

ON SOME IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. This paper deals with some impulsive fractional differential equations in Banach spaces. Utilizing the Leray-Schauder fixed point theorem and the impulsive nonlinear singular version of the Gronwall inequality, the existence of *PC*-mild solutions for some fractional differential equations with impulses are obtained under some easily checked conditions. At last, an example is given for demonstration.

Keywords: fractional differential equations with impulses, nonlinear impulsive singular version of the Gronwall inequality, *PC*-mild solutions, existence.

Mathematics Subject Classification: 45N05, 93C25.

1. INTRODUCTION

During the past decades, impulsive differential equations have attracted much interest since it is much richer than the corresponding theory of differential equations (see for instance [14, 25, 54] and references therein). Recently, impulsive evolution equations and their optimal control problems in infinite dimensional spaces have been investigated by many authors including Ahmed, Benchohra, Ntouyas, Liu, Nieto and us (see for instance [1–5, 8, 9, 29], [38, 39, 50–53] and references therein). Specially, we also studied the impulsive periodic system in infinite dimensional spaces (see [43–47]).

On the other hand, the fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [16, 19–21, 33, 37]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas *et al.* [24], Miller and Ross [31], Podlubny [41], Lakshmikantham *et al.* [28], and the papers [6, 7, 10, 15–19, 22, 24, 26, 27, 34, 35] and the references therein.

However, to our knowledge, the theory for impulsive fractional differential equations in Banach spaces has not yet been sufficiently developed. Very recently, Benchohra *et al.* [10, 12] applied the Banach contraction principle, Schaefer's fixed point theorem and the nonlinear alternative of the Leray-Schauder type or measure of noncompactness to a class of impulsive fractional differential equations without unbounded operator. A class of initial value problem for impulsive fractional differential equations with variable times is also considered in [13]. Balachandran *et al.* [13] using fractional calculus and fixed point theorems for a class of impulsive fractional evolution equations with bounded time-varying linear operator. Mophou *et al.* [36], Wang *et al.* [49], apply semigroup theory and fixed point theorems to study the impulsive fractional differential equations with an unbounded operator in Banach spaces.

Motivated by the above work including [34–36, 48, 49], the main purpose of this paper is to consider the following fractional differential equations with impulses

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + t^n f(t, x(t)), \quad \alpha \in (0, 1], \quad n \in Z^+, \quad t \in J = [0, b], \quad t \neq t_k, \\ x(0) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)) = x(t_k^+) - x(t_k^-), \quad k = 1, 2, \dots, \delta, \quad 0 < t_1 < t_2 < \dots < t_\delta < b, \end{cases} \quad (1.1)$$

where $A: D(A) \subset X \rightarrow X$ is the generator of a C_0 -semigroup $\{T(t), t \geq 0\}$ on a Banach space X , D_t^α is the Caputo fractional derivative, $f: J \times X \rightarrow X$ is specified later, x_0 is an element of X , $I_k: X \rightarrow X$ is a nonlinear map which determines the size of the jump at t_k , $0 = t_0 < t_1 < t_2 < \dots < t_\delta < t_{\delta+1} = b$, $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$ and $x(t_k^-) = x(t_k)$ represents respectively the right and left limits of $x(t)$ at $t = t_k$.

In order to obtain the existence of solutions for impulsive fractional differential equations, some authors use Krasnoselskii's fixed point theorem or contraction mapping principle. It is obvious that the conditions for Krasnoselskii's fixed point theorem are not easily verified sometimes and the conditions for the contraction mapping principle are too strong. Some authors give the prior estimate of the solutions for impulsive fractional differential equations, however, the condition on f is a little strong.

Here, we use the Leray-Schauder fixed point theorem to obtain the existence of PC -mild solutions for system (1.1) under some easily checked conditions. First, we construct an operator H for system (1.1), then use a generalized Ascoli-Arzelà theorem (see Theorem 2.5) and overcome some difficulties to show the compactness of H which is very important. With the help of an impulsive nonlinear singular version of the Gronwall inequality (see Theorem 2.7), the key estimate of the fixed point set $\{x = \sigma Hx, \sigma \in [0, 1]\}$ can be established successfully. Therefore, the existence of PC -mild solutions for system (1.1) is shown. Our methods are different from previous work and we give a new way to show the existence of solutions for impulsive fractional differential equations.

The paper is organized as follows. In Section 2, we introduce the PC -mild solution of system (1.1) and recall some basis results including the impulsive nonlinear singular version of the Gronwall inequality. In Section 3, the existence of PC -mild solutions for system (1.1) is proved under some easily checked conditions. Finally, an example is given to demonstrate the applicability of our result.

