

MULTI-VALUED CODENSING RANDOM OPERATORS AND FUNCTIONAL RANDOM INTEGRAL INCLUSIONS

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Abstract. In this paper, some random fixed point theorems for continuous and condensing multi-valued random operators are proved and they are further applied to the random integral inclusions for proving the existence of the solutions via the priori bound method.

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1. INTRODUCTION

Let (Ω, \mathcal{A}) be a measurable space and let X be a separable Banach space with norm $\|\cdot\|$. Let β_X denote the Borel algebra of open subsets of X . A function $x : \Omega \rightarrow X$ is called measurable if

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A} \quad (1.1)$$

for all $B \in \beta_X$. The set of all measurable functions from the set Ω into X is denoted by $\mathcal{M}(\Omega, X)$. Let $\mathcal{P}(X)$ denote the class of all subsets of X , called the power set of X . Denote

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has the property } p\}. \quad (1.2)$$

Here, p may be p =closed (in short cl) or p =convex (in short cv) or p =bounded (in short bd) or p =compact (in short cp). Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cl}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of X . Similarly, $\mathcal{P}_{cl,bd}(X)$ and $\mathcal{P}_{cv,cp}(X)$ denote, respectively, the classes of closed-bounded and compact-convex subsets of X .

A correspondence $Q : X \rightarrow \mathcal{P}_p(X)$ is called a multi-valued mapping or multi-valued operator on X into X . A point $u \in X$ is called a fixed point of Q if $u \in Qu$ and the set of all fixed points of Q in X is denoted by \mathcal{F}_Q .

Let $Q : X \rightarrow \mathcal{P}_p(X)$ be a multi-valued map. Q is called bounded if $\bigcup Q(S)$ is a bounded subset of X for all bounded subsets S of X . Q is called compact if $\overline{Q(X)}$ is a compact subset of X . Again, Q is called totally bounded if $Q(S)$ is a totally bounded subset of X for all bounded sets S in X . It is clear that every compact map is totally bounded, but the converse may not be true. Q is called an upper semi-continuous at $x \in X$ if for each open set V in X containing $f(x)$, there exists a neighborhood $N(x)$ in X such that $\bigcup Q(N(x)) \subset V$. Q is called upper semi-continuous on X if it is upper semi-continuous at each point of X . An upper semi-continuous multi-valued map Q on X is also called a closed multi-valued map on X . Finally, Q is called completely continuous on X if it is upper semi-continuous and totally bounded on X . It is known that if Q is a closed multi-valued map with compact values on X , then if we have sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \rightarrow x_*$, $y_n \rightarrow y_*$ and $y_n \in Qx_n$, $n \in \mathbb{N}$, then $y_* \in Qx_*$. The converse of this statement holds if Q is a compact multi-valued map on X . The details of all these definitions appear in Hu and Papageorgiou [7].

A multi-valued mapping $Q : \Omega \rightarrow \mathcal{P}_p(X)$ is called measurable (respectively weakly measurable) if

$$Q^-(B) = \{\omega \in \Omega \mid Q(\omega) \cap B \neq \emptyset\} \in \mathcal{A} \quad (1.3)$$

for all closed (respectively open) subsets B in X . A multi-valued mapping $Q : \Omega \rightarrow \mathcal{P}_p(X)$ is called a multi-valued random operator if $Q(\cdot, x)$ is measurable for each $x \in X$, and we write $Q(\omega, x) = Q(\omega)x$. A measurable function $\xi : \Omega \rightarrow X$ is called a random fixed point of the multi-valued random operator $Q(\omega)$ if $\xi(\omega) \in Q(\omega)\xi(\omega)$ for all $\omega \in \Omega$. The set of all random fixed points of the multi-valued random operator $Q(\omega)$ is denoted by $\mathcal{F}_{Q(\omega)}$. A multi-valued random operator $Q : \Omega \times X \rightarrow \mathcal{P}_p(X)$ is called bounded, totally bounded, compact, closed and completely continuous if the multi-valued map $Q(\omega, \cdot)$ is bounded, totally bounded, compact, closed and completely continuous for each $\omega \in \Omega$.

The Kuratowski measure $\alpha(S)$ and the Hausdorff measure $\beta(S)$ of noncompactness of a bounded set S in the Banach space X are the nonnegative real numbers defined by

$$\alpha(S) = \inf \left\{ r > 0 : S \subset \bigcup_{i=1}^n S_i, \text{ and } \text{diam}(S_i) \leq r, \forall i \right\} \quad (1.4)$$

and

$$\beta(S) = \inf \left\{ r > 0 : S \subset \bigcup_{i=1}^n \mathcal{B}_i(x_i, r), \text{ for some } x_i \in X \right\}, \quad (1.5)$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The details of Hausdorff measure of noncompactness and its properties appear in Deimling [2], Hu and Papageorgiou [7] and the references therein.

Definition 1.1. A multi-valued mapping $Q : X \rightarrow \mathcal{P}_{bd}(X)$ is called **condensing** (resp. countably condensing) if for any bounded (countable and bounded) $S \in \mathcal{P}_p(X)$, we have that $\beta(Q(S)) < \beta(S)$ whenever $\beta(S) > 0$.

It is known that compact and contraction multi-valued maps are β -condensing, but the converse may not be true. The following results are well-known in the literature.

Lemma 1.2 (Akhmerov *et al.* [1]). *Let α and β be respectively the Kuratowski and Hausdorff measure of noncompactness in a Banach space X , then for any bounded set S in X , we have $\alpha(S) \leq 2\beta(S)$.*

Lemma 1.3 (Akhmerov *et al.* [1]). *If $A : X \rightarrow X$ is a single-valued Lipschitz map with the Lipschitz constant k , then we have $\alpha(A(S)) \leq k\alpha(S)$ for any bounded subset S of X .*

The study of random fixed point theorems for the continuous and condensing multi-valued random self-mappings on the closed subsets of a Banach space have been obtained in Itoh [9] using the separability of the Banach spaces. In the present work, we establish a random version of the fixed point theorem due to Martelli [11] for continuous and condensing nonself multi-valued random operators on separable Banach spaces under suitable conditions. We also apply our random fixed point theorems to random integral inclusions for proving the existence results under certain compactness type conditions. We claim that the results of this paper are new to the literature on random fixed point theory and random integral inclusions.

2. MULTI-VALUED RANDOM FIXED POINT THEORY

We need the following definitions in the sequel.

Definition 2.1. A subset A of X is called countable if there exists a one-to-one correspondence $f : \mathbb{N} \rightarrow A$, where \mathbb{N} is a set of natural numbers. The element $a = f(1) \in A$ is called the first element of A . A multi-valued mapping $Q : X \rightarrow \mathcal{P}_{cl}(X)$ is said to satisfy **Condition \mathcal{D}** if for any countable subset A of X ,

$$A \subseteq \overline{\text{conv}}(\{a\} \cup Q(A)) \implies \bar{A} \text{ is compact,} \quad (2.1)$$

where a is a first element of A .

Definition 2.2. A multi-valued mapping $Q : X \rightarrow \mathcal{P}_{cl}(X)$ is called **Chandrabhan** if

$$A \subseteq \overline{\text{conv}}(\mathcal{C} \cup Q(A)) \implies \bar{A} \text{ is compact} \quad (2.2)$$

for any countable subset A of X , where \mathcal{C} is a relatively compact subset of X called the **support set** of Q in X .

Notice that *condensing mappings* \implies *countably condensing* \implies *Condition \mathcal{D}* \implies *Chandrabhan map*, but the converse may not be true.

