)$, are closed sets for all open neighborhoods V of $G_u \cap K_0$, respectively $G^u \cap K_0$.*

Then any function $h : G \to E$ which vanishes outside K and is continuous on K_0W_0, respectively W_0K_0, is left, respectively right, "uniformly continuous on fibres".

Proof. If we set

$$f : G^{(2)} \to E,\ f(s,y) = h(ys) - h(y),$$

then f is a continuous function in every point of $(W_{0r} \times_d K_0)$. Let $\varepsilon > 0$. By Lemma 2.2,

$$L = \left\{s \in W_0 : |h(ys) - h(y)| < \varepsilon \text{ for all } y \in K_0 \cap G_{r(s)}\right\}$$

is an open subset of W_0, and consequently, of G. Let us note that $G^{(0)} \subset L$. If $s \in L$ and $y \in G_{r(s)}$, then $y \in K \cap G_{r(s)}$ and $|h(ys) - h(y)| < \varepsilon$, or $y \in G_{r(s)} \setminus \left(K \cap G_{r(s)}\right)$ and in this case $y, ys \notin K$, therefore $|h(sy) - h(y)| = 0$. Hence h is left "uniformly continuous on fibres".

Similarly, h is right "uniformly continuous on fibres". □

Definition 2.4. Let G be a topological groupoid. By a *family of "abstractly compact" subsets* of G we mean a family $\mathcal{K}$ of subsets of G satisfying the following conditions:

1. For every $K \in \mathcal{K}$, $K^{-1} \in \mathcal{K}$.
2. For every $K_1, K_2 \in \mathcal{K}$, there is $K_3 \in \mathcal{K}$ such that $K_1K_2 \subset K_3$.
3. For every $K_1, K_2 \in \mathcal{K}$, $K_1 \cup K_2 \in \mathcal{K}$.
4. For every $u \in G^{(0)}$ and every $K \in \mathcal{K}$, $K \cap G^u$ and $K \cap G_u$ are compact.
   Any $K \in \mathcal{K}$ will be called a *"abstractly compact" subset* of G.

Definition 2.5. Let G be a topological groupoid such that:

1. The points are closed in G (or equivalently, G is a T_1-space).
2. For every $u \in G^{(0)}$, G^u and G_u are Hausdorff.

Then G is said to be a *"locally abstractly compact" groupoid* with respect to $\mathcal{K}$ if $\mathcal{K}$ is a family of "abstractly compact" subsets (in the sense of Definition 2.4) such that each point has a neighborhood basis of sets K belonging to $\mathcal{K}$.

Remark 2.6. Let G be a topological groupoid with the property that the points are closed in G. If $G^{(0)}$ is Hausdorff, then for every $u \in G^{(0)}$, G^u and G_u are Hausdorff. As in [12], the set

$$Z = \{(x, y) \in G^u \times G^u : \ d(x) = d(y)\}$$

is closed in $G^u \times G^u$, being the set where two continuous maps (to a Hausdorff space) coincide. Let $\varphi : Z \to G$ defined by $\varphi(x, y) = xy^{-1}$ for all $(x, y) \in Z$. Since $\{u\}$ is closed in G, $\varphi^{-1}(\{u\})$ is closed in Z. Furthermore, Z being closed in G, it follows that $\varphi^{-1}(\{u\})$, which is the diagonal of $G^u \times G^u$, is closed in G.

If G is a "locally abstractly compact" groupoid with respect to $\mathcal{K}$, then for every $u \in G^{(0)}$, G^u and G_u are locally compact Hausdorff subspaces of G. Indeed, each point x in G^u (respectively, in G_u) has a neighborhood $V \in \mathcal{K}$ in G. Then $V \cap G^u$ (respectively, $V \cap G_u$) is a compact neighborhood of x in G^u (respectively, in G_u).