2. PRELIMINARIES

Let $\mathcal{L}_b(X)$ be the Banach space of all linear and bounded operators on X . For a C_0 -semigroup $\{T(t), t \geq 0\}$ on X , we set $M \equiv \sup_{t \in J} \|T(t)\|_{\mathcal{L}_b(X)}$. Let $C(J, X)$ be the Banach space of all X -valued continuous functions from $J = [0, b]$ into X endowed with the norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$. We also introduce the set of functions $PC(J, X) \equiv \{x : J \rightarrow X \mid x \text{ is continuous at } t \in J \setminus \{t_1, t_2, \dots, t_\delta\}, \text{ and } x \text{ is continuous from left and has right hand limits at } t \in \{t_1, t_2, \dots, t_\delta\}\}$. Endowed with the norm

$$\|x\|_{PC} = \max \left\{ \sup_{t \in J} \|x(t+0)\|, \sup_{t \in J} \|x(t-0)\| \right\},$$

$(PC(J, X), \|\cdot\|_{PC})$ is a Banach space.

Let us recall the following definitions. For more details see [41].

Definition 2.1. A real function $f(t)$ is said to be in the space C_α , $\alpha \in R$ if there exists a real number $\kappa > \alpha$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. If the function $f \in C_{-1}^\zeta$, $\zeta \in N$, the fractional derivative of order $\alpha > 0$ of a function $f(t)$ in the Caputo sense is given by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\zeta - \alpha)} \int_0^t (t-s)^{\zeta-\alpha-1} f^{(\zeta)}(s) ds, \quad \zeta - 1 < \alpha \leq \zeta.$$

Based on [36] (Definition 3.2 and Lemma 3.3), we use the following definition of a PC -mild solution for system (1.1).

Definition 2.4. By a PC -mild solution of the system (1.1) we mean the function $x \in PC(J, X)$ which satisfies

$$\begin{aligned} x(t) = & T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) s^n f(s, x(s)) ds + \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s)) ds + \\ & + \sum_{0 < t_k < t} T(t-t_k) I_k(x(t_k)). \end{aligned} \tag{2.1}$$

The following results will be used later.

Lemma 2.5 (Generalized Ascoli-Arzela theorem, Theorem 2.1, [50]). *Suppose $\mathcal{W} \subset PC(J, X)$ be a subset. If the following conditions are satisfied:*

- (1) \mathcal{W} is a uniformly bounded subset of $PC(J, X)$.
- (2) \mathcal{W} is equicontinuous in (t_k, t_{k+1}) , $k = 0, 1, 2, \dots, \delta$, where $t_0 = 0$, $t_{\delta+1} = b$.
- (3) $\mathcal{W}(t) \equiv \{x(t) \mid x \in \mathcal{W}, t \in J \setminus \{t_1, \dots, t_\delta\}\}$, $\mathcal{W}(t_k + 0) \equiv \{x(t_k + 0) \mid x \in \mathcal{W}\}$ and $\mathcal{W}(t_k - 0) \equiv \{x(t_k - 0) \mid x \in \mathcal{W}\}$ are relatively compact subsets of $PC(J, X)$.

Then \mathcal{W} is a relatively compact subset of $PC(J, X)$.

Lemma 2.6 (Lemma 2.1, [42]). *For all $\beta > 0$ and $\vartheta > -1$,*

$$\int_0^t (t-s)^{\beta-1} s^\vartheta ds = C(\beta, \vartheta) t^{\beta+\vartheta},$$

where

$$C(\beta, \vartheta) = \frac{\Gamma(\beta)\Gamma(\vartheta+1)}{\Gamma(\beta+\vartheta+1)}.$$

Lemma 2.7 (Impulsive nonlinear singular version of the Gronwall inequality, Theorem 3.1, [42]). *Let $x \in PC([0, \infty), X)$ and satisfy the following inequality*

$$x(t) \leq a(t) + b(t) \int_0^t (t-s)^{\alpha-1} s^\gamma F_1(s) x^m(s) ds + d(t) \sum_{0 < t_k < t} \eta_k x(t_k), t \geq 0, \quad (2.2)$$

where $a(t)$, $b(t)$, $d(t)$ and $F_1(t)$ are nonnegative continuous functions, $\eta_k \geq 0$ are constants.