A nonlinear alternative for condensing multi-valued mappings useful in applications to differential and integral inclusions is the following variant of a fixed point theorem of Dhage [3].

Theorem 2.3. *Let X be a Banach space, $K \subset X$ a closed convex, and $U \subset K$ an open and bounded set in K . Let $Q : \bar{U} \rightarrow \mathcal{P}_{cp,cv}(K)$ be an upper semi-continuous and Chandrabhan map with support set \mathcal{C} in \bar{U} . In addition assume that*

$$x \notin (1 - \lambda)\overline{\text{conv}}(\mathcal{C}) + \lambda Q(x) \quad \text{for all } x \in \partial U \text{ and } \lambda \in [0, 1]. \quad (2.3)$$

Then Q has a fixed point.

As a consequence of Theorem 2.3, we obtain

Corollary 2.4. *Let U and \bar{U} be the open and closed subsets of a Banach space X , respectively, such that $0 \in U$ and let $Q : \bar{U} \rightarrow P_{cp,cv}(X)$ be an upper semi-continuous and Chandrabhan mapping with support set $C = \{0\}$. Then either:*

- (i) Q has a fixed point, or
- (ii) there exists an element $u \in \partial U$ such that $\lambda u \in Qu$ for some $\lambda > 1$, where ∂U is the boundary of U in X .

The following general random fixed point theorem appears in Tan and Yuan [17]. We need the following definition in the sequel.

Definition 2.5. An operator $Q : S \subset X \rightarrow \mathcal{P}_{cl}(X)$ is called **hemi-compact** if every sequence $\{x_n\}$ in S has a convergent subsequence, whenever $d(x_n, Qx_n) \rightarrow 0$ as $n \rightarrow \infty$.

It is known that condensing mappings with the bounded ranges in the Banach space X are hemi-compact, but the converse may not be true (see Tan and Yuan [17]).

Theorem 2.6. *Let (Ω, \mathcal{A}) be a measurable space and X a non-empty separable complete subset of a metric space (X, d) . Suppose that the map $Q : \Omega \times X \rightarrow \mathcal{P}_{cp}(X)$ is a continuous and hemi-compact random operator. Then F has a deterministic fixed point if and only if F has a random fixed point.*

To prove the main multi-valued random fixed point theorem of this section, we need the following lemma in the sequel.

Lemma 2.7. *Let X be a Banach space and let $Q : X \rightarrow \mathcal{P}_{cp}(X)$ be a Chandrabhan mapping. Then for any sequence $\{x_n\}$ in X satisfying $d(x_n, Qx_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence. In particular, Q is hemi-compact on X .*

Proof. Let $\{x_n\}$ be a sequence in X such that $d(x_n, Qx_n) \rightarrow 0$ as $n \rightarrow \infty$. Let $M = \{x_n\}$ so that $Q(M) := \bigcup_{n=1}^{\infty} Q(x_n)$. For a given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$ there is a $y_n \in Q(x_n)$ such that $d(x_n, y_n) < \epsilon$. Now let $M' := \bigcup_{n=1}^{\infty} \{y_n \in Q(x_n) : d(x_n, y_n) < \epsilon\}$, and so, $\mathcal{B}(M', \epsilon)$ contains all but a finite set \mathcal{C} of the elements of M , where $\mathcal{B}(M', \epsilon) = \bigcup_{a \in M'} \mathcal{B}(a, \epsilon)$ and $\mathcal{B}(a, \epsilon)$ is an open ball in X centered at a of radius ϵ . Therefore, it follows that $M \subset \mathcal{C} \cup \mathcal{B}(M', \epsilon) \subset \mathcal{C} \cup \mathcal{B}(Q(M), \epsilon)$. Since ϵ is arbitrary, one has $M = \mathcal{C} \cup M' \subseteq \mathcal{C} \cup Q(M)$. Since Q is Chandrabhan mapping \bar{M} is compact and that $\{x_n\}$ has a convergent subsequence. Hence Q is hemi-compact on X and the proof of the lemma is complete. \square

Theorem 2.8. *Let (Ω, \mathcal{A}) be a measurable space and let U and \bar{U} be respectively the open and closed subsets of a separable Banach space X such that $0 \in U$. Let $Q : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp,cv}(X)$ be a continuous and Chandrabhan multi-valued random operator with support set $C = \{0\}$ satisfying for each $\omega \in \Omega$:*

- (a) *there does not exist a function $u : \Omega \rightarrow \partial U$ such that $\lambda(\omega)u \in Q(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ with $\lambda(\omega) > 1$ on Ω , where ∂U is the boundary of U in X .*

Then $Q(\omega)$ has a random fixed point in \overline{U} in X .

Proof. Let $\omega \in \Omega$ be fixed. Then by Corollary 2.4, the operator inclusion $u \in Q(\omega)u$ has a solution in \overline{U} . Again, the multi-valued $x \mapsto Q(\omega)x$ is Chandrabhan, so it is hemi-compact on \overline{U} . Hence, by Theorem 2.6, $Q(\omega)$ has a random fixed point in \overline{U} . This completes the proof. \square

As a consequence of Theorem 2.8 we obtain very interesting results which are fruitful in the formulation of random hybrid fixed point theory for multi-valued random operators on separable Banach spaces and Banach algebras.

Theorem 2.9. *Let (Ω, \mathcal{A}) be a measurable space and let U and \overline{U} be respectively the open and closed subsets of a separable Banach space X such that $0 \in U$. Let $Q : \Omega \times \overline{U} \rightarrow \mathcal{P}_{cp,cv}(X)$ be a continuous and countably condensing multi-valued random operator satisfying for each $\omega \in \Omega$:*

- (a) $Q(\omega, \overline{U})$ is bounded, and
- (b) there does not exist a function $u : \Omega \rightarrow \partial U$ such that $\lambda(\omega)u \in Q(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ with $\lambda(\omega) > 1$ on Ω , where ∂U is the boundary of U in X .

Then $Q(\omega)$ has a random fixed point in \overline{U} .

Remark 2.10. Note that hypothesis (a) above holds if U is a bounded subset of X .

Corollary 2.11. *Let (Ω, \mathcal{A}) be a measurable space and let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a separable Banach space X centered at the origin of radius r . Let $Q : \Omega \times \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_{cp,cv}(X)$ be a continuous and condensing multi-valued random operator. If there does not exist a function $u : \Omega \rightarrow \partial U$ with $\|u(\omega)\| = r$ such that $\lambda(\omega)u \in Q(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω , then $Q(\omega)$ has a random fixed point in $\overline{\mathcal{B}_r(0)}$.*

A special case of Theorem 2.9 is useful in applications to differential and integral inclusions

Theorem 2.12. *Let (Ω, \mathcal{A}) be a measurable space, X a separable Banach space and let $Q : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ be a continuous and condensing multi-valued random operator. Furthermore, if the set $\mathcal{E} = \{u : \Omega \rightarrow X \mid \lambda(\omega)u \in Q(\omega)u\}$ is bounded for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ with $\lambda(\omega) > 1$ on Ω , then $Q(\omega)$ has a random fixed point.*

Proof. Since the set \mathcal{E} is bounded, there is a constant $r > 0$ such that $\|u(\omega)\| \leq r$ for all $u \in \mathcal{E}$ and $\omega \in \Omega$. Now consider the open ball $\mathcal{B}_r(0)$ in X centered at the origin of radius r . Then the multi-valued random operator $Q : \Omega \times \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_{cp}(X)$ satisfies all the conditions of Corollary 2.4. Hence $Q(\omega)$ has a random fixed point in $\overline{\mathcal{B}_r(0)}$. This completes the proof. \square

3. HYBRID RANDOM FIXED POINT THEORY

Let $A, B \in \mathcal{P}_p(X)$ and denote

$$A \pm B = \{a \pm b \mid a \in A \text{ and } b \in B\},$$

$$\lambda A = \{\lambda a \mid a \in A \text{ and } \lambda \in \mathbb{R}\}$$

and

$$\|A\| = \{\|a\| \mid a \in A\},$$

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| \mid a \in A\}.$$

For $x \in X$ and $Y, Z \in \mathcal{P}_p(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$. Define a function $d_H : \mathcal{P}_{cl}(X) \times \mathcal{P}_p(X) \rightarrow \mathbb{R}^+$ by

$$d_H(Y, Z) = \max\{\rho(Y, Z), \rho(Z, Y)\}. \quad (3.1)$$

The function d_H is called a Hausdorff metric on X . Note that $\|Y\|_{\mathcal{P}} = d_H(Y, \{0\})$.