Lemma 2.7. *Let G be a topological groupoid, K a subset of G and F a closed subset of G. If FK and KF are closed in G, then $r\left(K \cap F^{-1}\right)$ and $d\left(K \cap F^{-1}\right)$ are closed subsets of $G^{(0)}$.*

Proof. It suffices to note that

$$r\left(K \cap F^{-1}\right) = KF \cap G^{(0)} \text{ and } d\left(K \cap F^{-1}\right) = FK \cap G^{(0)}$$

and therefore $r\left(K \cap F^{-1}\right)$ and $d\left(K \cap F^{-1}\right)$ are closed subsets of $G^{(0)}$. □

Proposition 2.8. *Let G be a topological groupoid such that:*

1. *The points are closed in G (or equivalently, G is a T_1-space).*
2. *For every $u \in G^{(0)}$, G^u and G_u are Hausdorff.*
3. *Each point has a neighborhood basis of sets K with the properties:*
   (i) *for every closed subset F of G, FK and KF are closed in G,*
   (ii) *for every $u \in G^{(0)}$, $K \cap G^u$ and $K \cap G_u$ are compact.*

If

$$\mathcal{K} = \{K : \textit{for every closed set } F \subset G, \ FK \textit{ and } KF \textit{ are closed and}$$
$$\textit{for every } u \in G^{(0)}, \ K \cap G^u \textit{ and } K \cap G_u \textit{ are compact}\},$$

then G is a "locally abstractly compact" groupoid with respect to $\mathcal{K}$.

Proof. We have to prove that for $K_1, K_2 \in \mathcal{K}$, there is $K_3 \in \mathcal{K}$ such that $K_1K_2 \subset K_3$. Let us note that for every open set V and $K \in \mathcal{K}$,

$$r(K \setminus V) = KF^{-1} \cap G^{(0)} \text{ and } d(K \setminus V) = F^{-1}K \cap G^{(0)}$$

with $F = G \setminus V$ a closed set. Thus $r(K \setminus V)$ and $d(K \setminus V)$ are closed subsets of $G^{(0)}$.

Let $u \in G^{(0)}$ and $x \in G^u$. For every $y \in G^{d(x)}$ there is a neighborhood $K_y \in \mathcal{K}$ of xy and there are two open sets U_y and V_y such that $(x, y) \in U_y \times V_y$ and $xy \in U_yV_y \subset K_y$. Since $\{V_y\}_{y \in K_2 \cap r^{-1}(\{d(x)\})}$ is an open covering of the compact set $K_2 \cap G^{d(x)}$, it follows that there are $y_1, y_2, \ldots, y_n \in K_2$ such that $\bigcup_{i=1,n} V_{y_i} \supset K_2 \cap G^{d(x)}$. Let us denote $U_x = \bigcap_{i=1,n} U_{y_i}$, $V_x = \bigcup_{i=1,n} V_{y_i}$ and $K_x = \bigcup_{i=1,n} K_{y_i}$. Then U_x is an open neighborhood of x, V_x is an open neighborhood of $G^{d(x)} \cap K_2$ and $K_x \subset \mathcal{K}$. Furthermore $L_x = U_x \cap \left(G \setminus d^{-1}\left(r\left(K_2 \setminus V_x\right)\right)\right)$ is an open neighborhood of x and $L_x K_2 \subset K_x$. Since $\{L_x\}_{x \in K_1 \cap r^{-1}(\{u\})}$ is an open covering of the compact set $K_1 \cap G^u$, it follows that there are $x_1, x_2, \ldots, x_m \in K_1$ such that $\bigcup_{i=\overline{1,m}} L_{x_i} \supset K_1 \cap G^u$. Let us notice that $K_u = \bigcup_{i=\overline{1,m}} K_{x_i} \in \mathcal{K}$. Hence

$$G^u \cap K_1 K_2 = \left(G^u \cap K_1\right) K_2 \subset \bigcup_{i=\overline{1,m}} L_{x_i} K_2 \subset \bigcup_{i=\overline{1,m}} K_{x_i}$$

is relatively compact in G^u. Since G^u is closed $G^u K_1$ is closed and furthermore $\left(G^u K_1\right) K_2$ is closed. Thus $G^u \cap K_1 K_2 = \left(G^u \cap K_1\right) K_2$ is closed and therefore compact. Similarly, $G_u \cap K_1 K_2$ is compact.