- (1) If $\frac{1}{2} \geq \alpha > 0$, $-\frac{1}{2} \geq \gamma > -1$, then it holds that for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} x(t) \leq & \left[(k+3)^{q-1} f_p(t) \prod_{l=1}^k (1 + (k+2)^{q-1} \eta_l^q f(t_l)) \right]^{\frac{1}{q}} \times \\ & \times \left[1 - (m-1) \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (i+2)^{q-1} m \prod_{j=1}^{i-1} (1 + (j+2)^{q-1} \eta_j^q f(t_j))^m F_1^q(s) f_p^m(s) ds - \right. \\ & \left. - (m-1)(k+3)^{(q-1)m} \times \prod_{j=1}^k (1 + (j+2)^{q-1} \eta_j^q f(t_j))^m \int_{t_k}^t F_1^q(s) f_p^m(s) ds \right]^{\frac{1}{q(1-m)}} \end{aligned} \quad (2.3)$$

as long as the expression between the second brackets is positive, that is, on $(0, T_p)$, T_p is the sup of all values of t for which

$$\begin{aligned} & \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (i+2)^{(q-1)m} \prod_{j=1}^{i-1} (1+(j+2)^{q-1}) \eta_j^q f(t_j)^m F_1^q(s) f_p^m(s) ds - \\ & - (m-1)(k+3)^{(q-1)m} \times \prod_{j=1}^k (1+(j+2)^{q-1}) \eta_j^q f(t_j)^m \int_{t_k}^t F_1^q(s) f_p^m(s) ds < \\ & < \frac{1}{m-1}; \end{aligned}$$

(2) If $\frac{1}{2} < \alpha$, $-\frac{1}{2} < \gamma$, then it holds that for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} x(t) & \leq \left[(k+3)f(t) \prod_{l=1}^k (1+(k+2)) \eta_l^2 f(t_l) \right]^{\frac{1}{2}} \times \\ & \times \left[1 - (m-1) \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (i+2)^m \prod_{j=1}^{i-1} (1+(j+2)) \eta_j^2 f(t_j)^m F_1^2(s) f^m(s) ds - \right. \\ & \left. - (m-1)(k+3)^m \times \prod_{j=1}^k (1+(j+2)) \eta_j^2 f(t_j)^m \int_{t_k}^t F_1^2(s) f^m(s) ds \right]^{\frac{1}{2(1-m)}} \end{aligned}$$

as long as the expression between the second brackets is positive, that is, on $(0, T_2)$, T_2 is the sup of all values of t for which

$$\begin{aligned} & \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (i+2)^m \prod_{j=1}^{i-1} (1+(j+2)) \eta_j^2 f(t_j)^m F_1^2(s) f^m(s) ds - \\ & - (m-1)(k+3)^m \times \prod_{j=1}^k (1+(j+2)) \eta_j^2 f(t_j)^m \int_{t_k}^t F_1^2(s) f^m(s) ds < \\ & < \frac{1}{m-1}, \end{aligned}$$

where

$$f_p(t) = \sup \left\{ a^q(t), C^{\frac{q}{p}}(p\alpha - p + 1, p\gamma)b^q(t)t^{q(\alpha+\gamma)-1}, d^q(t) \right\},$$

$$p \text{ and } q \text{ such that } \frac{1}{p} + \frac{1}{q} = 1,$$

$$f(t) \equiv f_2(t) = \sup \left\{ a^2(t), C(2\alpha - 2 + 1, 2\gamma)b^2(t)t^{2(\alpha+\gamma)-1}, d^2(t) \right\},$$

$$\text{if } p = q = 2,$$

$$C(p\alpha - p + 1, p\gamma) = \frac{\Gamma(p\alpha - p + 1)\Gamma(p\gamma + 1)}{\Gamma(p\alpha - p + 1 + p\gamma + 1)}.$$

3. EXISTENCE OF MILD SOLUTIONS

In this section, we will derive the existence result concerning the *PC*-mild solution for the system (1.1) under some easily checked conditions.

We make the following assumptions.

[HA]: A is the infinitesimal generator of a compact C_0 -semigroup $\{T(t), t \geq 0\}$ on X with domain $D(A)$.

[Hf]: (1) $f : J \times X \rightarrow X$ is strongly measurable with respect to t on J and for any $x, y \in X$ satisfying $\|x\|, \|y\| \leq \rho$ there exists a positive constant $L_f(\rho) > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq L_f(\rho)\|x - y\|.$$

(2) There exists a positive constant $M_f > 0$ such that

$$\|f(t, x)\| \leq M_f(1 + \|x\|^m) \text{ for all } t \in J, x \in X, \text{ some } m > 1.$$

[HI]: (1) The nonlinear map $I_k : X \rightarrow X$, $I_k(X)$ is a bounded subset of X , $k = 1, 2, \dots, \delta$.