A multi-valued operator $Q : \Omega \rightarrow \mathcal{P}_{cl}(X)$ is called a step multi-valued operator or simple multi-valued operator if $Q(\omega) = \sum \chi_{A_i} C_i$, where A_i 's are disjoint sets in \mathcal{A} such that $\bigcup A_i = \Omega$ and C_i are the closed sets in X . A random multi-valued operator $Q : \Omega \rightarrow \mathcal{P}_{cl}(X)$ is called strongly measurable if there exists a sequence $\{Q_n(\omega)\}$ of step multi-valued operators such that $d_H(Q_n(\omega), Q(\omega)) \rightarrow 0$ as $n \rightarrow \infty$. It is known that if X is a separable Banach space and the measure space $(\Omega, \mathcal{A}, \mu)$ is complete, then strong measurability of the multi-valued mapping F implies the measurability of F on Ω , i.e. $Q^-(B) \in \mathcal{A}$ for every $B \in \beta_X$ (see Deimling [2, page 11]). To prove the next multi-valued random fixed point theorems, we need the following lemmas in the sequel.

Lemma 3.1 (Petruşel [14]). *Let $(\mathcal{P}_{cl}(X), d_H)$ be the generalized metric space. Then*

$$d_H(A + B, C + D) \leq d_H(A, C) + d_H(B, D)$$

for all $A, B, C, D \in \mathcal{P}_{cl}(X)$.

Lemma 3.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let X be a separable Banach space. If $S, T : \Omega \rightarrow \mathcal{P}_{cl}(X)$ are two random multi-valued operators, then the sum $S + T$ defined by $(S + T)(\omega) = S(\omega) + T(\omega)$ is again a random multi-valued operator on Ω .*

Proof. Since X is a separable Banach space and the measure space $(\Omega, \mathcal{A}, \mu)$ is complete, we assume that S and T are strongly measurable multi-valued mappings on Ω . Then there are sequences $\{S_n(\omega)\}$ and $\{T_n(\omega)\}$ of step multi-valued operators such that

$$d_H(S_n(\omega), S(\omega)) \rightarrow 0 \quad \text{and} \quad d_H(T_n(\omega), T(\omega)) \rightarrow 0$$

as $n \rightarrow \infty$. Note that by virtue of the step multi-valued operators, $S_n(\omega) + T_n(\omega)$ is well defined for each $n \in \mathbb{N}$. Now by Lemma 2.7,

$$d_H(S_n(\omega) + T_n(\omega), S(\omega) + T(\omega)) \leq d_H(S_n(\omega), S(\omega)) + d_H(T_n(\omega), T(\omega)) \rightarrow 0$$

as $n \rightarrow \infty$. This shows that $S(\omega) + T(\omega)$ is strongly measurable and hence a multi-valued random operator on Ω . \square

Definition 3.3. A multi-valued operator $Q : \Omega \times X \rightarrow \mathcal{P}_{cl}(X)$ is called **\mathcal{D} -Lipschitz** if there exists a continuous and nondecreasing function $\psi : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$d_H(Q(\omega)x, Q(\omega)y) \leq \psi_\omega(\|x - y\|) \quad (3.2)$$

for all $x, y \in X$, where $\psi_\omega(r) = \psi(\omega, r)$ and $\psi_\omega(0) = 0$ for all $\omega \in \Omega$. The function ψ is called a \mathcal{D} -function of Q on X . If $\psi_\omega(r) = k(\omega)r$ for some $k(\omega) > 0$ on Ω , then Q is called Lipschitz on X with the Lipschitz constant k . Further if $k(\omega) < 1$ on Ω , then Q is called a multi-valued contraction on X with contraction constant $k(\omega)$. Finally, if $\psi_\omega(r) < r$ for $r > 0$ and for all $\omega \in \Omega$, then Q is called a nonlinear \mathcal{D} -contraction on X .

Remark 3.4 (Deimling [2]). It is known that if $T : X \rightarrow \mathcal{P}_{cp}(X)$ is multi-valued **\mathcal{D} -Lipschitz** with \mathcal{D} -function ψ , then $\beta(T(S)) \leq \psi(\beta(S))$ for all $S \in \mathcal{P}_{cl, bd}(X)$. Similarly, if T is single-valued **\mathcal{D} -Lipschitz** with \mathcal{D} -function ϕ , then $\alpha(T(S)) \leq \psi(\beta(S))$ for $S \in \mathcal{P}_{bd}(X)$.

Theorem 3.5. Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let U and \bar{U} be the open-bounded and closed-bounded subsets of a separable Banach space X such that $0 \in U$. Let $A, B : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp, cv}(X)$ be two multi-valued random operators satisfying for each $\omega \in \Omega$:

- (a) $A(\omega)$ is a nonlinear \mathcal{D} -contraction,
- (b) $B(\omega)$ is compact and continuous, and
- (c) there does not exist a function $u : \Omega \rightarrow \partial U$ such that $\lambda(\omega)u \in A(\omega)u + B(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω , where ∂U is the boundary of U in X .

Then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution in \bar{U} .

Proof. Define a multi-valued mapping $Q : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cv, cp}(X)$ by

$$Q(\omega)x = A(\omega)x + B(\omega)x. \quad (3.3)$$

Since $(\Omega, \mathcal{A}, \mu)$ is a complete σ -finite measure space and X is a separable Banach space, by Lemma 3.1, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp}(X)$. As $A(\omega)$ is a multi-valued random contraction, it is continuous on \bar{U} . As a result, $Q(\omega)$ is a continuous multi-valued random operator on \bar{U} . Again U is bounded, so from the continuity of $Q(\omega)$ it follows that $Q(\omega)(\bar{U})$ is a bounded subset of X . We just show that $Q(\omega)$ is a condensing multi-valued operator on \bar{U} . Let $S \subset \bar{U}$. Then for all $\omega \in \Omega$, one has

$$\beta(Q(\omega)(S)) \leq \beta(A(\omega)(S)) + \beta(B(\omega)(S)) \leq \psi_\omega(\beta(S)) < \beta(S)$$

if $\beta(S) > 0$. This shows that $Q(\omega)$ is a condensing multi-valued random operator on \bar{U} . Now the desired conclusion follows by an application of Theorem 2.9. This completes the proof. \square

Corollary 3.6. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a separable Banach space X entered at the origin of radius r . Let $A, B : \Omega \times \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_{cp,cv}(X)$ be two multi-valued random operators satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ is a multi-valued contraction,
- (b) $B(\omega)$ is compact and continuous, and
- (c) there does not exist a function $u : \Omega \rightarrow X$ such that $\|u(\omega)\| = r$ and $\lambda(\omega)u \in A(\omega)u + B(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution in $\overline{\mathcal{B}_r(0)}$.

