ASYMPTOTIC BEHAVIOUR
AND APPROXIMATION OF EIGENVALUES
FOR UNBOUNDED BLOCK JACOBI MATRICES

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Abstract. The research included in the paper concerns a class of symmetric block Jacobi matrices. The problem of the approximation of eigenvalues for a class of a self-adjoint unbounded operators is considered. We estimate the joint error of approximation for the eigenvalues, numbered from 1 to \(N\), for a Jacobi matrix \(J\) by the eigenvalues of the finite submatrix \(J_n\) of order \(p n \times p n\), where \(N = \max \{k \in \mathbb{N} : k \leq rp n\}\) and \(r \in (0, 1)\) is suitably chosen. We apply this result to obtain the asymptotics of the eigenvalues of \(J\) in the case \(p = 3\).

Keywords: symmetric unbounded Jacobi matrix, block Jacobi matrix, tridiagonal matrix, point spectrum, eigenvalue, asymptotics.

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1. INTRODUCTION

Tridiagonal matrices are very useful in many problems in mathematics and in applications, and the theory and methods related to tridiagonal matrices are still developed and generalized (see [20]). In the context of advances and applications, block tridiagonal matrices are very interesting (see, e.g., [6] and [8]). This work is devoted to spectral properties of a class of block Jacobi matrices with discrete spectrum. The problem, when the linear operator defined by a Jacobi matrix has discrete spectrum, i.e., its spectrum consists of isolated eigenvalues of finite multiplicity, was already investigated and partially solved (see, e.g., [7, 10] and [12]). It is well known that sometimes it is possible to calculate exact formulas for eigenvalues of Jacobi matrices (see, e.g., [9, 18] and [11]), but it is not possible in general. So, asymptotic and approximate approaches have to be applied (see, e.g., [3–5, 12, 13, 17, 21] and [22]). Projective methods, that use finite submatrices to investigate spectral properties of operators given by infinite Jacobi matrices are applied successfully (see [1, 2, 10, 15, 16, 21]). In this paper we continue the research related to the approximation of the discrete
spectrum of selfadjoint operators in the Hilbert space $l^2(\mathbb{N})$ and generalize the results included in [16] and [15].

The paper is organized as follows. In Section 2 we introduce conditions that are needed to apply the projective method and obtain the result. The method, that is used in this paper, is based on the Volkmer’s results ([21]). Section 3 includes a generalization of the lemma, which come originally from [21], and other technical facts. In section 4 we formulate the main result of the article. There we estimate the joint error of approximation for the eigenvalues, numbered from 1 to $N$, of $J$ by the eigenvalues of the finite submatrix $J_n$ of order $pn \times pn$, where $N = \max \{k \in \mathbb{N} : k \leq rpn\}$ and $r \in (0, 1)$ is suitably chosen. Section 5 is devoted to an application of the main result to obtain asymptotic formulas for the eigenvalues of an operator that is defined by an infinite real symmetric 5-diagonal matrix and acts in the Hilbert space $l^2(\mathbb{N})$.

2. NOTATIONS AND PRELIMINARIES

The notations $(\cdot, \cdot)$ and $\| \cdot \|$ are used for an inner product and a norm, respectively, in the Euclidian space $\mathbb{C}^p$ as well as in any Hilbert spaces. Moreover, the notation $\| \cdot \|$ is also used for the operator norm.

Let $M_{k \times l}(\mathbb{C})$ be the set of complex matrices with $k$ rows and $l$ columns for any integers $k, l \geq 1$.

Next we introduce some concepts from abstract operator theory which we will need later. Let $H$ be a Hilbert space and $T : D(T) \subset H \rightarrow H$ be a self-adjoint operator in $H$. Assume that $T$ has a compact resolvent and is bounded from below in the sense that there exists $c \in \mathbb{R}$ such that $(Tf, f) \geq c \|f\|^2$ for $f \in D(T)$. Then the spectrum of $T$ consists of the eigenvalues that can be ordered non-decreasingly: $\lambda_1(T) \leq \lambda_2(T) \leq \lambda_3(T) \leq \ldots$. By the minimum-maximum principle, for all $k \in \mathbb{N}$, there holds

$$
\lambda_k(T) = \min_{E_k} \max \{(Tx, x) : x \in E_k, \|x\| = 1\},
$$

(2.1)

where the minimum is taken over all linear subspaces $E_k \subseteq D(T)$ of dimension $k$.

Denote by $x_k$ the eigenvector of $T$ associated with the eigenvalue $\lambda_k(T)$. We will assume that the system of eigenvectors $\{x_1, x_2, x_3, \ldots\}$ is orthonormal in $H$, so it forms an orthonormal basis of $H$.

Let $E_N$ be a $N$-dimensional subspace of $H$. Assume that $E_N \subset D(T)$. Denote by $P_N$ the orthogonal projection onto $E_N$ and $Q_N = I - P_N$. Let us consider the following operator on $E_N$:

$$
T_N : E_N \ni v \rightarrow P_NT v \in E_N.
$$

Denote by $\mu_i$, $1 \leq i \leq N$, the eigenvalues of $T_N$ by assuming that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$.

For any $k = 1, \ldots, N$, define

$$
L^{(k)} = (L_{i,j})_{i,j=1,\ldots,k} \in M_{k \times k}(\mathbb{C}) \quad \text{with} \quad L_{i,j} = (Q_Nx_i, x_j),
$$
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and

\[ M^{(k)} = (M_{i,j})_{i,j=1,...,k} \in M_{k \times k}(\mathbb{C}) \text{ with } M_{i,j} = ((P_NTP_N - T)x_i, x_j). \]

The following lemma is fundamental to obtain the results in this paper.

**Lemma 2.1** (Volkmer [21]). If \( \|L^{(k)}\| < 1 \) then

\[ 0 \leq \mu_k - \lambda_k(T) \leq \frac{\|M^{(k)} + \lambda_k(T)L^{(k)}\|}{1 - \|L^{(k)}\|}, \]

where \( 1 \leq k \leq n \).

Let \( p \geq 1 \) be an integer and also denote

\[ l^2(\mathbb{N}, \mathbb{C}^p) = \left\{ \{f_n\}_{n=1}^\infty : f_n \in \mathbb{C}^p, n \geq 1, \text{ and } \sum_{k=1}^\infty \|f_k\|^2 < +\infty \right\}. \]

Consider a Jacobi operator \( J \) in the Hilbert space \( l^2 = l^2(\mathbb{N}, \mathbb{C}^p) \) given by the symmetric block Jacobi matrix

\[
J = \begin{pmatrix}
D_1 & C_1^* & 0 & \cdots & \cdots \\
C_1 & D_2 & C_2^* & 0 & \cdots \\
0 & C_2 & D_3 & C_3^* & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}, \quad D_n = D_n^*, C_n \in M_{p \times p}(\mathbb{C}), \ n \geq 1, \quad (2.2)
\]

more exactly, \( J \) acts on the maximum domain

\[ D(J) = \left\{ \{f_n\}_{n=1}^\infty \in l^2 : (C_{n-1}f_{n-1} + D_nf_n + C_n^*f_{n+1})_{n=1}^\infty \in l^2 \right\}, \quad (2.3) \]

and it is defined by

\[ Jf = (C_{n-1}f_{n-1} + D_nf_n + C_n^*f_{n+1})_{n=1}^\infty \text{ for } f = \{f_n\}_{n=1}^\infty \in D(J), \]

where \( f_n \in \mathbb{C}^p, n \geq 1 \) and \( C_0 := 0 \).