(2) There exist constants $h_k > 0$, such that

$$\|I_k(x) - I_k(y)\| \leq h_k\|x - y\|, \quad \text{for all } x, y \in X, k = 1, 2, \dots, \delta.$$

Theorem 3.1. *Under the assumptions [HA], [Hf] and [HI], system (1.1) has at least a *PC*-mild solution on J .*

Proof. Let $x_0 \in X$ be fixed. Define an operator H on $PC(J, X)$ which is given by

$$\begin{aligned}
 (Hx)(t) = & T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_{t_k}^{t_k} (t_k - s)^{\alpha-1} T(t - s) s^n f(s, x(s)) ds + \\
 & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} T(t - s) s^n f(s, x(s)) ds + \\
 & + \sum_{0 < t_k < t} T(t - t_k) I_k(x(t_k)).
 \end{aligned}
 \tag{3.1}$$

Using [HA] and [Hf], one can verify that H is a continuous mapping from $PC(J, X)$ to $PC(J, X)$ for $x \in PC(J, X)$. In fact, for $0 \leq \tau < t \leq t_1$, it comes from [HA] and the following inequality

$$\begin{aligned}
 \|(Hx)(t) - (Hx)(\tau)\| \leq & \|T(t)x_0 - T(\tau)x_0\| + \\
 & + \frac{1}{\Gamma(\alpha)} \int_{\tau}^t \|(t - s)^{\alpha-1} T(t - s) s^n f(s, x(s))\| ds + \\
 & + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \|(t - s)^{\alpha-1} [T(t - s) - T(\tau - s)] s^n f(s, x(s))\| ds \leq \\
 \leq & M \|T(t - \tau)x_0 - x_0\| + \\
 & + \frac{t^n M \|f\|_{PC}}{\Gamma(\alpha)} \int_{\tau}^t (t - s)^{\alpha-1} ds + \\
 & + \frac{\tau^n M \|T(t - \tau) - I\| \|f\|_{PC}}{\Gamma(\alpha)} \int_0^{\tau} (t - s)^{\alpha-1} ds \leq \\
 \leq & M \|T(t - \tau)x_0 - x_0\| + \\
 & + \frac{t_1^n M \|f\|_{PC}}{\Gamma(\alpha + 1)} (t - \tau)^{\alpha} + \\
 & + \left[\frac{t_1^n M \|f\|_{PC}}{\Gamma(\alpha + 1)} |t^{\alpha} - (t - \tau)^{\alpha}| \right] \|T(t - \tau) - I\|
 \end{aligned}
 \tag{3.2}$$

that $Hx \in C([0, t_1], X)$.

With analogous arguments we can obtain $Hx \in C([t_k, t_{k+1}], X)$, $k = 0, 1, 2, \dots, \delta$. That is $Hx \in PC(J, X)$.

(1) H is a continuous operator on $PC(J, X)$.

Let $x_1, x_2 \in PC(J, X)$ and $\|x_1 - x_2\|_{PC} \leq 1$, then $\|x_2\|_{PC} \leq 1 + \|x_1\|_{PC} = \rho$. By assumptions [HA], [Hf] and [HI], we obtain

$$\begin{aligned}
& \| (Hx_1)(t) - (Hx_2)(t) \| \leq \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_0^{t_k} \| (t_k - s)^{\alpha-1} T(t-s) s^n [f(s, x_1(s)) - f(s, x_2(s))] \| ds + \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \| (t-s)^{\alpha-1} T(t-s) s^n [f(s, x_1(s)) - f(s, x_2(s))] \| ds + \\
& \quad + \sum_{0 < t_k < t} \| T(t-t_k) [I_k(x_1(t_k)) - I_k(x_2(t_k))] \| \leq \tag{3.3} \\
& \leq \frac{ML_f(\rho)}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_0^{t_k} (t_k - s)^{\alpha-1} s^n \| x_1(s) - x_2(s) \| ds + \\
& \quad + \frac{ML_f(\rho)}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} s^n \| x_1(s) - x_2(s) \| ds + M \sum_{0 < t_k < t} h_k \| x_1(t_k) - x_2(t_k) \| \leq \\
& \leq \left(\int_0^t (t-s)^{\alpha-1} s^n ds \right) \left[\frac{ML_f(\rho)}{\Gamma(\alpha)} + M \sum_{0 < t_k < t} h_k \right] \| x_1 - x_2 \|_{PC}.
\end{aligned}$$

Using Lemma 2.6, one can deduce that

$$\begin{aligned}
\| Hx_1 - Hx_2 \|_{PC} & \leq \frac{\Gamma(\alpha) \cdot \Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} \left[\frac{ML_f(\rho)}{\Gamma(\alpha)} + M \sum_{k=1}^{\delta} h_k \right] \| x_1 - x_2 \|_{PC} \leq \\
& \leq L \| x_1 - x_2 \|_{PC},
\end{aligned}$$

where

$$L = Mb^{\alpha+n} \frac{\Gamma(\alpha) \cdot \Gamma(n+1)}{\Gamma(\alpha+n+1)} \left[\frac{L_f(\rho)}{\Gamma(\alpha)} + \sum_{k=1}^{\delta} h_k \right].$$

(2) H is a compact operator on $PC(J, X)$.