Let F be a closed subset of G. Since $K_2 F$ is closed, if follows that $K_1 K_2 F = K_1 \left(K_2 F\right)$ is closed. Similarly, $F\left(K_1 K_2\right) = \left(F K_1\right) K_2$ is closed. □

Notation 2.9. Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5). Let us denote by:

1. $\mathcal{UF}(G)$ the space of complex valued functions on G which are left and right "uniformly continuous on fibres".
2. $\mathcal{UF}_{\mathcal{K}}(G)$ the subspace of $\mathcal{UF}(G)$ consisting of functions $f : G \to \mathbb{C}$ which vanish outside an abstractly compact set $K \in \mathcal{K}$.
3. $\mathcal{UF}_b(G) \subset \mathcal{UF}(G)$ the subspace of bounded functions.
4. $\mathcal{UF}_{\mathcal{K}b}(G) \subset \mathcal{UF}_{\mathcal{K}}(G)$ the subspace of bounded functions.

Remark 2.10. 1. If $f \in \mathcal{UF}(G)$, then for all $u \in G^{(0)}$, $f|_{G^u}$ and $f|_{G_u}$ are continuous. Indeed, let u be a unit, $x_0 \in G^u$ and $\varepsilon > 0$. Since f is left "uniformly continuous on fibres", there is a neighborhood W_ε of $G^{(0)}$ such that:

$$|f(x) - f(y)| < \varepsilon \text{ for all } (x, y) \in G_r \times_r G,\ x^{-1} y \in W_\varepsilon.$$

Since $y \mapsto x_0 y$ is a homeomorphism from $G^{d(x_0)}$ to G^u, it follows that $x_0 W_\varepsilon = x_0 \left(W_\varepsilon \cap G^{d(x_0)}\right)$ is an open subset of G^u containing x_0. But if $y \in x_0 W_\varepsilon$, then $x_0^{-1} y \in W_\varepsilon$ and therefore $|f(x_0) - f(y)| < \varepsilon$. Thus $f|_{G^u}$ is continuous in x_0. Similarly, $f|_{G_u}$ is continuous.

2. If $f \in \mathcal{UF}_{\mathcal{K}}(G)$, then for all $u \in G^{(0)}$, $f|_{G^u}$ and $f|_{G_u}$ are compactly supported continuous functions (because $f : G \to \mathbb{C}$ vanishes outside an abstractly compact set $K \in \mathcal{K}$ and $K \cap G^u$, respectively $K \cap G_u$ are compact).
3. If $f \in \mathcal{UF}(G)$, respectively $\mathcal{UF}_b(G), \mathcal{UF}_{\mathcal{K}}(G), \mathcal{UF}_{\mathcal{K}b}(G), \mathcal{UF}_I(G)$, then $|f|$, $\overline{f} \in \mathcal{UF}(G)$, respectively $\mathcal{UF}_b(G)$, $\mathcal{UF}_{\mathcal{K}}(G)$, $\mathcal{UF}_{\mathcal{K}b}(G)$, $\mathcal{UF}_I(G)$. If $f, g \in \mathcal{UF}_b(G)$, respectively $\mathcal{UF}_{\mathcal{K}b}(G)$, then $fg \in \mathcal{UF}_b(G)$, respectively $\mathcal{UF}_{\mathcal{K}b}(G)$.

Examples. A topological groupoid G is said to be locally compact if it is locally compact as a topological space (this means that every point $x \in G$ has a compact Hausdorff neighborhood). Thus any locally compact groupoid G (in the above sense) is locally Hausdorff, and therefore a T_1-space. A *conditionally compact* subset of a topological groupoid G is a subset K such that for every compact subset L of $G^{(0)}$, $K \cap r^{-1}(L)$ and $K \cap d^{-1}(L)$ are compact subsets of G ([9]).

1. Let G be a locally compact groupoid having a Hausdorff unit space. Then G endowed with the family of compact subsets and satisfies the conditions of the Definition 2.5.
2. If G is a locally conditionally compact groupoid in the sense of [9], then G endowed with the family of conditionally compact subsets (in the sense of [9]) is a "locally abstractly compact" groupoid in the sense of Definition 2.5.