Denote

\[ d_n^{\text{min}} = \inf \{(D_nf, f) : f \in \mathbb{C}^p, \|f\| = 1\}, \quad (2.4) \]

\[ d_n^{\text{max}} = \sup \{(D_nf, f) : f \in \mathbb{C}^p, \|f\| = 1\}. \quad (2.5) \]

We assume the following conditions:

(C1) \( D_n = D_n^* \) for \( n \geq 1 \) and there exist \( \alpha > 0, \delta_1 \geq \delta_2 > 0 \) and \( \{\epsilon_n\}_{n=1}^\infty \subset [0, +\infty) \), \( \lim_{n \to \infty} \epsilon_n = 0 \), such that

\[ \delta_2 n^\alpha (1 - \epsilon_n) \leq d_n^{\text{min}} \leq d_n^{\text{max}} \leq \delta_1 n^\alpha (1 + \epsilon_n), \ n \geq 1; \]
(C2) there exist \( \beta \in \mathbb{R} \) and \( S > 0 \) such that
\[
\|C_n\| \leq Su^n, \ n \geq 1;
\]

(C3) \( \alpha > \beta \).

**Proposition 2.2.** If (C1)–(C3) are satisfied then:

1. \( D(J) = \{\{f_n\}_{n=1}^\infty \in l^2 : \{D_n f_n\} \in l^2\}\),
2. \( J \) is a selfadjoint operator in \( l^2 \),
3. \( J \) is bounded from below,
4. \( (J - \lambda)^{-1} \) is compact for any \( \lambda \) belonging to the resolvent set of \( J \).

**Proof.** Let
\[
c = \inf\{d_n^{\text{min}} - 2Sn^\beta : n \geq 1\},
\]
then (C1)–(C3) yield \( c \in \mathbb{R} \). Denote
\[
A = \begin{pmatrix} D_1 & 0 & 0 & \cdots \\ 0 & D_2 & 0 & 0 & \cdots \\ 0 & 0 & D_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & C_1^* & 0 & \cdots \\ C_1 & 0 & C_2^* & 0 & \cdots \\ 0 & C_2 & 0 & C_3^* & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.
\]

Let \( \lambda \in (-\infty, c) \). For \( n \geq 1 \), \( D_n - \lambda \) is invertible and
\[
\|(D_n - \lambda)^{-1}\| \leq (d_n^{\text{min}} - \lambda)^{-1},
\]
because \( \lambda < d_n^{\text{min}} \). Moreover, the operator given by the matrix \( A - \lambda \) is also invertible, \( (A - \lambda)^{-1} \) is a compact operator on \( l^2 \) and
\[
\|(A - \lambda)^{-1}\| \leq \sup_{n \geq 1} (d_n^{\text{min}} - \lambda)^{-1} < +\infty,
\]
because
\[
\lim_{n \to \infty} (d_n^{\text{min}} - \lambda)^{-1} = 0.
\]

Next calculate
\[
B(A - \lambda)^{-1} = \begin{pmatrix} 0 & C_1^*(D_2 - \lambda)^{-1} & 0 & \cdots \\ C_1(D_1 - \lambda)^{-1} & 0 & C_2^*(D_3 - \lambda)^{-1} & 0 & \cdots \\ 0 & C_2(D_2 - \lambda)^{-1} & 0 & C_3^*(D_4 - \lambda)^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.
\]

The operator norm for the matrix \( B(A - \lambda)^{-1} \) is estimated as follows
\[
\|B(A - \lambda)^{-1}\| \leq 2 \sup\{Sn^\beta(d_n^{\text{min}} - \lambda)^{-1} : n \geq 1\},
\]
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Because

$$\|C_n(D_n - \lambda)^{-1}\| \leq S_n^\beta(d_n^{min} - \lambda)^{-1}, \quad n \geq 1,$$

and

$$\|C_{n-1}^*(D_n - \lambda)^{-1}\| \leq S_n^\beta(d_n^{min} - \lambda)^{-1}, \quad n \geq 2.$$  

Clearly, \(\lim_{n \to \infty} n^\beta(d_n^{min} - \lambda)^{-1} = 0\), so

$$2 \sup_{n \geq 1} \{ S_n^\beta(d_n^{min} - \lambda)^{-1} \} = 2 S_{n_0}^\beta(d_{n_0}^{min} - \lambda)^{-1}$$

denotes a \(n_0 \in \mathbb{N}\). Notice that

$$2 S_{n_0}^\beta(d_{n_0}^{min} - \lambda)^{-1} < 1 \iff \lambda < d_{n_0}^{min} - 2 S_{n_0}^\beta.$$

The last inequality is satisfied because \(\lambda < c\). Thus \(\|B(A - \lambda)^{-1}\| < 1\) and we observe that the infinite matrix \(I + B(A - \lambda)^{-1}\) acts as a bounded and boundedly invertible operator in \(l^2\).

Notice that the matrices \(J, A\) and \(B\) satisfy the following formal identity:

$$J - \lambda = A - \lambda + B = (I + B(A - \lambda)^{-1})(A - \lambda).$$

Consequently

$$D(J) = D(A) = \{ \{ f_n \} : (D_n f_n) \in l^2 \};$$

moreover,

$$(J - \lambda)^{-1} = (A - \lambda)^{-1}(I + B(A - \lambda)^{-1})^{-1}.$$  

Thus \((J - \lambda)^{-1}\) is compact for \(\lambda < c\) and, therefore, for all \(\lambda\) from the resolvent set. In particular, due to the fact that \(J\) is symmetric, it follows that \(J\), in fact, is a self-adjoint operator in \(l^2\). Consequently, we had proved that \(J\) is bounded from below by a lower bound \(c\) and \((Jf, f) \geq c\|f\|^2\) for \(f \in D(J)\) because \((-\infty, c)\) is included in the resolvent set of \(J\).