Let \mathfrak{B} be a bounded subset of $PC(J, X)$, there exists a constant $\mu > 0$ such that $\|x\|_{PC} \leq \mu$ for all $x \in \mathfrak{B}$. Using [HI], there exists a constant N such that $\|I_k(x(t))\| \leq N$ for all $x \in \mathfrak{B}$, $t \in J$, $k = 1, 2, \dots, \delta$. Also using [Hf], there exists a constant ω such that $\|f(t, x(t))\| \leq M_f(1 + \|x\|_{PC}^m) \leq M_f(1 + \mu^m) \equiv \omega$ for all $x \in \mathfrak{B}$,

$t \in J$. Further, $H\mathfrak{B}$ is a bounded subset of $PC(J, X)$. In fact, let $x \in \mathfrak{B}$, we have

$$\begin{aligned} \|(Hx)(t)\| &\leq M\|x_0\| + \frac{M\omega}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_{t_k}^{t_{k-1}} (t_k - s)^{\alpha-1} s^n ds + \\ &\quad + \frac{M\omega}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} s^n ds + M \sum_{0 < t_k < t} N \leq \\ &\leq M\|x_0\| + MN\delta + \frac{M\omega}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} s^n ds \leq \\ &\leq M\|x_0\| + MN\delta + \frac{\omega M\Gamma(n+1)}{\Gamma(\alpha+n+1)} b^{\alpha+n}. \end{aligned}$$

Hence $H\mathfrak{B}$ is bounded.

Define

$$\Pi = H\mathfrak{B} \quad \text{and} \quad \Pi(t) = \{(Hx)(t) \mid x \in \mathfrak{B}\} \text{ for } t \in J.$$

Clearly, $\Pi(0) = \{x_0\}$ is compact, hence, it is only necessary to check that $\Pi(t) = \{(Hx)(t) \mid x \in \mathfrak{B}\}$ for $t \in (0, b]$ is also compact. For $0 < \varepsilon < t \leq b$, define

$$\Pi_\varepsilon(t) \equiv (H_\varepsilon\mathfrak{B})(t) = \{(H_\varepsilon x)(t) \mid x \in \mathfrak{B}\} \tag{3.4}$$

and the operator H_ε is defined by

$$\begin{aligned} (H_\varepsilon x)(t) &= T(\varepsilon)T(t - \varepsilon)x_0 + \\ &\quad + T(\varepsilon) \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_{t_k}^{t_{k-1}} (t_k - s)^{\alpha-1} T(t - \varepsilon - s) s^n f(s, x(s)) ds + \\ &\quad + T(\varepsilon) \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t - s)^{\alpha-1} T(t - \varepsilon - s) s^n f(s, x(s)) ds + \\ &\quad + T(\varepsilon) \sum_{0 < t_k < t} T(t - \varepsilon - t_k) I_k(x(t_k)) = \\ &= T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_{t_k}^{t_k} (t_k - s)^{\alpha-1} T(t - s) s^n f(s, x(s)) ds + \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t - s)^{\alpha-1} T(t - s) s^n f(s, x(s)) ds + \\ &\quad + \sum_{0 < t_k < t} T(t - t_k) I_k(x(t_k)), \end{aligned} \tag{3.5}$$

from which implies that $\Pi_\varepsilon(t)$ is relatively compact for $t \in (\varepsilon, b]$ due to $\{T(t), t \geq 0\}$ is a compact semigroup.

For interval $(0, t_1]$, (3.4) reduces to

$$\Pi_\varepsilon(t) \equiv (H_\varepsilon \mathfrak{B})(t) = \{(H_\varepsilon x)(t) \mid x \in \mathfrak{B}\}.$$

Combine with (3.1) and (3.5), we can deduce

$$\begin{aligned} \sup_{x \in \mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t-\varepsilon}^t (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s)) ds \right\| \leq \\ &\leq \frac{t^n \omega M}{\Gamma(\alpha)} \int_{t-\varepsilon}^t (t-s)^{\alpha-1} ds \leq \\ &\leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

It shows that the set $\Pi(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set for $t \in (0, t_1]$. Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (0, t_1]$.