3. CONVOLUTION ALGEBRAS FOR GROUPOIDS WITH LOCALLY COMPACT FIBRES

The analogue notion of the Haar measure on locally compact groups that we consider in this section is a system of measures, called pre-Haar system, subject to a suitable left invariance condition.

Definition 3.1. Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5). A *left pre-Haar system* on G is a family of positive measures, $\left\{\nu^u,\ u \in G^{(0)}\right\}$, with the following properties:

(1) is a positive Radon measure on G^u for all $u \in G^{(0)}$,
(2) $\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$ for all $x \in G$ and all $f \in \mathcal{UF}_{\mathcal{K}}(G)$.

Definition 3.2. Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5). The pre-Haar system $\left\{\nu^u,\ u \in G^{(0)}\right\}$ on G is said to be *bounded on abstractly compact sets* if

$$\sup\left\{\nu^u(K),\ u \in G^{(0)}\right\} < \infty$$

for each $K \in \mathcal{K}$.

Notation 3.3. Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5) and $\left\{\nu^u,\ u \in G^{(0)}\right\}$ be a pre-Haar system on G. For each $f \in \mathcal{UF}(G)$, let us denote by

$$\|f\|_{I,r} = \sup\left\{\int |f(y)|\, d\nu^u(y),\ u \in G^{(0)}\right\},$$
$$\|f\|_{I,d} = \sup\left\{\int \left|f\left(y^{-1}\right)\right| d\nu^u(y),\ u \in G^{(0)}\right\},$$
$$\|f\|_I = \max\left\{\|f\|_{I,r},\ \|f\|_{I,d}\right\}$$

and let us consider $\mathcal{UF}_I(G) = \{f \in \mathcal{UF}(G) :\ \|f\|_I < \infty\}$.

For $f \in \mathcal{UF}(G)$, the *involution* is defined by

$$f^*(x) = \overline{f\left(x^{-1}\right)},\ x \in G.$$

Obviously, if $f \in \mathcal{UF}(G)$, respectively $\mathcal{UF}_b(G)$, $\mathcal{UF}_{\mathcal{K}}(G)$, $\mathcal{UF}_{\mathcal{K}b}(G)$, $\mathcal{UF}_I(G)$, then $f^* \in \mathcal{UF}(G)$, respectively $\mathcal{UF}_b(G)$, $\mathcal{UF}_{\mathcal{K}}(G)$, $\mathcal{UF}_{\mathcal{K}b}(G)$, $\mathcal{UF}_I(G)$.

For $f,\, g \in \mathcal{UF}_I\,(G)$ the *convolution* is defined by:

$$\begin{aligned} f * g\,(x) &= \int f\,(xy)\, g\left(y^{-1}\right) d\nu^{d(x)}\,(y) = \\ &= \int f\,(y)\, g\left(y^{-1}x\right) d\nu^{r(x)}\,(y)\,,\ x \in G. \end{aligned}$$

Theorem 3.4. *Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5) and $\left\{\nu^u,\ u \in G^{(0)}\right\}$ be a pre-Haar system on G. Then $\mathcal{UF}_I\,(G)$ and $\mathcal{UF}_{\mathcal{K}}\,(G)$ are $*$-algebras. If $\left\{\nu^u,\ u \in G^{(0)}\right\}$ is bounded on abstractly compact sets, then also $\mathcal{UF}_{\mathcal{K}b}\,(G)$ is closed under involution and convolution, and consequently it is a $*$-algebra.*

Proof. Let us prove if $f,\, g \in \mathcal{UF}_I\,(G)$, then $f * g \in \mathcal{UF}_I\,(G)$. Let $\varepsilon > 0$. Since g is left "uniformly continuous on fibres", there is a neighborhood W_ε of $G^{(0)}$ such that:

$$|g\,(x) - g\,(y)| < \varepsilon \ \text{ for all } \ (x,y) \in G_r \times_r G,\ x^{-1}y \in W_\varepsilon.$$