Let \(J\) be an operator given by (2.2) and assume (C1)–(C3). The spectrum of \(J\) consists of the sequence of the eigenvalues of finite multiplicities only:

$$\sigma(J) = \{ \lambda_k(J) : k = 1, 2, 3, \ldots \},$$

and we can assume

$$\lambda_1(J) \leq \lambda_2(J) \leq \lambda_3(J) \leq \ldots.$$  

Let \(x_i \in l^2\) be an eigenvector of \(J\), such that \(Jx_i = \lambda_i(J)x_i\) (\(i = 1, 2, 3, \ldots\)). Moreover, we can assume \(\{x_i : i = 1, 2, 3, \ldots\}\) is an orthonormal basis in \(l^2\). Let

$$x_i = \{ x_{i,n} \}_{n=1}^\infty,$$

where

$$x_{i,n} = (w_{i,(n-1)p+1}, w_{i,(n-1)p+2}, \ldots, w_{i,np})^T \in \mathbb{C}^p.$$
Then
\[ \|x_i\|^2 = \sum_{n=1}^{\infty} \|x_{i,n}\|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{p} |w_{i,(n-1)p+k}|^2 = 1. \]

Denote \( e_i = \{\Delta_{i,n}\}_{n=1}^{\infty} \) for \( i = 1, 2, 3, \ldots \), where \( \Delta_{i,n} \) is defined as follows. If \( i = (n-1)p + k \), where \( n \geq 1 \) and \( k \in \{1, 2, \ldots, p\} \), then
\[ \Delta_{i,m} = (0, 0, \ldots, 0)\top \in \mathbb{C}^p, \text{ for } m \neq n, \]
and
\[ \Delta_{i,n} = (\delta_{1,k}, \delta_{2,k}, \ldots, \delta_{p,k})\top \in \mathbb{C}^p, \text{ where } \delta_{t,s} = \begin{cases} 0, & t \neq s, \\ 1, & t = s. \end{cases} \]
The system \( \{e_i : i = 1, 2, 3, \ldots\} \) is the canonical orthonormal basis in \( l^2 = l^2(\mathbb{N}, \mathbb{C}^p) \).

Put
\[ E_n = \text{span}\{e_1, e_2, \ldots, e_{np}\}. \] (2.8)
then \( \dim E_n = np \). Let \( P_n \) be an orthogonal projection on \( E_n \), and let
\[ J_n : E_n \ni x \mapsto P_n Jx \in E_n. \] (2.9)
Then \( J_n \) is represented, with respect the canonical basis of \( E_n \), as the matrix
\[
\begin{pmatrix}
D_1 & C_1^* & 0 & 0 \\
C_1 & D_2 & C_2^* & 0 \\
0 & C_2 & D_3 & C_3^* \\
& & & \ddots \\
& 0 & C_{n-2} & D_{n-1} & C_{n-1}^* \\
& & & 0 & C_{n-1} & D_n
\end{pmatrix}.
\] (2.10)

Denote by
\[ \mu_{1,n} \leq \mu_{2,n} \leq \cdots \mu_{np-1,n} \leq \mu_{np,n} \]
the sequence of the eigenvalues of \( J_n \).

From the min-max principle we derive
\[ \lambda_k(J) \leq \mu_{k,n} \text{ and } \lambda_k(J) \leq \|J_n\| \leq Cn^{\alpha} \text{ for } k = 1, 2, \ldots, np. \]

3. AUXILIARY ESTIMATIONS

In this section we use the notations introduced in Section 2.

Denote
\[ Q_n = I - P_n. \] (3.1)

Let \( k \in \{1, \ldots, np\} \) and define the following \( k \times k \)-matrices:
\[ L^{(k,n)} = (L_{i,j}^{(n)})_{i,j=1,\ldots,k}, \text{ where } L_{i,j}^{(n)} = (Q_n x_i, x_j), \]
and
\[ M^{(k,n)} = (M^{(n)}_{i,j})_{i,j=1,...,k}, \text{ where } M^{(n)}_{i,j} = ((P_n J P_n - J) x_i, x_j). \]

**Lemma 3.1.** If \( n \in \mathbb{N} \) and \( k \in \{1,2,\ldots,np\} \), then

\[ \|L^{(k,n)}\| \leq \sum_{i=1}^{k} \|Q_n x_i\|^2; \]
\[ \|M^{(k,n)} + \lambda_k(J) L^{(k,n)}\| \leq \|C_n\| \left( \sum_{i=1}^{k} \|x_{i,n+1}\|^2 \right)^{1/2} \left( \sum_{j=1}^{k} \|x_{j,n}\|^2 \right)^{1/2} + \]
\[ \sum_{i=1}^{k} \left| \lambda_k(J) - \lambda_i(J) \right|^2 \|Q_n x_i\|^2 \right)^{1/2} \left( \sum_{j=1}^{k} \|Q_n x_j\|^2 \right)^{1/2}. \]

**Proof.** The proof follows the Volkmer’s method (see [21]). At first notice that
\[ |L^{(n)}_{i,j}| = |(Q_n x_i, x_j)| = |(Q_n x_i, Q_n x_j)| \leq \|Q_n x_i\| \|Q_n x_j\|; \] therefore, the operator norm \( \|L^{(k,n)}\| \) of the \( k \times k \) matrix can be estimated as above.

Next notice that
\[
JP_n x_j = \begin{pmatrix}
D_1 x_{i,1} + C_1^* x_{i,2} \\
C_1 x_{i,1} + D_2 x_{i,2} + C_1^* x_{i,3} \\
\vdots \\
C_{n-2} x_{i,n-2} + D_{n-1} x_{i,n-1} + C_{n-1}^* x_{i,n} \\
C_{n-1} x_{i,n-1} + D_n x_{i,n} \\
C_n x_{i,n} \\
0 \\
\vdots \\
0 \\
\end{pmatrix} = \begin{pmatrix}
\lambda_i(J) x_{i,1} \\
\lambda_i(J) x_{i,2} \\
\vdots \\
\lambda_i(J) x_{i,n-1} \\
\lambda_i(J) x_{i,n} - C_n^* x_{i,n+1} \\
C_n x_{i,n} \\
0 \\
\vdots \\
0 \\
\end{pmatrix} = \lambda_i(J) P_n x_i + \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
-C_n^* x_{i,n+1} \\
C_n x_{i,n} \\
0 \\
\vdots \\
\end{pmatrix}
\]

and
\[
P_n J P_n x_i - J x_i = \lambda_i(J) P_n x_i + \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
-C_n^* x_{i,n+1} \\
0 \\
\vdots \\
\end{pmatrix} - \lambda_i(J) x_i = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
-C_n^* x_{i,n+1} \\
0 \\
\vdots \\
\end{pmatrix} - \lambda_i(J) Q_n x_i.
\]
Then
\[ M_{ij}^{(n)} = (P_n J P_n x_i - J x_i, x_j) = -(C^*_n x_{i,n+1}, x_{j,n}) - \lambda_i(J)(Q_n x_i, x_j); \]
motor,
\[ |M_{ij}^{(n)} + \lambda_k L_{ij}^{(n)}| = |(P_n J P_n x_i - J x_i, x_j) + \lambda_k(Q_n x_i, x_j)| = \]
\[ \leq |(C^*_n x_{i,n+1}, x_{i,n})| + |\lambda_i(J) - \lambda_k(J)|(Q_n x_i, x_j)| \leq \]
\[ \leq \|C_n\| \|x_{i,n+1}\| \|x_{i,n}\| + |\lambda_i(J) - \lambda_k(J)| \|Q_n x_i\| \|Q_n x_j\|. \]

Finally, from the above estimation we derive the second inequality of the lemma. 