Theorem 3.7. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let $A, B : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ be two multi-valued random operators satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ is a multi-valued contraction,
- (b) $B(\omega)$ is compact and continuous, and
- (c) the set $\mathcal{E} = \{u : \Omega \rightarrow X \mid \lambda(\omega)u \in A(\omega)u + B(\omega)u\}$ is bounded for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution.

Remark 3.8. The conclusion of Theorem 3.5, Corollary 3.6 and Theorem 3.7 also remains true if we replace the multi-valued random operator $A(\omega)$ with a single valued random operator and the hypothesis (a) with “ $A(\omega)$ is single-valued contraction.”

Next we prove some hybrid fixed point theorems for multi-valued mappings in Banach algebras. Let X be a Banach algebras and let $A, B \in \mathcal{P}_p(X)$. Then we define

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

To prove the next multi-valued random fixed point theorems, we need the following lemmas due to Dhage [3] in the sequel.

Lemma 3.9. *Let X be a Banach algebra and let $(\mathcal{P}_{cl}(X), d_H)$ be the generalized metric space. Then*

$$d_H(AC, BC) \leq d_H(A, B) d_H(0, C) = \|C\|_{\mathcal{P}} d_H(A, B)$$

for all $A, B, C, D \in \mathcal{P}_{cl}(X)$.

Lemma 3.10. *Let X be a Banach algebra and let $(\mathcal{P}_{cl}(X), d_H)$ be the generalized metric space. Then for any $A \in \mathcal{P}_{cl}(X)$,*

$$d_H(\lambda A, \mu A) \leq |\lambda - \mu| \|A\|_{\mathcal{P}}$$

for all $\lambda, \mu \in \mathbb{R}$.

Lemma 3.11. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete and σ -finite measure space and let X be a separable Banach algebra. If $S, T : \Omega \rightarrow \mathcal{P}_{cl}(X)$ be two random multi-valued operators, then the multi-valued operator Q defined by $Q(\omega) = S(\omega)T(\omega)$ is again a random multi-valued operator on Ω .*

Proof. Since X is a separable Banach space and the measure space $(\Omega, \mathcal{A}, \mu)$ is complete, we assume that S and T are strongly measurable multi-valued maps on Ω . Then there are sequences $\{S_n(\omega)\}$ and $\{T_n(\omega)\}$ of step multi-valued mappings such that

$$d_H(S_n(\omega), S(\omega)) \rightarrow 0 \quad \text{and} \quad d_H(T_n(\omega), T(\omega)) \rightarrow 0$$

as $n \rightarrow \infty$. Note that by virtue of step multi-valued mappings, $S_n(\omega)T_n(\omega)$ is well defined for each $n \in \mathbb{N}$. Now by Lemma 3.9,

$$\begin{aligned} d_H(Q_n(\omega), Q(\omega)) &= d_H(S_n(\omega)T_n(\omega), S(\omega)T(\omega)) \leq \\ &\leq d_H(S_n(\omega)T_n(\omega), S(\omega)T_n(\omega)) + d_H(S(\omega)T_n(\omega), S(\omega)T(\omega)) \leq \\ &\leq d_H(S_n(\omega), S(\omega))d_H(0, T_n(\omega)) + d_H(T_n(\omega), T(\omega))d_H(0, S_n(\omega)) \leq \\ &\leq d_H(S_n(\omega), S(\omega))\|T_n(\omega)\|_{\mathcal{P}} + d_H(T_n(\omega), T(\omega))\|S_n(\omega)\|_{\mathcal{P}} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This shows that $Q(\omega)$ is strongly measurable and hence $Q(\omega)$ is a multi-valued random operator on Ω . This completes the proof. \square

Theorem 3.12. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let U and \bar{U} be the open-bounded and closed-bounded subsets of a separable Banach algebra X such that $0 \in U$. Let $A, B, C : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp,cv}(X)$ be three multi-valued random operators satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ and $C(\omega)$ are \mathcal{D} -Lipschitz with \mathcal{D} -functions ϕ_ω and ψ_ω respectively,
- (b) $B(\omega)$ is compact and continuous,
- (c) $A(\omega)x + B(\omega)x + C(\omega)x \in \mathcal{P}_{cl,cv}(X)$ for all $x \in \bar{U}$,
- (d) $M(\omega)\phi_\omega(r) + \psi_\omega(r) < r$ if $r > 0$, where $M(\omega) = \|\bigcup B(\omega)(\bar{U})\|_{\mathcal{P}}$, and
- (e) there does not exist a function $u : \Omega \rightarrow \partial U$ such that $\lambda(\omega)u \in A(\omega)u + B(\omega)u + C(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω , where ∂U is the boundary of U in X .

Then the random operator inclusion $x \in A(\omega)x + B(\omega)x + C(\omega)x$ has a random solution in \bar{U} .

Proof. Define a multi-valued mapping $Q : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp,cv}(X)$ by

$$Q(\omega)x = A(\omega)x + B(\omega)x + C(\omega)x. \quad (3.4)$$

Since $(\Omega, \mathcal{A}, \mu)$ is a complete σ -finite measure space and X is a separable Banach space, by Lemma 3.9, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times \bar{U} \rightarrow \mathcal{P}_{cp,cv}(X)$. As $A(\omega)$ and $C(\omega)$ are multi-valued \mathcal{D} -Lipschitz, they are continuous on \bar{U} in view of hypothesis (c). As a result, $Q(\omega)$ is a continuous multi-valued random operator on \bar{U} . Again U is bounded, so from the continuity of $Q(\omega)$, it follows that $Q(\omega)(\bar{U})$ is a

bounded subset of X . We just show that $Q(\omega)$ is a condensing multi-valued operator on \bar{U} . Let $S \subset \bar{U}$. Then for all $\omega \in \Omega$, one has

$$\begin{aligned} \beta(Q(\omega)(S)) &\leq \beta(A(\omega)(S)B(\omega)(S)) + \beta(C(\omega)(S)) \leq \\ &\leq \left\| \bigcup B(\omega)(S) \right\|_{\mathcal{P}} \beta(A(\omega)(S)) + \\ &\quad + \left\| \bigcup A(\omega)(S) \right\|_{\mathcal{P}} \beta(B(\omega)(S)) + \beta(C(\omega)(S)) \leq \\ &\leq [M(\omega) \phi_{\omega}(r) + \psi_{\omega}(r)] \beta(S) < \\ &< \beta(S) \end{aligned}$$

if $\beta(S) > 0$. This shows that $Q(\omega)$ is a condensing multi-valued random operator on \bar{U} . Now the desired conclusion follows by an application of Theorem 2.9. This completes the proof. \square

Corollary 3.13. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let $\mathcal{B}_r(0)$ and $\bar{\mathcal{B}}_r(0)$ be respectively the open and closed balls in a separable Banach algebra X centered at the origin of radius r . Let $A, C : \Omega \times \bar{\mathcal{B}}_r(0) \rightarrow X$ be two single-valued random operators and $B : \Omega \times \bar{\mathcal{B}}_r(0) \rightarrow \mathcal{P}_{cp,cv}(X)$ be multi-valued random operators satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ and $C(\omega)$ are Lipschitz with the Lipschitz constants $k_1(\omega)$ and $k_2(\omega)$,
- (b) $B(\omega)$ is compact and continuous,
- (c) $M(\omega)k_1(\omega) + k_2(\omega) < 1/2$, where $M(\omega) = \left\| \bigcup B(\omega)(\bar{\mathcal{B}}_r(0)) \right\|_{\mathcal{P}}$, and
- (d) there does not exist a $u : \Omega \rightarrow X$ such that $\|u(\omega)\| = r$ and $\lambda(\omega)u \in A(\omega)uB(\omega)u + C(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)xB(\omega)x + C(\omega)x$ has a random solution in $\bar{\mathcal{B}}_r(0)$.