Hence

$$\begin{aligned} |f * g\,(x) - f * g\,(y)| &= \left| \int f\,(z)\, g\left(z^{-1}x\right) d\nu^{r(x)}\,(z) - \int f\,(z)\, g\left(z^{-1}y\right) d\nu^{r(y)}\,(z) \right| \leq \\ &\leq \int |f\,(z)| \left| g\left(z^{-1}x\right) - g\left(z^{-1}y\right) \right| d\nu^{r(y)}\,(z) \leq \\ &\leq \varepsilon \left\| f \right\|_{I,r}, \end{aligned}$$

and therefore $f * g$ is left "uniformly continuous on fibres". Since f is right "uniformly continuous on fibres", there is a neighborhood W_ε of $G^{(0)}$ such that:

$$|f\,(x) - f\,(y)| < \varepsilon \ \text{ for all } \ (x,y) \in G_d \times_d G,\ xy^{-1} \in W_\varepsilon.$$

Hence

$$\begin{aligned} |f * g\,(x) - f * g\,(y)| &= \left| \int f\,(xz)\, g\left(z^{-1}\right) d\nu^{d(x)}\,(z) - \int f\,(yz)\, g\left(z^{-1}\right) d\nu^{d(y)}\,(z) \right| \leq \\ &\leq \int |f\,(xz) - f\,(yz)| \left| g\left(z^{-1}\right) \right| d\nu^{d(y)}\,(z) \leq \\ &\leq \varepsilon \left\| g \right\|_{I,d}, \end{aligned}$$

and therefore $f * g$ is right "uniformly continuous on fibres". For $u \in G^{(0)}$ we have

$$\begin{aligned} \int |f * g\,(x)|\, d\nu^u\,(x) &\leq \int \int \left| f\,(y)\, g\left(y^{-1}x\right) \right| d\nu^{r(x)}\,(y)\, d\nu^u\,(x) = \\ &= \int \int \left| f\,(y)\, g\left(y^{-1}x\right) \right| d\nu^u\,(x)\, d\nu^u\,(y) = \\ &= \int \int |f\,(y)\, g\,(x)|\, d\nu^{d(y)}\,(x)\, d\nu^u\,(y) = \\ &= \int |f\,(y)| \int |g\,(x)|\, d\nu^{d(y)}\,(x)\, d\nu^u\,(y) \end{aligned}$$

and

$$\begin{aligned}\int \left|f*g\left(x^{-1}\right)\right| d\nu^{u}(x) &\le \int\int \left|f\left(x^{-1}y\right)g\left(y^{-1}\right)\right| d\nu^{r(x)}(y)\,d\nu^{u}(x) = \\ &= \int\int \left|f\left(x^{-1}y\right)g\left(y^{-1}\right)\right| d\nu^{u}(x)\,d\nu^{u}(y) = \\ &= \int\int \left|f\left(x^{-1}\right)g\left(y^{-1}\right)\right| d\nu^{d(y)}(x)\,d\nu^{u}(y) = \\ &= \int \left|g\left(y^{-1}\right)\right| \int \left|f\left(x^{-1}\right)\right| d\nu^{d(y)}(x)\,d\nu^{u}(y).\end{aligned}$$

Thus $\|f*g\|_{I,r} \le \|f\|_{I,r}\|g\|_{I,r}$ and $\|f*g\|_{I,d} \le \|f\|_{I,d}\|g\|_{I,d}$. Therefore under the involution and convolution defined above, $\mathcal{UF}_I(G)$ becomes a $*$-algebra.