Define
\[ p_n = \max\{\epsilon_k k^n : k \leq n\}, \quad q_n = \max\{S_n^\beta, S\}, \quad n \geq 1. \quad (3.2) \]

**Lemma 3.2.** Under assumptions (C1) and (C2), the sequences \{p_n\} and \{q_n\} are non-decreasing and
\[ \lim_{n \to \infty} \frac{p_n}{n^\alpha} = 0. \]

**Proof.** By definition \( p_n = \epsilon_k k^n\), for some \( k_n \leq n \). Assume that \{p_n\} is unbounded, then \( \lim_{n \to \infty} k_n = +\infty \). So,
\[ \frac{|p_n|}{n^\alpha} = \frac{\epsilon_k k^n}{n^\alpha} \leq \epsilon_k \to 0, \quad n \to \infty, \]
because \( \lim_{n \to \infty} \epsilon_k = 0. \)

The following estimates are satisfied for the eigenvalues of \( J \).

**Proposition 3.3.** Assume that (C1)–(C3) are fulfilled. Let \( j \geq 1, l \in \{1, 2, \ldots, p\} \) and \( i = (j - 1)p + l \), then
\[ \lambda_i(J) \leq \|J_j\| \leq \delta_j j^\alpha + p_j + 2q_j. \]

**Proof.** Notice that \( i \leq pj \). By applying the minimum-maximum principle (2.1) and using (C1)–(C3), we derive the following estimate
\[ \lambda_i \leq \mu_{i,j} \leq \|J_j\| \leq \max_{1 \leq k \leq j} \|C_k\| \leq \max_{1 \leq k \leq j - 1} \|C_k\| \leq \|J_j\| \leq \delta_j j^\alpha + p_j + 2q_j. \]

Let \( 0 < r < r' < (\delta_2/\delta_1)^{1/\alpha}, 1 \leq j \leq r'k \) and \( i = (j-1)p+l \), where \( l \in \{1, 2, \ldots, p\} \). Next, follows Volkmer ([21]), we define
\[ f_{i,k} = \frac{\|C_{k-1}\|}{\alpha k^{\min} - \|J_j\| - \|C_k\|}, \quad k \geq n. \]
If \( k \geq n \) then \( j \leq r'k \), so from Lemma 3.2 and Proposition 3.3

\[
 f_{i,k} \leq \frac{Sk^\beta}{\delta_2 k^n (1 - \epsilon_k) - \delta_1 j^\alpha - \delta_1 p_j - 2S j^\beta - Sk^\beta} \leq \frac{Sk^\beta}{\delta_2 k^n (r'k)^\alpha - \delta_2 k^n \epsilon_k - \delta_1 p_k - 3Sk^\beta = \frac{\delta_1 k^n (\delta_2 / \delta_1 - r'^\alpha)}{\epsilon_k}}
\]

where \( \tilde{\epsilon}_k = o(k^n) \), \( k \to \infty \), i.e.,

\[
 \lim_{k \to \infty} \frac{\tilde{\epsilon}_k}{K^{\alpha}} = 0.
\]

Therefore,

\[
 f_{i,k} \leq \frac{c}{K^{\alpha - 3}} \leq \frac{1}{2} \text{ for } k \geq K_0,
\]

where \( K_0 \) is large enough and \( c > 0 \) is a constant independent of \( i \) and \( k \).

**Lemma 3.4.** Assume \((C1)–(C3)\). If \( n \geq K_0, 1 \leq j \leq r' n, i = (j - 1)p + l, 1 \leq l \leq p, \) then

\[
 ||x_{i,n}|| \leq f_{i,n} ||x_{i,n-1}||.
\]

**Proof.** If \( \lambda_i(J) \) is an eigenvalue of \( J \) and \( x_i \) is a normalized eigenvector associated to \( \lambda_i(J) \), then

\[
 C_{k-1}x_{i,k-1} + (D_k - \lambda_i(J))x_{i,k} + C_k^*x_{i,k+1} = 0, \quad k \geq 2.
\]

There exists \( k \geq n \) such that \( ||x_{i,k+1}|| \leq ||x_{i,k}|| \). Then

\[
 C_{k-1}x_{i,k-1} = -(D_k - \lambda_i(J))x_{i,k} - C_k^*x_{i,k+1},
\]

so

\[
 (C_{k-1}x_{i,k-1}, x_{i,k}) = -((D_k - \lambda_i(J))x_{i,k}, x_{i,k}) - (C_k^*x_{i,k+1}, x_{i,k}),
\]

\[
 ||C_{k-1}x_{i,k-1}, x_{i,k}|| \geq \|(D_k - \lambda_i(J))x_{i,k}, x_{i,k}\| - ||C_k^*x_{i,k+1}, x_{i,k}||
\]

and

\[
 ||C_{k-1}|| ||x_{i,k-1}|| ||x_{i,k}|| \geq d_k^{\min} ||x_{i,k}||^2 - \lambda_i(J) ||x_{i,k}||^2 - ||C_k|| ||x_{i,k+1}|| ||x_{i,k}||.
\]

Assume \( ||x_{i,k}|| \neq 0 \). Then

\[
 ||C_{k-1}|| ||x_{i,k-1}|| \geq (d_k^{\min} - \lambda_i(J)) ||x_{i,k}|| - ||C_k|| ||x_{i,k+1}|| \geq (d_k^{\min} - \lambda_i(J)) ||x_{i,k}|| - ||C_k|| ||x_{i,k}|| = (d_k^{\min} - ||J|| - ||C_k||)||x_{i,k}||.
\]

Obviously, \( k \geq K_0 \), so \( d_k^{\min} - ||J|| - ||C_k|| > 0 \),

\[
 ||x_{i,k}|| \leq \frac{||C_{k-1}||}{d_k^{\min} - ||J|| - ||C_k||} ||x_{i,k-1}|| \leq \frac{1}{2} ||x_{i,k-1}|| \leq ||x_{i,k-1}||
\]

and

\[
 ||x_{i,k}|| \leq f_{i,k} ||x_{i,k-1}||.
\]

If \( k > n \) then we can repeat this procedure to obtain \( ||x_{i,n}|| \leq f_{i,n} ||x_{i,n-1}|| \).
4. APPROXIMATION FOR EIGENVALUES
OF UNBOUNDED SELF-ADJOINT JACOBI MATRICES
WITH MATRIX ENTRIES BY THE USE OF FINITE SUBMATRICES

The main result of this article is formulated as the following theorem.