For interval $(t_1, t_2]$, define

$$\begin{aligned} \Pi(t_1+0) &\equiv \Pi(t_1-0) + I_1(\Pi(t_1-0)) = \Pi(t_1) + I_1(\Pi(t_1)) = \\ &= \{(Hx)(t_1) + I_1(x(t_1)) \mid x \in \mathfrak{B}\}. \end{aligned}$$

By assumption [HI], one can verify that $I_1(\Pi(t_1))$ is relatively compact. Hence, $\Pi(t_1+0)$ is relatively compact. Then (3.4) reduces to

$$\begin{aligned} \Pi_\varepsilon(t) &\equiv (H_\varepsilon \mathfrak{B})(t) = \\ &= \left\{ (Hx)(t_1+0) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s)) ds \mid x \in \mathfrak{B} \right\}. \end{aligned}$$

By elementary computation again, we have

$$\sup_{x \in \mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| \leq \frac{\omega M}{\Gamma(\alpha)} \int_{t-\varepsilon}^t (t-s)^{\alpha-1} s^n ds \leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha+1)}.$$

Hence, $\Pi(t)$ itself is relatively compact set for $t \in (t_1, t_2]$.

In general, for any given t_k , $k = 1, 2, \dots, \delta$, we define that $x(t_i+0) = x_i$, and

$$\begin{aligned} \Pi(t_k+0) &\equiv \Pi(t_k-0) + I_k(\Pi(t_k-0)) = \\ &= \Pi(t_k) + I_k(\Pi(t_k)) = \\ &= \{(Hx)(t_k) + I_k(x(t_k)) \mid x \in \mathfrak{B}\}, \quad k = 1, 2, \dots, \delta. \end{aligned}$$

By assumption [HI] again, $I_k(\Pi(t_k))$ is relatively compact and the associated $\Pi_\varepsilon(t)$ over the interval $(t_k, t_{k+1}]$ is given by

$$\begin{aligned} \Pi_\varepsilon(t) &\equiv (H_\varepsilon \mathfrak{B})(t) = \\ &= \left\{ (Hx)(t_k + 0) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s)) ds \mid x \in \mathfrak{B} \right\}. \end{aligned}$$

Thus, we have

$$\sup_{x \in \mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| \leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha + 1)}.$$

Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (t_k, t_{k+1}]$.

Now, we repeat the procedures till the time interval which is expanded. Thus, we can obtain that the set $\Pi(t)$ itself is relatively compact for $t \in J \setminus \{t_1, t_2, \dots, t_\delta\}$ and $\Pi(t_k + 0)$ is relatively compact for $t_k, k = 1, 2, \dots, \delta$.

(3) Π is equicontinuous on the interval $(t_k, t_{k+1}), k = 1, 2, \dots, \delta$.

For interval $(0, t_1)$, we note that for $t_1 > h > 0$,

$$\|(Hx)(h) - (Hx)(0)\| \leq \|T(h) - I\| \|x_0\| + \omega M \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} h^{\alpha+n},$$

and for $t_1 \geq t+h \geq t \geq \gamma \geq 0, \gamma < h$ and $x \in \mathfrak{B}$,

$$\begin{aligned} (Hx)(t+h) - (Hx)(t) &= (T(t+h) - T(t))x_0 + \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^{t+h} (t+h-s)^{\alpha-1} T(t+h-s) s^n f(s, x(s)) ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^t (t+h-s)^{\alpha-1} [T(t+h-s) - T(t-s)] s^n f(s, x(s)) ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^t [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T(t-s) s^n f(s, x(s)) ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t-\gamma} (t+h-s)^{\alpha-1} [T(t+h-s) - T(t-s)] s^n f(s, x(s)) ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t-\gamma} [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T(t-s) s^n f(s, x(s)) ds, \end{aligned}$$

hence,

$$\begin{aligned}
\|(Hx)(t+h) - (Hx)(t)\| &\leq M\|T(h) - I\|\|x_0\| + \\
&+ \frac{\omega M t_1^n}{\Gamma(\alpha+1)} h^\alpha + \\
&+ \omega M t_1^n \frac{|h^\alpha - (h+\gamma)^\alpha|}{\Gamma(\alpha+1)} \|T(h) - I\| + \\
&+ \omega M t_1^n \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^t |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| ds + \quad (3.6) \\
&+ \omega M t_1^n \frac{|(t+h)^\alpha - (\gamma+h)^\alpha|}{\Gamma(\alpha+1)} \|T(h) - I\| + \\
&+ \omega M t_1^n \frac{1}{\Gamma(\alpha)} \int_0^{t-\gamma} |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| ds.
\end{aligned}$$

Since $\|T(h) - I\| \rightarrow 0$, $|(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| \rightarrow 0$ as $h \rightarrow 0$, thus the right hand side of (3.6) can be made as small as desired by choosing h sufficiently small. Hence, $\Pi(t)$ is equicontinuous in interval $(0, t_1)$.