It is also easy to see that $\mathcal{UF}_{\mathcal{K}}(G)$ is closed under involution and convolution (if f vanishes outside K_1 and g vanishes outside K_2, then $f*g$ vanishes outside $K_1K_2 \subset K_3 \in \mathcal{K}$). □

Theorem 3.5. *Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5) and $\left\{\nu^u,\ u \in G^{(0)}\right\}$ be a pre-Haar system on G. Let $\mathcal{C}$ be a class of complex valued functions on $G^{(0)}$ closed to addition and scalar multiplication (for instance, the class of functions continuous, or the class of Borel functions, etc.). Let*

$$\begin{aligned}\mathcal{A}_{\mathcal{K}}^{\mathcal{C}} = \Big\{ f \in \mathcal{UF}_{\mathcal{K}}(G),\ u \mapsto \int f(y)\varphi(d(y))\,d\nu^u(y) \in \mathcal{C} \text{ for all } \varphi \in \mathcal{C} \text{ and} \\ u \mapsto \int \overline{f\left(y^{-1}\right)}\varphi(d(y))\,d\nu^u(y) \in \mathcal{C} \text{ for all } \varphi \in \mathcal{C}\Big\},\\ \mathcal{A}_{I}^{\mathcal{C}} = \Big\{ f \in \mathcal{UF}_{I}(G),\ u \mapsto \int f(y)\varphi(d(y))\,d\nu^u(y) \in \mathcal{C} \text{ for all } \varphi \in \mathcal{C} \text{ and} \\ u \mapsto \int \overline{f\left(y^{-1}\right)}\varphi(d(y))\,d\nu^u(y) \in \mathcal{C} \text{ for all } \varphi \in \mathcal{C}\Big\}.\end{aligned}$$

Then $\mathcal{A}_I^{\mathcal{C}}$ and $\mathcal{A}_{\mathcal{K}}^{\mathcal{C}}$ are $*$-algebras.

Proof. It suffices to prove that $\mathcal{A}_{\mathcal{K}}^{\mathcal{C}}$ and $\mathcal{A}_I^{\mathcal{C}}$ are closed to convolution. But this is obvious, since for $f, g \in \mathcal{A}_K$ (respectively, $\mathcal{A}_I^{\mathcal{C}}$) we have

$$\begin{aligned}\int f*g(x)\varphi(d(x))\,d\nu^u(x) &= \int\int f(y)\,g\left(y^{-1}x\right)\varphi(d(x))\,d\nu^{r(x)}(y)\,d\nu^u(x) = \\ &= \int\int f(y)\,g\left(y^{-1}x\right)\varphi(d(x))\,d\nu^{u}(x)\,d\nu^u(y) = \\ &= \int\int f(y)\,g(x)\,\varphi(d(x))\,d\nu^{d(y)}(x)\,d\nu^u(y) = \\ &= \int f(y)\int g(x)\,\varphi(d(x))\,d\nu^{d(y)}(x)\,d\nu^u(y) = \\ &= \int f(y)\,\varphi_g(d(y))\,d\nu^u(y),\end{aligned}$$

where $\varphi_g(u) = \int g(x)\varphi(d(x))\,d\nu^u(x)$, and

$$\int \overline{f * g(x^{-1})}\varphi(d(x))\,d\nu^u(x) = \int (f*g)^*(x)\varphi(d(x))\,d\nu^u(x) =$$
$$= \int g^* * f^*(x)\varphi(d(x))\,d\nu^u(x) =$$
$$= \int g^*(y)\varphi_{f^*}(d(y))\,d\nu^u(y). \qquad \square$$

Theorem 3.6. *Let G be a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Definition 2.5) and $\{\nu^u,\ u \in G^{(0)}\}$ be a pre-Haar system on G. Let $\mathcal{C}_{\mathcal{K}}(G)$ be the space of complex valued functions on G spanned by functions f with the property that there are two symmetric open neighborhoods W and W_0 of $G^{(0)}$ and two sets K, $K_0 \in \mathcal{K}$ such that:*

1. $KW \cup WK \subset K_0$.
2. *For all $u \in G^{(0)}$, $r(K_0 \setminus V)$ and $d(K_0 \setminus V)$ are closed sets for all open sets V.*
3. *f vanishes outside K and is continuous on $K_0W_0 \cup W_0K_0$.*

If we assume that for every $f, g \in \mathcal{C}_{\mathcal{K}}(G)$, $f*g \in \mathcal{C}_{\mathcal{K}}(G)$, then $\mathcal{C}_{\mathcal{K}}(G)$ is a $*$-subalgebra of $\mathcal{UF}_{\mathcal{K}}(G)$.