**Theorem 4.1.** Let $J$ be an operator in the Hilbert space $l^2$ defined by the infinite matrix (2.2) satisfying (C1)–(C3). Then for every $\gamma > 0$ and $r \in (0, (\delta_2/\delta_1)^{1/\alpha})$ there exists $C > 0$ such that

$$\sup_{1 \leq k \leq rnp} |\mu_{k,n} - \lambda_k(J)| \leq Cn^{-\gamma} \text{ for } n > r^{-1},$$

where $\lambda_k(J)$ is the $k$-th eigenvalue of $J$ and $\mu_{1,n} \leq \mu_{2,n} \leq \ldots \leq \mu_{pn,n}$ are the eigenvalues of the matrix $J_n$ given by (2.10).

**Proof.** Let $s \in \mathbb{N}$ be such that

$$2s(\alpha - \beta) - \alpha - 1 \geq \gamma, \quad (4.1)$$

and choose $r < r' < (\delta_2/\delta_1)^{1/\alpha}$ and $K_0 \in \mathbb{N}$ for which (3.3) is satisfied, and put

$$N_0 = \max\{K_0 + s, r' - r\}.$$

For $n \geq N_0$, $1 \leq j \leq rn$, $i = (j-1)p + l$, where $l \in \{1, 2, \ldots, p\}$, and $m > n$, by using Lemma 3.4, we deduce

$$\|x_{i,m}\| \leq f_{i,m} \cdot \|x_{i,m-1}\| \leq f_{i,m} \cdot f_{i,m-1} \cdot \ldots \cdot f_{i,n+1} \cdot \|x_{i,n}\| \leq \left(\frac{1}{2}\right)^{m-n} \|x_{i,n}\|.$$

If $j \leq rn$ and $n \geq N_0$ then $j \leq r'(n-s)$, and then

$$\|x_{i,n}\| \leq f_{i,n} \cdot f_{i,n-1} \cdot \ldots \cdot f_{i,n-s} \cdot \|x_{i,n-s}\| \leq \frac{e^s}{n(n-1)\ldots(n-s+1)} \leq \frac{M}{n^{s(\alpha-\beta)}},$$

where $M = M(s, \alpha, \beta)$ is a positive constant independent of $i$ and $n$. Now, we use Lemma 3.1 to continue the proof. At first notice that

$$\|Q_n x_i\|^2 = \sum_{m=n+1}^{\infty} \|x_{i,m}\|^2 \leq \sum_{m=n+1}^{\infty} \left(\frac{1}{2}\right)^{m-n} \|x_{i,n}\|^2 \leq \frac{M^2}{n^{2s(\alpha-\beta)}}.$$

Let $k \leq rnp$. Then

$$\|L^{(k,n)}\| \leq \sum_{i=1}^{k} \|Q_n x_i\|^2 \leq prn \frac{M^2}{n^{2s(\alpha-\beta)}} = \frac{C}{n^{2s(\alpha-\beta)-1}}.$$

Since the sequence $\{\lambda_m(J)\}$ is non-decreasing and since

$$\lim_{m \to \infty} \lambda_m(J) = +\infty,$$
it follows
\[ \max\{|\lambda_m(J)| : \lambda_m(J) < 0\} = \mu < +\infty, \]
and then by using Proposition 3.3, we obtain
\[ \lambda_k(J) - \lambda_i(J) \leq \lambda_k(J) + \mu \leq C_0 n^\alpha \quad \text{for} \quad i \leq k. \]

Thus
\[
\sum_{i=1}^k |\lambda_k(J) - \lambda_i(J)|^2 \|Q_n x_i\|^2 \leq pr n C_0^2 n^{2\alpha} M^2 \frac{M'}{n^{2\alpha(\alpha-\beta)}} = \frac{M'}{n^{2\alpha(\alpha-\beta)-2\alpha-1}}.
\]

Next,
\[
\sum_{i=1}^k \|x_{i,n}\|^2 \leq M^2 pr \frac{M'}{n^{2\alpha(\alpha-\beta)-1}},
\]
\[
\sum_{i=1}^k \|x_{i,n+1}\|^2 \leq \sum_{i=1}^k f_{i,n} \|x_{i,n}\|^2 \leq \frac{c^2}{n^{2(\alpha-\beta)}} \sum_{i=1}^k \|x_{i,n}\|^2 \leq \frac{c^2 M^2 pr}{n^{2(\alpha+1)(\alpha-\beta)-1}}
\]
and, finally, from Lemma 3.1 we derive
\[
\|M^{(k,n)} + \lambda_k(J)L^{(k,n)}\| \leq \frac{cM^2 pr}{n^{2\alpha(\alpha-\beta)+\alpha-\beta-1}} + \frac{CM'}{n^{2\alpha(\alpha-\beta)-\alpha-1}} \leq M'' \frac{M''}{n^{\gamma}}.
\]

Assume
\[
\frac{pr M^2}{n^{2\alpha(\alpha-\beta)-1}} \leq \frac{1}{2} \quad \text{for} \quad n \geq N_1,
\]
where \(N_1\) is large enough and \(N_1 > N_0\). Then
\[
\|L^{(k,n)}\| \leq \frac{1}{2} < 1
\]
and, by Lemma 2.1,
\[
\mu_{k,n} - \lambda_k(J) \leq \frac{2M''}{n^{\gamma}} \quad \text{for} \quad k \leq rnp.
\]

Finally,
\[
\sup_{1 \leq k \leq rnp} |\mu_{k,n} - \lambda_k(J)| \leq \frac{2M''}{n^{\gamma}}
\]
for \(n \geq N_1\), and the proof is complete.

Theorem 4.1 generalizes the results included in [16] and [15].
5. ASYMPTOTICS

Theorem 4.1 can be applied to obtain an asymptotic behaviour of the discrete spectrum for a concrete class of operators acting in $l^2(N)$. Let us consider a 5-diagonal symmetric infinite matrix

$$ J = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & \cdots \\ \beta_1 & \alpha_2 & \beta_2 & \gamma_2 & 0 \\ \gamma_1 & \beta_2 & \alpha_3 & \beta_3 & \gamma_3 \\ 0 & \gamma_2 & \beta_3 & \alpha_4 & \beta_4 \\ 0 & 0 & \gamma_3 & \beta_4 & \alpha_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \quad (5.1) $$

We identify this matrix with a block Jacobi matrix with $3 \times 3$-matrix entries

$$ \begin{pmatrix} D_1 & C_1^* & 0 & \cdots \\ C_1 & D_2 & C_2^* & 0 \\ 0 & C_2 & D_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, $$

where

$$ D_n = \begin{pmatrix} \alpha_{3n-2} & \beta_{3n-2} & \gamma_{3n-2} \\ \beta_{3n-2} & \alpha_{3n-1} & \beta_{3n-1} \\ \gamma_{3n-2} & \beta_{3n-1} & \alpha_{3n} \end{pmatrix}, \quad C_n = \begin{pmatrix} 0 & \gamma_{3n-1} & \beta_{3n-2} \\ 0 & 0 & \gamma_{3n} \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.2) $$

We introduce the following conditions:

(A1) $\alpha_n \in \mathbb{R}$ and $\alpha_n = \delta n^\alpha (1 + \Delta_n)$, $n \geq 1$, where $\alpha > 0$ and $\lim_{n \to \infty} \Delta_n = 0$;

(A2) $\beta_n, \gamma_n \in \mathbb{R}$ and there exist $\beta \in \mathbb{R}$ and $B > 0$ such that $|\beta_n|, |\gamma_n| \leq Bn^\beta$, $n \geq 1$;

(A3) $\alpha > \beta$;

(A4) $\alpha > \beta + 1$, $\alpha \geq 1$ and $\Delta_n - \Delta_{n-1} = o(\frac{1}{n^2})$, $n \to \infty$.