Theorem 3.14. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let X be a separable Banach algebra. Let $A, C : \Omega \times X \rightarrow X$ be two single-valued and $B : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ be a multi-valued random operator satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ and $C(\omega)$ are Lipschitz with the Lipschitz constants $k_1(\omega)$ and $k_2(\omega)$ respectively,
- (b) $B(\omega)$ is compact and continuous,
- (c) $M(\omega)k_1(\omega) + k_2(\omega) < 1/2$, where $M(\omega) = \left\| \bigcup B(\omega)(X) \right\|_{\mathcal{P}}$, and
- (d) the set $\mathcal{E} = \{u : \Omega \rightarrow X \mid \lambda(\omega)u \in A(\omega)uB(\omega)u + C(\omega)u\}$ is bounded for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)xB(\omega)x + C(\omega)x$ has a random solution.

Corollary 3.15. *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let $\mathcal{B}_r(0)$ and $\bar{\mathcal{B}}_r(0)$ be respectively the open and closed balls in a separable Banach algebra X centered at the origin of radius r . Let $A, B : \Omega \times \bar{\mathcal{B}}_r(0) \rightarrow \mathcal{P}_{cp,cv}(X)$ be two multi-valued random operators satisfying for each $\omega \in \Omega$:*

- (a) $A(\omega)$ is Lipschitz with the Lipschitz constant $k(\omega)$,

- (b) $B(\omega)$ is compact and continuous,
- (c) $A(\omega)x B(\omega)x \in \mathcal{P}_{cl,cv}(X)$ for each $x \in \overline{\mathcal{B}_r(0)}$,
- (d) $M(\omega)k(\omega) < 1$, where $M(\omega) = \|\bigcup B(\omega)(\overline{\mathcal{B}_r(0)})\|_{\mathcal{P}}$, and
- (e) there does not exist a $u : \Omega \rightarrow X$ such that $\|u(\omega)\| = r$ and $\lambda(\omega)u \in A(\omega)u B(\omega)u$ for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x B(\omega)x$ has a random solution in $\overline{\mathcal{B}_r(0)}$.

Theorem 3.16. Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let X be a separable Banach algebra. Let $A, B : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ be two multi-valued random operators satisfying for each $\omega \in \Omega$:

- (a) $A(\omega)$ is Lipschitz with the Lipschitz constant $k(\omega)$,
- (b) $B(\omega)$ is compact and continuous,
- (c) $A(\omega)x B(\omega)x \in \mathcal{P}_{cl,cv}(X)$ for each $x \in X$,
- (d) $M(\omega)k(\omega) < 1$, where $M(\omega) = \|\bigcup B(\omega)(X)\|_{\mathcal{P}}$, and
- (e) the set $\mathcal{E} = \{u : \Omega \rightarrow X \mid \lambda(\omega)u \in A(\omega)u B(\omega)u\}$ is bounded for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x B(\omega)x$ has a random solution.

The following results are sometimes useful for applications to quadratic random differential and integral inclusions and hence we state them separately.

Corollary 3.17. Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a separable Banach algebra X centered at the origin of radius r . Let $A : \Omega \times \overline{\mathcal{B}_r(0)} \rightarrow X$ be a single-valued and $B : \Omega \times \overline{\mathcal{B}_r(0)} \rightarrow \mathcal{P}_{cp,cv}(X)$ a multi-valued random operator satisfying for each $\omega \in \Omega$:

- (a) $A(\omega)$ is Lipschitz with the Lipschitz constant $k(\omega)$,
- (b) $B(\omega)$ is compact and continuous,
- (c) $M(\omega)k(\omega) < 1/2$, where $M(\omega) = \|\bigcup B(\omega)(\overline{\mathcal{B}_r(0)})\|_{\mathcal{P}}$, and
- (d) there does not exist a function $u : \Omega \rightarrow X$ such that $\|u(\omega)\| = r$ and $\lambda(\omega)u \in A(\omega)u B(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x B(\omega)x$ has a random solution in $\overline{\mathcal{B}_r(0)}$.

Theorem 3.18. Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let X be a separable Banach algebra. Let $A : \Omega \times X \rightarrow X$ be a single-valued and $B : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$ a multi-valued random operator satisfying for each $\omega \in \Omega$:

- (a) $A(\omega)$ is Lipschitz with the Lipschitz constant $k(\omega)$,
- (b) $B(\omega)$ is compact and continuous,
- (c) $M(\omega)k(\omega) < 1/2$, where $M(\omega) = \|\bigcup B(\omega)(X)\|_{\mathcal{P}}$, and
- (d) the set $\mathcal{E} = \{u : \Omega \rightarrow X \mid \lambda(\omega)u \in A(\omega)u B(\omega)u\}$ is bounded for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \rightarrow \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω .

Then the random operator inclusion $x \in A(\omega)x B(\omega)x$ has a random solution.

4. FUNCTIONAL RANDOM INTEGRAL INCLUSIONS

First, we discuss the Volterra type random integral inclusions for existence as well as existence of the extremal random solutions between the given strict lower and strict upper random solutions.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ -finite measure space and let \mathbb{R} be the real and let $J = [0, T]$ be a closed and bounded interval in \mathbb{R} . Consider the functional random integral inclusion (in short FRII),

$$x(t, \omega) \in q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) G(s, x(\eta(s), \omega), \omega) ds \quad (4.1)$$

for all $\omega \in \Omega$, where $q : J \times \Omega \rightarrow \mathbb{R}$, $k : J \times J \times \Omega \rightarrow \mathbb{R}$, $G : J \times \mathbb{R} \times \Omega \rightarrow \mathcal{P}_p(\mathbb{R})$ and the functions $\sigma, \eta : J \rightarrow J$ are continuous.

By a *random solution* of the FRII (4.1) on $J \times \Omega$ we mean a measurable function $x : \Omega \rightarrow C(J, \mathbb{R})$ satisfying for each $\omega \in \Omega$, $x(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v(s) ds$ for some measurable $v : J \rightarrow L^1(J, \mathbb{R})$ with $v(t) \in F(t, x(\eta(t), \omega), \omega)$ a.e. $t \in J$, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J .

The RII (3.1) seems to be new and includes several known random differential inclusions already studied as special cases. The special case in the form of differential inclusion

$$\left. \begin{aligned} x'(t, \omega) &\in G(t, x(t, \omega), \omega) \text{ a.e. } t \in J \\ x(0, \omega) &= q(\omega) \end{aligned} \right\} \quad (4.2)$$

for all $\omega \in \Omega$, where $q : \Omega \rightarrow \mathbb{R}$ is measurable and $G : J \times \mathbb{R} \times \Omega \rightarrow \mathcal{P}_p(\mathbb{R})$ has been discussed in the literature for various aspects of the solutions. See Papageorgiou [12, 13] and the reference therein.