Proof. According to Proposition 2.3 each $f \in \mathcal{C}_{\mathcal{K}}(G)$ is left and right "uniformly continuous on fibres", therefore $f \in \mathcal{UF}_{\mathcal{K}}(G)$. $\square$

Remark 3.7. Let G be a topological groupoid such that:

1. The points are closed in G (or equivalently, G is a T_1-space).
2. For every $u \in G^{(0)}$, G^u and G_u are Hausdorff.
3. Each point has a neighborhood basis of sets K with the property that for every closed subset F of G, FK and KF are closed in G and for every $u \in G^{(0)}$, $K \cap G^u$ and $K \cap G_u$ are compact.

Let

$$\mathcal{K} = \{K : \text{for every closed set } F \subset G,\ FK \text{ and } KF \text{ are closed and}$$
$$\text{for every } u \in G^{(0)},\ K \cap G^u \text{ and } K \cap G_u \text{ are compact } \}.$$

Then G is a "locally abstractly compact" groupoid with respect to $\mathcal{K}$ (Proposition 2.8).

In this case the space $\mathcal{C}_{\mathcal{K}}(G)$ from the above theorem is the space of complex valued functions on G spanned by functions f with the property that there are two symmetric open neighborhoods W and W_0 of $G^{(0)}$ and two sets K, $K_0 \in \mathcal{K}$ such that:

1. $KW \cup WK \subset K_0$.
2. f vanishes outside K and is continuous on $K_0W_0 \cup W_0K_0$.

In particular let us consider a locally compact topological groupoid G whose unit space $G^{(0)}$ is paracompact. Let $\mathcal{K}$ be the family of compact subsets of G. Let $\mathcal{C}_c(G)$

be the space of complex valued functions on G spanned by functions f which vanishe outside a compact set K contained in an open Hausdorff subset U of G and being continuous on U. Let $\left\{\nu^u,\ u \in G^{(0)}\right\}$ be a left Haar system on G. This means that $\left\{\nu^u,\ u \in G^{(0)}\right\}$ is a left pre-Haar system on G and the map

$$u \mapsto \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbb{C}\right]$$

is continuous is for all $f \in \mathcal{C}_c(G)$. Then $\mathcal{C}_c(G)$ is a $*$-subalgebra of $\mathcal{UF}_{\mathcal{K}}(G)$.

Examples.

Let us consider the following special cases in Theorem 3.5:

(i) $\mathcal{C}(\tau) = \left\{\varphi : G^0 \to \mathbb{C},\ \varphi \text{ continuous with respect to } \tau\right\}$, where τ is a topology possible weaker than the topology induced on $G^{(0)}$ from G.

(ii) $\mathcal{C}(\mathcal{B}) = \left\{\varphi : G^0 \to \mathbb{C},\ \varphi \text{ measurable with respect to } \sigma\text{-algebra } \mathcal{B}\right\}$- we can work with a Borel structure weaker than the Borel structure generated by the open sets.

Let us consider now two classes of groupoids: the groups and the sets (as co-trivial groupoids):

1. **Groups:** If G is a locally compact Hausdorff group, then G (as a groupoid) admits an essentially unique (left) pre-Haar system $\{\lambda\}$ where λ is a Haar measure on G. Let $C_c(G)$ be space of complex valued continuous functions with compact support on G and let $\mathcal{U}(G)$ be the space of complex valued left and right uniformly continuous functions on G. The convolution of $f,\ g \in C_c(G)$ (G is seen as a groupoid) is given by:

$$f * g(x) = \int f(xy)\, g\left(y^{-1}\right) d\lambda(y) \text{ (usual convolution on } G)$$

and the involution by

$$f^*(x) = \overline{f\left(x^{-1}\right)}.$$

The involution defined above is slightly different from the usual involution on groups. In fact if Δ is the modular function of the Haar measure λ, then $f \mapsto \Delta^{1/2} f$ is $*$-isomorphism from $C_c(G)$ for G seen as a groupoid to $C_c(G)$ for G seen as a group.