In this section we use the standard notations $o(a_n)$ and $O(a_n)$, as $n \to \infty$.

Apply formulas (2.4) and (2.5) to (5.2) and notice that

$$ d_n^{\min} \geq \min\{\alpha_{3n-2}, \alpha_{3n-1}, \alpha_{3n}\} - 6B(3n)^\beta \geq \delta(3n-2)^\alpha - \delta(3n-2)^\alpha \max\{\Delta_{3n}, |\Delta_{3n-1}|, |\Delta_{3n-2}|\} - 6B3^3n^3 = $$
Asymptotic behaviour and approximation of eigenvalues...

\[ = 3^\alpha \delta n^\alpha (1 + \epsilon'_n), \quad \text{where } \epsilon'_n = o(1), \]

and

\[ d^{\max}_n \leq \max\{\alpha_{3n-2}, \alpha_{3n-1}, \alpha_{3n}\} + 6B(3n)^3 = 3^\alpha \delta n^\alpha (1 + \epsilon''_n), \quad \text{where } \epsilon''_n = o(1). \]

Thus, \( \epsilon_n = \max\{|\epsilon'_n|, |\epsilon''_n|\} = o(1), \ n \to \infty, \) and

\[ 3^\alpha \delta n^\alpha (1 - \epsilon_n) \leq d_{\min}^n \leq d_{\max}^n \leq 3^\alpha \delta n^\alpha (1 + \epsilon_n), \ n \geq 1. \]

It is easy to verify that

\[ \|C_n\| \leq 3^{1+\beta} B^{\beta} n^\beta, \quad n \geq 1. \]

Therefore, (A1)–(A3) yield (C1)–(C3), where \( \delta_1 = \delta_2 = 3^\alpha \delta. \) Then \( J \) defines an operator in \( l^2(N, \mathbb{C}) \) which is identified with a Jacobi operator in \( l^2(N, \mathbb{C}^3). \) Moreover, we can apply Theorem 4.1 with any \( r \in (0, 1) \) and \( \gamma > 0 \) to the operator given by the matrix \( J. \)

Define the Gerschgorin radius (see [19])

\[ R_n = |\beta_n| + |\gamma_n| + |\beta_{n-1}| + |\gamma_{n-2}| \quad (5.3) \]

and let

\[ K_n = \{x \in \mathbb{R} : |\alpha_n - x| \leq R_n\}. \quad (5.4) \]

**Lemma 5.1.** If (A1)–(A4) are satisfied then

1. \( \alpha_{n+1} - \alpha_n = R_{n+1} - R_n = \delta n^{\alpha-1} + o(n^{\alpha-1}), \ n \to \infty; \)

2. there exists \( n_0 > 1 \) such that \( K_n \cap \left( \bigcup_{m \neq n} K_m \right) = \emptyset \) for \( n \geq n_0. \)

Denote

\[ J^k_l = \begin{pmatrix} \alpha_k & \beta_k & \gamma_k & 0 \\ \beta_k & \alpha_{k+1} & \beta_{k+1} & \gamma_{k+1} \\ \gamma_k & \beta_{k+1} & \alpha_{k+2} & \beta_{k+2} \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_{l-2} & \beta_{l-1} & \alpha_l \end{pmatrix}, \quad k \leq l, \]

and

\[ J_n = J^1_{3n}. \quad (5.5) \]

Let \( \mu_{1,n}, \mu_{2,n}, \ldots, \mu_{3n,n} \) be the non-decreasingly arranged sequence of the eigenvalues of the matrix \( J_n. \)

**Lemma 5.2.** Let \( \gamma > 0. \) If (A1)–(A3) are satisfied then

\[ \lambda_n(J) = \mu_{n,n} + O(n^{-\gamma}), \ n \to \infty. \]
Proof. Notice that $p = 3$. Let $r = \frac{1}{3}$, then $rnp = n$. From Theorem 4.1 we have
\[ \sup_{1 \leq i \leq n} |\mu_{i,n} - \lambda_i(J)| \leq Cn^{-\gamma}, \]
where $C$ is independent of $n$ and $i$. Thus
\[ |\mu_{n,n} - \lambda_n(J)| \leq Cn^{-\gamma}. \]

Remark 5.3. We apply the Gerschgorin theorem (see [19]) and the generalized Gerschgorin theorem, which is given in the book of Saad (see Theorem 3.12, [19]), to the symmetric matrix $J_n$, and we observe that if $n_0 < i \leq 3n$ then $\mu_{i,n} \in K_i$, where $n_0$ is given in Lemma 5.1 and $K_i$ is defined by (5.4). Moreover, from Theorem 4.1 we derive
\[ \lambda_i(J) = \lim_{n \to \infty} \mu_{i,n} \in K_i, \quad i > n_0. \]

Lemma 5.4 (Lütkepohl [14]). Let $A \in M_{k \times k}$, $D \in M_{l \times l}$, $B, C^\top \in M_{l \times k}$. Then
\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{cases} 
\det A \det(D - CA^{-1}B), & \text{if } A \text{ is invertible,} \\
\det D \det(A - BD^{-1}C), & \text{if } D \text{ is invertible.}
\end{cases}
\]

Theorem 5.5. Let $J$ be an operator defined in the Hilbert space $l^2(\mathbb{N})$ by the matrix (5.1). Under (A1)–(A4) the following asymptotic formula for the discrete spectrum of $J$ is satisfied:
\[ \lambda_n(J) = \alpha_n - \frac{\beta_n^2}{\alpha_{n-1} - \alpha_n} - \frac{\gamma_n^2}{\alpha_{n-2} - \alpha_n} + O\left(\frac{\gamma_n^2}{n^{2(\alpha-1)}}\right), \]
as $n \to \infty$.