Let $M(J, \mathbb{R})$, $B(J, \mathbb{R})$ and $C(J, \mathbb{R})$ denote the spaces of measurable, bounded and continuous real-valued functions on J respectively. We will seek the random solutions of FRII (3.1) in the function space $C(J, \mathbb{R})$ defined on $J \times \Omega$. Define a norm $\| \cdot \|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|. \quad (4.3)$$

Clearly, $C(J, \mathbb{R})$ becomes a separable Banach space. Similarly, let $L^1(J, \mathbb{R})$ be the Banach space of Lebesgue integrable real-valued functions on J with the norm $\| \cdot \|$ defined by

$$\|x\| = \int_0^T |x(t)| dt.$$

Let $\beta : J \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a multi-valued mapping. Then for any function $x \in C(J, \mathbb{R})$, let $S_\beta, S_\beta^1 : C(J, \mathbb{R}) \rightarrow \mathcal{P}_p(L^1(J, \mathbb{R}))$ be two multi-valued operators defined by

$$S_\beta(\omega)(x) = \{v \in M(J, \mathbb{R}) \mid v(t) \in \beta(t, x(t, \omega), \omega) \text{ a.e. } t \in J\}, \quad (4.4)$$

$$S_\beta^C(\omega)(x) = \{v \in C(J, \mathbb{R}) \mid v(t) \in \beta(t, x(t, \omega), \omega) \text{ a.e. } t \in J\} \quad (4.5)$$

and

$$S_{\beta}^1(\omega)(x) = \{v \in L^1(J, \mathbb{R}) \mid v(t) \in \beta(t, x(t, \omega), \omega) \text{ a.e. } t \in J\} \quad (4.6)$$

for all $\omega \in \Omega$. This is our set of *selection functions* for β on $J \times \mathbb{R} \times \Omega$. When there is no confusion, we denote $S_{\beta}^1(\omega)(x) = S_{\beta}^1(\omega)(y)$, where $y(t, \omega) = x(\theta(t), \omega)$ for some continuous function $\theta : J \rightarrow J$. The integral of the random multi-valued function β is defined as

$$\int_0^t \beta(s, x(s, \omega), \omega) ds = \left\{ \int_0^t v(s) ds : v \in S_{\beta}^1(\omega)(x) \right\}.$$

Furthermore, if the integral $\int_0^t \beta(s, x(s, \omega), \omega) ds$ exists for every measurable function $x : \Omega \rightarrow C(J, \mathbb{R})$, then we say the multi-valued mapping β is Lebesgue integrable on J .

We need the following definitions in the sequel.

Definition 4.1. A multi-valued mapping $\beta : \Omega \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $\omega \mapsto d(y, \beta(\omega)) = \inf\{|y - x| : x \in \beta(\omega)\}$ is measurable.

Definition 4.2. A multi-valued function $\beta : J \times \mathbb{R} \times \Omega \rightarrow \mathcal{P}_{cp}(\mathbb{R})$ is called Carathéodory if for each $\omega \in \Omega$:

- (i) $(t, \omega) \mapsto \beta(t, x, \omega)$ is jointly measurable for each $x \in \mathbb{R}$, and
- (ii) $x \mapsto \beta(t, x, \omega)$ is Hausdorff continuous almost everywhere for $t \in J$.

Again, a Carathéodory multi-valued function β is called strong L^1 -Carathéodory if:

- (iii) for each real number $r > 0$ there exists a measurable function $h_r : \Omega \rightarrow L^1(J, \mathbb{R})$ such that for each $\omega \in \Omega$

$$\|\beta(t, x, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x, \omega)\} \leq h_r(t, \omega) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Finally, a Carathéodory multi-valued β is called strong $L^1_{\mathbb{R}}$ -Carathéodory if:

- (iv) there exists a measurable function $h : \Omega \rightarrow L^1(J, \mathbb{R})$ such that

$$\|\beta(t, x, \omega)\|_{\mathcal{P}} \leq h(t, \omega) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of β on $J \times \mathbb{R} \times \Omega$.

Then, we have the following lemmas which are well-known in the literature.

Lemma 4.3 (Lasota and Opial [8]). *Let E be a Banach space. If $\dim(E) < \infty$ and $\beta : J \times E \rightarrow \mathcal{P}_{cp}(E)$ is L^1 -Carathéodory, then $S_\beta^1(x) \neq \emptyset$ for each $x \in E$.*

Lemma 4.4 (Lasota and Opial [8]). *Let E be a Banach space, F a Carathéodory multi-valued operator with $S_\beta^1 \neq \emptyset$ and let $\mathcal{L} : L^1(J, E) \rightarrow C(J, E)$ be a continuous linear mapping. Then the composite operator*

$$\mathcal{L} \circ S_\beta^1 : C(J, E) \rightarrow \mathcal{P}_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

Lemma 4.5 (Carathéodory theorem [7]). *Let E be a Banach space. If $\beta : J \times E \rightarrow \mathcal{P}_{cp}(E)$ is Carathéodory, then the multi-valued mapping $t \mapsto \beta(t, x(t))$ is measurable for each measurable function $x : J \rightarrow E$.*

We consider the following set of hypotheses in the sequel.

- (A₀) The single-valued mapping $q : \Omega \rightarrow C(J, \mathbb{R})$ is measurable and bounded. Moreover, $Q = \text{ess sup}_{\omega \in \Omega} \|q(\omega)\|$.
- (A₁) The single-valued mapping $k : \Omega \rightarrow C(J \times J, \mathbb{R})$ is measurable and bounded. Moreover, $K_1 = \text{ess sup}_{\omega \in \Omega} K(\omega)$.
- (A₂) $G(t, x, \omega)$ is a compact-convex subset of \mathbb{R} for all $(t, x, \omega) \in J \times \mathbb{R} \times \Omega$.
- (A₂) G is L^1 -Carathéodory.
- (A₃) There exists a measurable and bounded function $\gamma : \Omega \rightarrow L^1(J, \mathbb{R})$ with $\gamma(t, \omega) > 0$ a.e. $t \in J$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \rightarrow (0, \infty)$ such that for each $\omega \in \Omega$,

$$G(t, x, \omega) \|\mathcal{P} \leq \gamma(t, \omega) \psi(|x|) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$. Moreover, $c = \text{ess sup}_{\omega \in \Omega} \|\gamma(\omega)\|_{L^1}$.

Theorem 4.6. *Assume that the hypotheses (A₀) – (A₄) hold. Furthermore, if $\sigma(t) \leq t$, $\eta(t) \leq t$ for all $t \in J$ and*

$$\int_Q \frac{dr}{\psi(r)} > K_1 \|\gamma(\omega)\|_{L^1} \quad (4.7)$$

for all $\omega \in \Omega$, then the RDI (4.1) has a random solution in $C(J, \mathbb{R})$ defined on $J \times \Omega$.

Proof. Let $X = C(J, \mathbb{R})$. Define a multi-valued operator $Q : \Omega \times X \rightarrow \mathcal{P}_p(X)$ by

$$Q(\omega)x = \left\{ u \in X \mid u(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega)v(s) ds, t \in J \text{ and } v \in S_G^1(\omega)(x) \right\} \\ = (\mathcal{L} \circ S_G^1(\omega))(x), \quad (4.8)$$

where $\mathcal{L} : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}v(t) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega)v(s) ds. \quad (4.9)$$

Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (H_2) . We shall show that $Q(\omega)$ satisfies all the conditions of Theorem 2.8.

Step I. First, we show that Q is a closed valued multi-valued random operator on $\Omega \times X$. Observe that the operator $Q(\omega)$ is equivalent to the composition $\mathcal{L} \circ S_G^1(\omega)$ of two operators on $L^1(J, \mathbb{R})$, where $\mathcal{L} : L^1(J, \mathbb{R}) \rightarrow X$ is the continuous operator defined by (4.9).