It is easy to see that $\mathcal{K}$ is a family of "abstractly compact" subsets (in the sense of Definition 2.4) such that each point of G has a neighborhood basis of sets K belonging to $\mathcal{K}$ if and only if $\mathcal{K}$ is the family of compact subsets of G. Therefore,

$$\mathcal{A}_{\mathcal{K}}^{\mathcal{C}(\tau)} = \mathcal{A}_{\mathcal{K}}^{\mathcal{C}(\mathcal{B})} = \mathcal{UF}_{\mathcal{K}}(G) = C_c(G),$$
$$\mathcal{A}_{I}^{\mathcal{C}(\tau)} = \mathcal{A}_{I}^{\mathcal{C}(\mathcal{B})} = \mathcal{U}_I(G),$$

where $\mathcal{U}_I(G)$ is the space of complex valued left and right uniformly continuous functions on G which are integrable with respect to $\Delta^{-1/2}\nu$.

2. **Sets:** If X is a topological space which is a T_1-space, then X seen as a co-trivial groupoid is a *"locally abstractly compact" groupoid* with respect to $\mathcal{K}$ if
a) $\mathcal{K}$ is a family of subsets of X closed at union.
b) each point of X has a neighborhood basis of sets K belonging to $\mathcal{K}$.
The system of measures $\{\alpha(x)\varepsilon_x,\ x \in X\}$ is a pre-Haar system on X $(\alpha(x)\varepsilon_x(f) = \alpha(x) f(x))$, where α is a positive function on X. The convolution of $f,\ g \in C_c(G)$ is given by:

$$\begin{aligned} f * g(x) &= \int f(xy)\, g\left(y^{-1}\right) \alpha(x)\, d\varepsilon_x(y) = \\ &= f(xx)\, g\left(x^{-1}\right) \alpha(x) = \\ &= f(x)\, g(x)\, \alpha(x) \end{aligned}$$

and the involution by

$$f^*(x) = \overline{f(x)}.$$

In this framework:
a) $\mathcal{UF}(X)$ is the space of all complex valued functions on X.
b) $\mathcal{UF}_b(X)$ is the space of all complex valued bounded functions on X.
c) $\mathcal{UF}_{\mathcal{K}}(X)$ is the space of functions which vanish outside a set $K \in \mathcal{K}$.
d) $\mathcal{UF}_{\mathcal{K}b}(X)$ is the space of functions which vanish outside a set $K \in \mathcal{K}$ and are bounded on X.
e) $\mathcal{UF}_I(X)$ is the space of functions f such that $x \mapsto f(x)\alpha(x)$ is bounded on X.
f) $\mathcal{A}_{\mathcal{K}}^{\mathcal{C}(\tau)}$ is the space of functions f which vanish outside a set $K \in \mathcal{K}$ and such that $x \mapsto f(x)\alpha(x)$ is τ-continuous.
g) $\mathcal{A}_{\mathcal{K}}^{\mathcal{C}(\mathcal{B})}$ is the space of functions f which vanish outside a set $K \in \mathcal{K}$ and such that $x \mapsto f(x)\alpha(x)$ is measurable with respect to the σ-algebra $\mathcal{B}$.
h) $\mathcal{A}_I^{\mathcal{C}(\tau)}$ is the space of functions f such that $x \mapsto f(x)\alpha(x)$ is bounded and is τ-continuous on X.
i) $\mathcal{A}_I^{\mathcal{C}(\mathcal{B})}$ is the space of functions f such that $x \mapsto f(x)\alpha(x)$ is bounded and measurable with respect to the σ-algebra $\mathcal{B}$ on X.

When X is a locally compact Hausdorff space, τ is the topology of X, $\alpha \equiv 1$ and $\mathcal{K}$ is the the family of compact subsets of X, then $\mathcal{A}_{\mathcal{K}}^{\mathcal{C}(\tau)} = C_c(X)$, the space of continuous functions with compact support on X and $\mathcal{A}_I^{\mathcal{C}(\tau)} = C_b(X)$, the space of continuous bounded functions on X .

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