Proof. Let $n > n_0 + 2$, where $n_0$ is given in Lemma 5.1, $N = 3n$ and $\lambda = \mu_{n,n}$ be the $n$-th eigenvalue of $J_n = J^\lambda_N \in M_{N \times N}$. Then
\[ J^\lambda_N - \lambda = \begin{pmatrix} J^\lambda_{n-2} - \lambda & E^*_n \\ E_n & J^\lambda_{n-1} - \lambda \end{pmatrix}, \]
where
\[ E_n = \begin{pmatrix} 0 & \cdots & \gamma_n - 3 & \beta_n - 2 \\ 0 & \cdots & 0 & \gamma_n - 2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{(N-n+2) \times (n-2)}. \]

$J^\lambda_{n-2}$ is a real symmetric matrix, so
\[ ||J^\lambda_{n-2}|| = \max\{\mu \in \mathbb{R} : \mu \text{ is an eigenvalue of } J^\lambda_{n-2} \} \in K_{n-2}, \]
\[ ||J^\lambda_{n-2}|| \leq \alpha_{n-2} + R_{n-2} < \alpha_n - R_n \leq \mu_{n,n} = \lambda; \]
therefore, $J^\lambda_{n-2} - \lambda$ is invertible and, from Lemma 5.4, we derive
\[ \det(J^\lambda_N - \lambda) = \det(J^\lambda_{n-2} - \lambda) \det(J^\lambda_{n-1} - \lambda - E_n(J^\lambda_{n-2} - \lambda)^{-1}E^*_n). \]
Denote

\[(J_{n-2}^1 - \lambda)^{-1} = (m_{i,j}(\lambda))_{i,j=1}^{n-2}.\]  

Then

\[E_n(J_{n-2}^1 - \lambda)^{-1}E_n^* = \begin{pmatrix} a(\lambda) & b(\lambda) & 0 & \cdots \\ b(\lambda) & d(\lambda) & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & 0 \end{pmatrix} \in M_{(N-n+2) \times (N-n+2)},\]

where

\[a(\lambda) = \gamma_{n-3}m_{n-3,n-3}(\lambda) + \beta_{n-2}^2m_{n-2,n-2}(\lambda) + 2\gamma_{n-3}\beta_{n-3}m_{n-2,n-3}(\lambda),\]  

\[b(\lambda) = \gamma_{n-2}\gamma_{n-3}m_{n-3,n-2}(\lambda) + \gamma_{n-2}\beta_{n-2}m_{n-2,n-2}(\lambda),\]  

\[d(\lambda) = \gamma_{n-2}^2m_{n-2,n-2}(\lambda).\]  

Applying Lemma 5.4, we deduce

\[\det(J_{N-1}^n - \lambda - E_n(J_{n-2}^1 - \lambda)^{-1}E_n^*) = \det(J_{n+1}^{n+1} - \lambda - E(\lambda) - E_n^* (J_{N+2}^n - \lambda)^{-1}E_{n+1}'),\]

where

\[E(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) & 0 \\ b(\lambda) & d(\lambda) & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_{3 \times 3}\]

and

\[E_{n+1}' = \begin{pmatrix} 0 & \gamma_n & \beta_{n+1} \\ 0 & 0 & \gamma_{n+1} \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \end{pmatrix} \in M_{(N-n-1) \times 3}.\]

Notice that \(J_{N+2}^n - \lambda\) is invertible because from (5.4) and Remark 5.3

\[\lambda = \mu_{n,n} \leq \alpha_n + R_n < \alpha_{n+2} - R_{n+2} \leq \min\{\mu : \mu \text{ is an eigenvalue of } J_{N+2}^n\}.\]

Let

\[(J_{N+2}^n - \lambda)^{-1} = (s_{i,j}(\lambda))_{i,j=1}^{N-n-1}.\]  

Thus

\[E_{n+1}' (J_{N+2}^n - \lambda)^{-1} E_{n+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d'(\lambda) & b'(\lambda) \\ 0 & b'(\lambda) & c'(\lambda) \end{pmatrix} \in M_{3 \times 3},\]
where
\[ d'(\lambda) = \gamma_n^2 s_{1,1}(\lambda), \quad (5.13) \]
\[ c'(\lambda) = \beta_{n+1}^2 s_{1,1}(\lambda) + \gamma_n^2 s_{2,2}(\lambda) + 2\beta_{n+1}\gamma_{n+1} s_{1,2}(\lambda), \quad (5.14) \]
\[ b'(\lambda) = \gamma_n(\beta_{n+1} s_{1,1}(\lambda) + \gamma_{n+1} s_{2,1}(\lambda)). \quad (5.15) \]

From (5.6) and (5.11) we deduce
\[ \det(J^1 - \lambda) = \det(J^1_{n-2} - \lambda)\det(J^2_{n-2} - \lambda)\det A_n(\lambda), \]
where
\[
A_n(\lambda) = \begin{pmatrix}
\alpha_{n-1} - \lambda - a(\lambda) & \beta_{n-1} - b(\lambda) & \gamma_{n-1} \\
\beta_{n-1} - b(\lambda) & \alpha_{n-1} - \lambda - d(\lambda) - d'(\lambda) & \beta_n - b'(\lambda) \\
\gamma_{n-1} & \beta_n - b'(\lambda) & \alpha_{n+1} - \lambda - c'(\lambda)
\end{pmatrix}.
\]

The matrices \( J^1_{n-2} - \lambda \) and \( J^2_{n-2} - \lambda \) are invertible and \( \lambda = \mu_{n,n} \in K_n \) is an eigenvalue of \( J^1_{n-2} \), so \( \det(J^1_{n-2} - \lambda) = 0 \), or, equivalently, \( \det A_n(\lambda) = 0 \), or also
\[ \lambda = \alpha_n - d(\lambda) - d'(\lambda) + F_n(\lambda) + G_n(\lambda), \quad (5.16) \]
where
\[
F_n(\lambda) = \frac{-(\beta_{n-1} - b(\lambda))^2}{\alpha_{n-1} - \lambda - a(\lambda)} + \frac{-(\beta_n - b'(\lambda))^2}{\alpha_{n+1} - \lambda - c'(\lambda)} + \frac{2\gamma_{n-1}(\beta_{n-1} - b(\lambda))(\beta_n - b'(\lambda))}{(\alpha_{n-1} - \lambda - a(\lambda))(\alpha_{n+1} - \lambda - c'(\lambda))}
\]
and
\[
G_n(\lambda) = \frac{\gamma_{n-1}^2 F_n(\lambda)}{(\alpha_{n-1} - \lambda - a(\lambda))(\alpha_{n+1} - \lambda - c'(\lambda)) - \gamma_{n-1}^2}.
\]