Next, we show that $Q(\omega)$ is a multi-valued random operator on X . First, we show that the multi-valued map $(\omega, x) \mapsto S_G^1(\omega)(x)$ is measurable. Let $f \in L^1(J, \mathbb{R})$ be arbitrary. Then we have

$$\begin{aligned} d(f, S_G^1(\omega)(x)) &= \inf\{\|f - h\|_{L^1} : h \in S_G^1(\omega)(x)\} = \\ &= \inf\left\{\int_0^T |f(t) - h(t)| dt : h \in S_G(\omega)(x)\right\} = \\ &= \int_0^T \inf\{|f(t) - z| : z \in G(t, x(\eta(t), \omega), \omega)\} dt = \\ &= \int_0^T d(f(t), G(t, x(\eta(t), \omega), \omega)) dt. \end{aligned}$$

But by hypothesis (A_0) , the map $(t, x, \omega, f) \mapsto d(f(t), G(t, x(\eta(t), \omega), \omega))$ is measurable from $J \times X \times \Omega \times L^1(J, \mathbb{R})$ into \mathbb{R}^+ . Now the integral is the limit of the finite sum of measurable functions, and so, $d(f, S_G^1(\omega)(x))$ is measurable. As a result, the multi-valued mapping $(\cdot, \cdot) \rightarrow S_{F(\cdot)}^1(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J \times X \times \Omega$ by

$$\phi(t, x, \omega) = (\mathcal{L} \circ S_G^1(\omega))(x)(t) = \int_0^{\sigma(t)} k(t, s, \omega)G(s, x(\eta(s), \omega), \omega) ds.$$

We shall show that $\phi(t, x, \omega)$ is continuous in t in the Hausdorff metric on \mathbb{R} . Let $\{t_n\}$ be a sequence in J converging to $t \in J$. Then we have:

$$\begin{aligned}
d_H(\phi(t_n, x, \omega), \phi(t, x, \omega)) &= \\
&= d_H\left(\int_0^{\sigma(t_n)} k(t_n, s, \omega)G(s, x(\eta(s), \omega), \omega) ds, \int_0^{\sigma(t)} k(t, s, \omega)G(s, x(\eta(s), \omega), \omega) ds)\right) = \\
&= d_H\left(\int_0^{\sigma(t_n)} k(t_n, s, \omega)G(s, x(\eta(s), \omega), \omega) ds, \int_0^{\sigma(t)} k(t_n, s, \omega)G(s, x(\eta(s), \omega), \omega) ds)\right) = \\
&\quad + d_H\left(\int_0^{\sigma(t)} k(t, s, \omega)G(s, x(\eta(s), \omega), \omega) ds, \int_0^{\sigma(t)} k(t_n, s, \omega)G(s, x(\eta(s), \omega), \omega) ds)\right) = \\
&= d_H\left(\int_J \chi_{[0, \sigma(t_n)]}(s)k(t, s, \omega)G(s, x(\eta(s), \omega), \omega) ds, \right. \\
&\quad \left. \int_J \chi_{[0, \sigma(t)]}(s)k(t, s, \omega)G(s, x(s, \omega), \omega) ds\right) + \\
&\quad + \int_0^{\sigma(t)} d_H\left(k(t_n, s, \omega)G(s, x(s, \omega), \omega), k(t, s, \omega)G(s, x(s, \omega), \omega)\right) ds = \\
&= \int_J |\chi_{[0, \sigma(t_n)]}(s) - \chi_{[0, \sigma(t)]}(s)| |k(t, s, \omega)| \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds + \\
&\quad + \int_0^{\sigma(t)} |k(t_n, s, \omega) - k(t, s, \omega)| \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds = \\
&= \int_J |\chi_{[0, \sigma(t_n)]}(s) - \chi_{[0, \sigma(t)]}(s)| K(\omega) \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds + \\
&\quad + \int_0^T |k(t_n, s, \omega) - k(t, s, \omega)| \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds = \\
&= \int_J |\chi_{[0, \sigma(t_n)]}(s) - \chi_{[0, \sigma(t)]}(s)| \gamma(s, \omega) \psi(|x(\eta(s), \omega)|) ds + \\
&\quad + \int_0^T |k(t_n, s, \omega) - k(t, s, \omega)| \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds = \\
&= \int_J |\chi_{[0, \sigma(t_n)]}(s) - \chi_{[0, \sigma(t)]}(s)| \gamma(s, \omega) \psi(\|x(\omega)\|) ds + \\
&\quad + \int_0^T |k(t_n, s, \omega) - k(t, s, \omega)| \|G(s, x(\eta(s), \omega), \omega)\|_{\mathcal{P}} ds \rightarrow \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Thus the multi-valued map $t \mapsto \phi(t, x, \omega)$ is continuous and hence, by Lemma 3.9, the map $(t, x, \omega) \mapsto \int_0^{\sigma(t)} k(t, s, \omega)G(s, x(\eta(s), \omega), \omega) ds$ is measurable. Again, since

the sum of two measurable multi-valued functions is measurable, the map

$$(t, x, \omega) \mapsto q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) G(s, x(\eta(s)), \omega), \omega) ds$$

is measurable. Consequently, $Q(\omega)$ is a random multi-valued operator on $[a, b]$.

Step II. Next, we show that $Q(\omega)$ is compact and continuous for each $\omega \in \Omega$. First, we show that $Q(\omega)$ is a continuous multi-valued random operator on X . Let $\{x_n\}$ be a sequence in X converging to a point x . Then by H-continuity of the multi-valued mapping $G(t, x, \omega)$ in x , we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} Q(\omega)x_n(t) &= q(t, \omega) + \lim_{n \rightarrow \infty} \int_0^{\sigma(t)} k(t, s, \omega) G(s, x_n(\eta(s)), \omega), \omega) ds = \\ &= q(t, \omega) + \int_0^{\sigma(t)} \lim_{n \rightarrow \infty} k(t, s, \omega) G(s, x_n(\eta(s)), \omega), \omega) ds = \\ &= q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) G(s, x(\eta(s)), \omega), \omega) ds = Q(\omega)x(t) \end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a Hausdorff continuous multi-valued random operator on X .

Next we show that $Q(\omega)$ is a compact operator on X for each $\omega \in \Omega$. Let S be a bounded set in X . Then there is a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. Let $\{y_n(\omega)\}$ be a sequence in $\cup Q(\omega)(S)$ for some fixed $\omega \in \Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X .

Case I. First, we show that $\{y_n(\omega)\}$ is a uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n \in S_G^1(\omega)(x_n)$ for some $x_n : \Omega \rightarrow S$ such that

$$y_n(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v_n(s) ds, \quad t \in J.$$