In general, for time interval (t_k, t_{k+1}) , $k = 1, 2, \dots, \delta$, we similarly obtain the following inequalities

$$\begin{aligned}
\|(Hx)(t+h) - (Hx)(t)\| &\leq M\|T(h) - I\|\|x_k\| + \\
&+ \frac{\omega M t_{k+1}^n}{\Gamma(\alpha+1)} h^\alpha + \\
&+ \omega M t_{k+1}^n \frac{|h^\alpha - (h+\gamma)^\alpha|}{\Gamma(\alpha+1)} \|T(h) - I\| + \\
&+ \omega M t_{k+1}^n \frac{1}{\Gamma(\alpha)} \int_{t-\gamma}^t |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| ds + \\
&+ \omega M t_{k+1}^n \frac{|(t+h)^\alpha - (\gamma+h)^\alpha|}{\Gamma(\alpha+1)} \|T(h) - I\| + \\
&+ \omega M t_{k+1}^n \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\gamma} |(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}| ds.
\end{aligned}$$

With analogous arguments, one can verify that Π is also equicontinuous on the interval (t_k, t_{k+1}) , $k = 1, 2, \dots, \delta$.

Now, we repeat the procedures till the time interval which is expanded. Thus we obtain that the set $\Pi(t)$ itself is relatively compact for $t \in J \setminus \{t_1, \dots, t_\delta\}$ and $\Pi(t_k+0)$ is relatively compact for $t_k \in \{t_1, \dots, t_\delta\}$.

(4) H has a fixed point in $PC(J, X)$.

According to Leray-Schauder fixed point theorem, it suffices to show the following set

$$\{x \in PC(J, X) \mid x = \sigma Hx, \sigma \in [0, 1]\}$$

is a bounded subset of $PC(J, X)$. In fact, let $x \in \{x \in PC(J, X) \mid x = H(\sigma x), \sigma \in [0, 1]\}$, we have

$$\begin{aligned} \|x(t)\| &= \|H(\sigma x(t))\| \leq \\ &\leq \|T(t)(\sigma x_0)\| + \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t_{k-1}} \int_{t_{k-1}}^{t_k} \|(t_k - s)^{\alpha-1} T(t - s) s^n f(s, \sigma x(s))\| ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t \|(t - s)^{\alpha-1} T(t - s) s^n f(s, \sigma x(s))\| ds + \\ &+ \sum_{0 < t_k < t} \|T(t - t_k) I_k(\sigma x(t_k))\| \leq \\ &\leq \sigma M \|x_0\| + MM_f \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} s^n (1 + \sigma \|x(s)\|^m) ds + \\ &+ M \sum_{0 < t_k < t} (\|I_k(0)\| + \sigma h_k \|x(t_k)\|) \leq \\ &\leq M \left(\|x_0\| + M_f \frac{\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_k(0)\| \right) + \\ &+ \frac{MM_f}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} s^\gamma s^{n-\gamma} \|x(s)\|^m ds + \\ &+ M \sum_{0 < t_k < t} h_k \|x(t_k)\| \leq \\ &\leq M \left(\|x_0\| + M_f \frac{\Gamma(n + 1)}{\Gamma(\alpha + n + 1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_k(0)\| \right) + \\ &+ t^{n-\gamma} \frac{MM_f}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} s^\gamma \|x(s)\|^m ds + \\ &+ M \sum_{0 < t_k < t} h_k \|x(t_k)\|. \end{aligned}$$

Denote

$$\begin{aligned}
 f_p(t) &= \sup \left\{ M^q \left(\|x_0\| + M_f \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_k(0)\| \right)^q, \right. \\
 &\quad \left. C^{\frac{q}{p}}(p\alpha - p + 1, p\gamma) t^{(n-\gamma)q} \left(\frac{MM_f}{\Gamma(\alpha)} \right)^q t^{q(\alpha+\gamma)-1}, M^q \right\}, \\
 p \text{ and } q &\text{ such that } \frac{1}{p} + \frac{1}{q} = 1, \\
 f(t) \equiv f_2(t) &= \sup \left\{ M^2 \left(\|x_0\| + M_f \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_k(0)\| \right)^2, \right. \\
 &\quad \left. C(2\alpha - 2 + 1, 2\gamma) t^{(n-\gamma)2} \left(\frac{MM_f}{\Gamma(\alpha)} \right)^2 t^{2(\alpha+\gamma)-1}, M^2 \right\}.
 \end{aligned}$$

(i) If $\frac{1}{2} \geq \alpha > 0, -\frac{1}{2} \geq \gamma > -1$, by (1) of Lemma 2.7, it holds that for each $(t_k, t_{k+1}]$,

$$\|x(t)\| \leq \sup_{t \in [0, T_p]} \left\{ \left[(k+3)^{q-1} f_p(t) \prod_{l=1}^k (1 + (k+2)^{q-1} h_l^q f(t_l)) \right]^{\frac{1}{q}} \right\} \equiv M_k^{1*},$$

where T_p is the sup of all values of t for which

$$\begin{aligned}
 &\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (i+2)^{(q-1)m} \prod_{j=1}^{i-1} (1 + (j+2)^{q-1} h_j^q f(t_j))^m f_p^m(s) ds - \\
 &- (m-1)(k+3)^{(q-1)m} \times \prod_{j=1}^k (1 + (j+2)^{q-1} h_j^q f(t_j))^m \int_{t_k}^t f_p^m(s) ds < \frac{1}{m-1}.
 \end{aligned}$$