Observe that, under conditions (A1)–(A4), if \( 1 \leq k \leq n - 2 \), \( |\lambda - \alpha_n| \leq R_n \) and \( x \in \mathbb{R}^k \), then
\[
\|(J^1_k - \lambda)x\| \geq \lambda\|x\| - \|J^1_k x\| \geq \lambda\|x\| - \|J^2_k\||\|x\| \geq (\alpha_n - R_n - \alpha_k - R_k)\|x\| \geq cn^{\alpha - 1}\|x\|,
\]
for a constant \( c > 0 \); therefore,
\[
\|(J^1_k - \lambda)^{-1}\| \leq (cn^{\alpha - 1})^{-1}.
\]
Then, by Lemma 5.4, from (5.7) we derive
\[
m_{n-3,n-3}(\lambda) = \frac{1}{\alpha_{n-3} - \alpha_n} + O\left(\frac{n^\beta}{n^2(\alpha-1)}\right),
m_{n-2,n-2}(\lambda) = \frac{1}{\alpha_{n-2} - \alpha_n} + O\left(\frac{n^\beta}{n^2(\alpha-1)}\right),
m_{n-2,n-3}(\lambda) = m_{n-3,n-2}(\lambda) = O\left(\frac{n^\beta}{n^2(\alpha-1)}\right),
\]
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if $|\lambda - \alpha_n| = O(n^\beta)$. Then we calculate the following asymptotic equalities

\[ a(\lambda) = \frac{\gamma_{n-3}}{\alpha_{n-3} - \alpha_n} + \frac{\beta_{n-2}^2}{\alpha_{n-2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \quad (5.19) \]

\[ b(\lambda) = \frac{\beta_{n-2}^2 \gamma_{n-2}}{\alpha_{n-2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \quad (5.20) \]

\[ d(\lambda) = \frac{\gamma_{n-2}^2}{\alpha_{n-2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \quad (5.21) \]

for $\lambda = \mu_{n,n} \in K_n$, where $K_n$ is given by (5.4).

From (A1)–(A4) we derive also $||(J^k_N - \lambda)^{-1}|| \leq (cn^{n-1})^{-1}$ for $k \geq n + 2$ and $\lambda = \mu_{n,n}$. Then from Lemma 5.4 and equation (5.12) we deduce

\[ s_{1,1}(\lambda) = \frac{1}{\alpha_{n+2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \quad (5.22) \]

\[ s_{2,2}(\lambda) = \frac{1}{\alpha_{n+3} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \]

and

\[ s_{1,2}(\lambda) = s_{2,1}(\lambda) = O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right). \]

Then

\[ d'(\lambda) = \frac{\gamma_n^2}{\alpha_{n+2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right), \quad (5.23) \]

\[ c'(\lambda) = \frac{\beta_{n+1}^2}{\alpha_{n+2} - \alpha_n} + \frac{\gamma_{n+2}^2}{\alpha_{n+3} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \]

and

\[ u'(\lambda) = \frac{\gamma_n \beta_{n+1}}{\alpha_{n+2} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right). \quad (5.24) \]

Notice that if $\lambda = \mu_{n,n}$ then $\lambda = \alpha_n + O(n^\beta)$ and using (5.19)–(5.21), (5.22)–(5.24), (5.17) and (5.18) we have

\[ F_n(\lambda) = -\frac{(\beta_{n-1} - \gamma_n \beta_{n-2}/(\alpha_{n-1} - \alpha_n))^2}{\alpha_{n-1} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \]

\[ = -\frac{\beta_{n-1}^2}{\alpha_{n-1} - \alpha_n} + O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right) \]

and

\[ G_n(\lambda) = O\left(\frac{n^{3\beta}}{n^{2(\alpha-1)}}\right). \]
Thus
\[ \lambda = \mu_{n,n} = \alpha_n - \frac{\beta_{n-1}^2}{\alpha_{n-1} - \alpha_n} - \frac{\beta_n^2}{\alpha_{n+1} - \alpha_n} + \frac{\gamma_{n-2}^2}{\alpha_{n-2} - \alpha_n} + \frac{\gamma_n^2}{\alpha_{n+2} - \alpha_n} + O \left( \frac{n^{3\beta}}{n^{2(\alpha - 1)}} \right). \] (5.25)

Notice that the above estimate is satisfied under conditions (A1)–(A4).

Finally we apply Lemma 5.5 with a constant \( \gamma > \max \{0, 2(\alpha - 1) - 3\beta\} \) to obtain the asymptotic formula for the eigenvalues of the operator \( J \):
\[ \lambda_n(J) = \alpha_n - \frac{\beta_{n-1}^2}{\alpha_{n-1} - \alpha_n} - \frac{\beta_n^2}{\alpha_{n+1} - \alpha_n} + \frac{\gamma_{n-2}^2}{\alpha_{n-2} - \alpha_n} + \frac{\gamma_n^2}{\alpha_{n+2} - \alpha_n} + O \left( \frac{n^{3\beta}}{n^{2(\alpha - 1)}} \right), \]
as \( n \to \infty \).

**Remark 5.6.** The asymptotic formula for \( \lambda_n(J) \) from Theorem 5.5 and formula (5.25) for \( \mu_{n,n} \) are more precise then the estimates mentioned in Remark 5.3 even if we do not assume additional conditions on the sign of the expression \( 2(\alpha - 1) - 3\beta \).

**Example 5.7.** Let consider a non-symmetric tridiagonal operator \( T \) on \( L^2(\mathbb{N}) \)
\[
\begin{pmatrix}
1 & a_1 & 0 & \cdots & \\
& b_1 & 4 & a_2 & 0 & \cdots \\
& & 0 & b_2 & 9 & a_3 & \cdots \\
& & & 0 & 0 & b_3 & 16 & \\
& & & & & \cdots & \cdots & \\
& & & & & & \cdots & \cdots
\end{pmatrix},
\]
where \( \{a_n\} \) and \( \{b_n\} \) are bounded real sequences. If \( J = T^* T \) then \( J \) is symmetric 5-diagonal operator and the infinite matrix, associated with \( J \), has the entries determined by the sequences
\[
\alpha_n = n^4 + a_{n-1}^2 + b_n^2, \quad \text{for} \quad n \geq 2, \quad \alpha_1 = 1 + b_1^2,
\]
\[
\beta_n = n^2 a_n + (n + 1)^2 b_n, \quad \gamma_n = a_n b_n, \quad \text{for} \quad n \geq 1.
\]
The above sequences satisfy (A1)–(A4) with \( \alpha = 4 \) and \( \beta = 2 \), so we apply Theorem 5.5 to obtain
\[
\lambda_n(T^* T) = \lambda_n(J) = n^4 + \frac{\beta_{n-1}^2}{n^4 - (n - 1)^2 - \rho_n} - \frac{\beta_n^2}{(n + 1)^4 - n^4 + \rho_{n+1}} + O(1) =
\]
\[
= n^4 + \frac{n((a_{n-1} + b_{n-1})^2 - (a_n + b_n)^2)}{4} + O(1), \quad n \to \infty,
\]
\( (\rho_n = a_{n-1}^2 + b_n^2 - a_n^2 - b_{n-1}^2). \)

From the above result we deduce easily the asymptotic formula for the singular numbers of \( T \) as follow
\[
s_n(T) = (\lambda_n(T^* T))^\frac{1}{2} = n^2 + \frac{(a_{n-1} + b_{n-1})^2 - (a_n + b_n)^2}{8n} + O\left( \frac{1}{n^2} \right), \quad n \to \infty.
\]
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