Therefore,

$$\begin{aligned} |y_n(t, \omega)| &\leq |q(t, \omega)| + \int_0^{\sigma(t)} |k(t, s, \omega)| |v_n(s)| ds \leq \\ &\leq \|q(\omega)\| + \int_0^{\sigma(t)} |k(t, s, \omega)| \|G(s, x_n(\eta(s)), \omega), \omega\|_{\mathcal{P}} ds \leq \\ &\leq \|q(\omega)\| + \int_0^{\sigma(t)} K(\omega) \gamma(s, \omega) \psi(\|x(\omega)\|) \leq \\ &\leq \|q(\omega)\| + K(\omega) \|\gamma(\omega)\|_{L^1} \psi(r) \leq Q + cK_1 \psi(r) \end{aligned}$$

for all $t \in J$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \leq Q + cK_1\psi(r),$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)(X)$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X)$. Let $t, \tau \in J$. Then, for each $\omega \in \Omega$, we have

$$\begin{aligned} |y_n(t, \omega) - y_n(\tau, \omega)| &= \left| \int_0^{\sigma(t)} k(t, s, \omega)v_n(s) ds - \int_0^{\sigma(\tau)} k(\tau, s, \omega)v_n(s) ds \right| \leq \\ &\leq \left| \int_0^{\sigma(t)} k(t, s, \omega)v_n(s) ds - \int_0^{\sigma(t)} k(\tau, s, \omega)v_n(s) ds \right| + \\ &\quad + \left| \int_0^{\sigma(\tau)} k(\tau, s, \omega)v_n(s) ds - \int_0^{\sigma(\tau)} k(\tau, s, \omega)v_n(s) ds \right| \leq \\ &\leq \int_0^{\sigma(t)} |k(t, s, \omega) - k(\tau, s, \omega)| |v_n(s)| ds + \\ &\quad + \left| \int_{\sigma(\tau)}^{\sigma(t)} |k(\tau, s, \omega)| |v_n(s)| ds \right| \leq \\ &\leq \int_0^T |k(t, s, \omega) - k(\tau, s, \omega)| \|G(s, x_n(s, \omega), \omega)\|_{\mathcal{P}} ds + \\ &\quad + \left| \int_{\sigma(\tau)}^{\sigma(t)} k(\tau, s, \omega) \|G(s, x_n(s, \omega), \omega)\|_{\mathcal{P}} ds \right| \leq \\ &\leq \int_0^T |k(t, s, \omega) - k(\tau, s, \omega)| \gamma(s, \omega) \psi(\|x(\omega)\|) ds + \\ &\quad + \left| \int_{\sigma(\tau)}^{\sigma(t)} K(\omega) \gamma(s, \omega) \psi(\|x(\omega)\|) ds \right| \leq \\ &\leq \int_0^T |k(t, s, \omega) - k(\tau, s, \omega)| \gamma(s, \omega) \psi(r) ds + \\ &\quad + |p(t, \omega) - p(\tau, \omega)|, \end{aligned}$$

where $p(t, \omega) = K_1 \int_0^{\sigma(t)} \gamma(s, \omega) \psi(r) ds$. From the above inequality, it follows that

$$|y_n(t, \omega) - y_n(\tau, \omega)| \rightarrow 0 \quad \text{as } t \rightarrow \tau$$

uniformly for all $n \in \mathbb{N}$. This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(X)$. Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega \in \Omega$, so it has a cluster point in view of the Arzelà-Ascoli theorem. As a result, $Q(\omega)$ is a compact multi-valued random operator on X . Thus $Q(\omega)$ is continuous and compact and hence a completely continuous multi-valued random operator on X .

Step III. Next, we show that $Q(\omega)$ has convex values on X for each fixed $\omega \in \Omega$. Again, let $u_1, u_2 \in Q(\omega)x$. Then there are $v_1, v_2 \in S_G^1(\omega)(x)$ such that

$$u_1(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v_1(s) ds, \quad t \in J,$$

and

$$u_2(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v_2(s) ds, \quad t \in J.$$

Now for any $\lambda \in [0, 1]$,

$$\begin{aligned} \lambda u_1(t, \omega) + (1 - \lambda) u_2(t, \omega) &= \lambda \left(q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v_1(s) ds \right) + \\ &\quad + (1 - \lambda) \left(q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) v_2(s) ds \right) = \\ &= q(t, \omega) + \int_0^{\sigma(t)} k(t, s, \omega) [v_1(s) + (1 - \lambda)v_2(s)] ds. \end{aligned}$$

Since $S_G^1(\omega)$ has convex values on X (because G has convex values), we have that $v(t, \omega) = \lambda v_1(t) + (1 - \lambda)v_2(t) \in S_G^1(\omega)(x)(t)$ for all $t \in J$. Hence, $\lambda u_1 + (1 - \lambda)u_2 \in Q(\omega)x$ and consequently $Q(\omega)x$ is convex for each $x \in X$. As a result, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times X \rightarrow \mathcal{P}_{cp,cv}(X)$.

Step IV. Finally, we show that the set \mathcal{E} is bounded. Let $u : \Omega \rightarrow C(J, \mathbb{R})$ be a function such that $\lambda u(t, \omega) \in Q(\omega)u(t)$ on $J \times \Omega$ for all $\lambda > 1$. Then there is a $v \in S_G^1(\omega)(u)$ such that

$$u(t, \omega) = \lambda^{-1} q(t, \omega) + \lambda^{-1} \int_0^{\sigma(t)} k(t, s, \omega) v(s) ds$$

for all $t \in J$ and $\omega \in \Omega$. Therefore,

$$\begin{aligned}
|u(t, \omega)| &\leq |q(t, \omega)| + \int_0^{\sigma(t)} |k(t, s, \omega)| |v(s)| ds \leq \\
&\leq \|q(\omega)\| + \int_0^{\sigma(t)} K(\omega) \|G(s, u(\eta(s), \omega))\|_{\mathcal{P}} ds \leq \\
&\leq \|q(\omega)\| + \int_0^t K(\omega) \gamma(s, \omega) \psi(|u(\eta(s), \omega)|) ds \leq \\
&\leq Q + K_1 \int_0^t \gamma(s, \omega) \psi(|u(\eta(s), \omega)|) ds
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$.

Let $m(t, \omega) = \sup_{s \in [0, t]} |u(s, \omega)|$. Then, we have $|u(t, \omega)| \leq m(t, \omega)$ and $|u(\eta(t), \omega)| \leq m(t, \omega)$ for all $(t, \omega) \in J \times \Omega$. Furthermore, there is a point $t^* \in [0, t]$ such that $m(t, \omega) = |u(t^*, \omega)|$. Hence we have

$$\begin{aligned}
m(t, \omega) &= |u(t^*, \omega)| \leq \\
&\leq Q + K_1 \int_0^{t^*} \gamma(s, \omega) \psi(|u(\eta(s), \omega)|) ds \leq \\
&\leq Q + K_1 \int_0^t \gamma(s, \omega) \psi(m(s, \omega)) ds.
\end{aligned}$$

Put

$$w(t, \omega) = Q + K_1 \int_0^t \gamma(s, \omega) \psi(m(s, \omega)) ds.$$

Differentiating w.r.t. t ,

$$\left. \begin{aligned} w'(t, \omega) &= K_1 \gamma(t, \omega) \psi(m(t, \omega)) \\ w(0, \omega) &= Q \end{aligned} \right\} \quad (4.10)$$

for all $t \in J$ and $\omega \in \Omega$.

From the above expression we obtain

$$\left. \begin{aligned} \frac{w'(t, \omega)}{\psi(w(t, \omega))} &\leq K_1 \gamma(t, \omega) \\ w(0, \omega) &= Q \end{aligned} \right\}. \quad (4.11)$$

Integrating the above inequality from 0 to t ,

$$\int_0^t \frac{w'(s, \omega)}{\psi(w(s, \omega))} ds \leq \int_0^t K_1 \gamma(s, \omega) ds.$$

By change of variables,

$$\int_Q^{w(t, \omega)} \frac{dr}{\psi(r)} \leq K_1 \|\gamma(\omega)\|_{L^1} < \int_Q^\infty \frac{dr}{\psi(r)}$$

for all $\omega \in \Omega$. Now an application of the mean value theorem yields that there is a constant $M > 0$ such that

$$|u(t, \omega)| \leq m(t, \omega) \leq w(t, \omega) \leq M$$

for all $t \in J$ and $\omega \in \Omega$. Hence, by Theorem 2.12, the RII (4.1) has a random solution on J . This completes the proof. \square

Remark 4.7. The results of this paper also remain valid for $\omega \in \Omega$ almost surely and not just only for all $\omega \in \Omega$.

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