(ii) If $\frac{1}{2} < \alpha < 1, -\frac{1}{2} < \gamma$, by (2) of Lemma 2.7, it holds that for each $(t_k, t_{k+1}]$,

$$\|x(t)\| \leq \sup_{t \in [0, T_2]} \left\{ \left[(k+3) f(t) \prod_{l=1}^k (1 + (k+2) h_l^2 f(t_l)) \right]^{\frac{1}{2}} \right\} \equiv M_k^{2*},$$

where T_2 is the sup of all values of t for which

$$\begin{aligned}
 &\sum_{i=1}^k \int_{t_{i-1}}^{t_i} ((i+2)^m) \prod_{j=1}^{i-1} (1 + (j+2) h_j^2 f(t_j))^m f^m(s) ds - \\
 &- (m-1)(k+3)^m \times \prod_{j=1}^k (1 + (j+2) h_j^2 f(t_j))^m \int_{t_k}^t f^m(s) ds < \frac{1}{m-1}.
 \end{aligned}$$

Set $M^{1*} = \max\{M_1^{1*}, M_2^{1*}, \dots, M_\delta^{1*}\}$, $M^{2*} = \max\{M_1^{2*}, M_2^{2*}, \dots, M_\delta^{2*}\}$. We denote $M^* = \max\{M^{1*}, M^{2*}\}$. Then we have

$$\|x\|_{PC} \leq M^* \text{ for all } x \in \{x \in PC(J, X) \mid x = \sigma Hx, \sigma \in [0, 1]\}.$$

Thus, $\{x \in PC(J, X) \mid x = \sigma Hx, \sigma \in [0, 1]\}$ is a bounded subset of $PC(J, X)$. By the Leray-Schauder fixed point theorem, we obtain that H has a fixed point in $PC(J, X)$. This completes the system (1.1) has at least a PC -mild solution on J . \square

At last, an example is given to illustrate our theory. Consider the following impulsive fractional differential equations

$$\begin{cases} D_t^{\frac{1}{2}} x(t, y) = \frac{\partial^2}{\partial y^2} x(t, y) + tx^2(t, y) + t \sin(t, y), & t \in (0, 1] \setminus \{\frac{1}{2}\}, \\ \Delta x(t_1, y) = -x(t_1, y), & t_1 = \frac{1}{2}, y \in \Omega = (0, \pi), \\ x(t, y)|_{y \in \partial\Omega} = 0, & t > 0, x(0, y) = 0, y \in \Omega. \end{cases} \tag{3.7}$$

Let $X = L^2([0, \pi])$. Define

$$D(A) = \left\{ x \in X \mid \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \text{ and } x(0) = x(\pi) = 0 \right\} \text{ and } Ax = -\frac{\partial^2}{\partial y^2} x \text{ for } x \in D(A)$$

which can determine a compact C_0 -semigroup $\{T(t), t \geq 0\}$ in $L^2([0, \pi])$ such that $\|T(t)\| \leq 1$.

Denote $x(\cdot)(y) = x(\cdot, y)$, $\sin(\cdot)(y) = \sin(\cdot, y)$, $f(\cdot, x(\cdot))(y) = x^2(\cdot, y) + \sin(\cdot, y)$, $I_1(x(t_1))(y) = -x(t_1, y)$. Thus, problem (3.7) can be rewritten as

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + t^n f(t, x(t)), & \alpha = \frac{1}{2} \in (0, \frac{1}{2}], n = 1, t \in (0, 1] \setminus \{t_k\}, \\ \Delta x(t_k) = I_k(x(t_k)), & t_k = \frac{1}{2}, k = 1, \\ x(0) = 0. \end{cases} \tag{3.8}$$

Obviously, all the assumptions in Theorem 3.1 are satisfied. Our results can be used to solve problem (3.7).

Acknowledgments

JinRong Wang acknowledge support from the National Natural Science Foundation of Guizhou Province (2010, No. 2142) and the Introducing Talents Foundation for the Doctor of Guizhou University (2009, No. 031).

W. Wei acknowledge support from the National Natural Science Foundation of China (No. 10961009).

YanLong Yang acknowledge support from the Youth Teachers Natural Science Foundation of Guizhou University (2009, No. 083).

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Received: March 25, 2010.

Revised: April 12, 2010.

Accepted: April 17, 2010.