Abstract. Let $k \geq 1$ be an integer. A set $S$ of vertices of a graph $G = (V(G), E(G))$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $N(v)$ is the neighborhood of $v$. The subset $S$ is a $k$-dominating set of $G$ if every vertex in $V(G) - S$ has at least $k$ neighbors in $S$. The global offensive $k$-alliance number $\gamma_k^o(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$ and the $k$-domination number $\gamma_k(G)$ is the minimum cardinality of a $k$-dominating set of $G$. For every integer $k \geq 1$ every graph $G$ satisfies $\gamma_k^o(G) \geq \gamma_k(G)$. In this paper we provide for $k \geq 2$ a characterization of trees $T$ with equal $\gamma_k^o(T)$ and $\gamma_k(T)$.

Keywords: global offensive $k$-alliance number, $k$-domination number, trees.

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1. INTRODUCTION

We begin with some terminology. For a vertex $v$ of a simple graph $G = (V(G), E(G))$, the open neighborhood of $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the degree of $v$, denoted by $\deg_G(v)$, is $|N_G(v)|$. By $n(G)$ and $\Delta(G) = \Delta$ we denote the order and the maximum degree of the graph $G$, respectively. Specifically, for a vertex $v$ in a rooted tree $T$, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of $v$, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

In [9] Kristiansen, Hedetniemi, and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive $k$-alliances given by Shafique and Dutton [10, 11]. Let $k \geq 1$ be an integer. A set $S$ of vertices of a graph $G$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$ for $1 \leq k \leq \Delta$. The global offensive $k$-alliance number $\gamma_k^o(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. If $S$ is a global offensive...
k-alliance of $G$ and $|S| = \gamma^k_0(G)$, then we say that $S$ is a $\gamma^k_0(G)$-set. Note that a global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. Recently, Fernau, Rodríguez and Sigarreta showed in [5] that the problem of finding optimal global offensive $k$-alliances is NP-complete, and Chellali, Haynes, Randerath and Volkmann presented in [3] several bounds on the global offensive $k$-alliance number.

For a positive integer $k$, a set of vertices $D$ in a graph $G$ is said to be a $k$-dominating set if each vertex of $G$ not in $D$ has at least $k$ neighbors in $D$. The order of the smallest $k$-dominating set of $G$ is called the $k$-domination number, and it is denoted by $\gamma_k(G)$. The concept of $k$-domination was introduced by Fink and Jacobson in [6, 7], and is studied, for example, in [4, 8] and elsewhere.

Clearly, if $S$ is any global offensive $k$-alliance, then every vertex of $V(G) - S$ has at least $k$ neighbors in $S$. Thus $S$ is a $k$-dominating set of $G$, and hence $\gamma_k(G) \leq \gamma^k_0(G)$.

In this paper, we provide a characterization of trees with equal global offensive $k$-alliance and $k$-domination numbers for every integer $k \geq 2$. Note that a characterization of trees $T$ with $\gamma_1(T) = \gamma^k_0(T)$ has been given by Bouzefrane and Chellali [2].

2. MAIN RESULT

We begin by introducing the following trees defined in [1] by Blidia, Chellali and Volkmann. For a positive integer $p$, a nontrivial tree $T$ is called an $N^p_T$-tree if $T$ contains a vertex, say $w$, of degree at least $p - 1$ and $\deg_T(x) \leq p - 1$ for every vertex of $x \in V(T) - \{w\}$. The vertex $w$ will be called the special vertex of $T$. An $N^p_T$-tree with special vertex $w$ is called exact if $\deg_T(w) = p - 1$.

For the purpose of characterizing trees $T$ with $\gamma_k(T) = \gamma^k_0(T)$ for $k \geq 2$ we define the family $\mathcal{F}_k$ of all trees $T$ that can be obtained from a sequence $T_1$, $T_2$, $\ldots$, $T_p$ ($p \geq 1$) of trees, where $T_1$ is an $N^k_T$-tree with special vertex $w$ of degree at least $k - 1$, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the operations listed below.

- Operation $O_1$: Attach an $N^k_T$-tree with special vertex $x$ of degree at least $k + 1$ by adding an edge from $x$ to any vertex $u$ of $T_i$ with the condition that if $u$ does not belong to a $\gamma^k_0(T_i)$-set $D$, then $|N_T(u) \cap D| > |N_T(u) - D| + k$.
- Operation $O_2$: Attach an $N^k_T$-tree with special vertex $x$ of degree $k - 1$ or $k$ by adding an edge from $x$ to a vertex $u$ of $T_i$ that belongs to a $\gamma^k_0(T_i)$-set.
- Operation $O_3$: Attach an exact $N^k_T$-tree with special vertex $x$ and $q \geq 1$ new trees, all vertices of degree at most $k - 1$ and join $x$ and a vertex of each new tree by an edge to a vertex $z$ of $T_i$ of degree exactly $k - 1$.

The following observations will be useful for the next.

**Observation 2.1.** For every graph $G$ and positive integer $k$, every vertex with degree at most $k - 1$ belongs to every $\gamma^k_0(G)$-set and to every $\gamma_k(G)$-set.
Observation 2.2. Let $k \geq 2$ be an integer and $T$ a tree obtained from an $N_k$-tree $H$ with special vertex $w$ by adding an edge between $w$ and a vertex $v$ of a tree $T'$. Then $\gamma^k_w(T') \leq \gamma^k_w(T) - |V(H)| + 1$ with equality if:

1) $v$ belongs to a $\gamma^k_w(T')$-set.
2) $\deg_H(w) \geq k + 1$ and $v$ satisfies $|N_T(v) \cap D| > |N_T(v) - D| + k$, where $D$ is a $\gamma^k_w(T')$-set such that $v \notin D$.

Proof. Let $Q$ be a $\gamma^k_w(T)$-set. Then by Observation 2.1, $Q$ contains $V(H) - \{w\}$ and, without loss of generality, $w \notin Q$ (else replace $w$ in $Q$ by $v$) and hence $v \in Q$. Thus $Q \cap V(T')$ is a global offensive $k$-alliance of $T'$, and so $\gamma^k_w(T') \leq \gamma^k_w(T) - |V(H)| + 1$. Now let $D'$ be a $\gamma^k_w(T')$-set. If $v \in D'$, then $D' \cup (V(H) - \{w\})$ is a global offensive $k$-alliance of $T'$ if $\deg_H(w) \geq k + 1$, $v \notin D'$ and $v$ satisfies $|N_T(v) \cap D'| > |N_T(v) - D'| + k$, then $D' \cup (V(H) - \{w\})$ is a global offensive $k$-alliance of $T'$ too. In both cases $\gamma^k_w(T) \leq \gamma^k_w(T') + |V(H)| - 1$ and the equality follows. 

By using a similar proof we obtain the following

Observation 2.3. Let $k \geq 2$ be an integer and $T$ a tree obtained from an $N_k$-tree $H$ with special vertex $w$ by adding an edge between $w$ and a vertex $v$ of a tree $T'$. Then $\gamma_k(T') \leq \gamma_k(T) - |V(H)| + 1$ with equality if either $\deg_H(w) \geq k$ or $v$ belongs to a $\gamma_k(T')$-set.

We state a lemma.

Lemma 2.4. If $k \geq 2$ and $T \in F_k$, then $\gamma^k_w(T) = \gamma_k(T)$.

Proof. Assume that $k \geq 2$ and let $T$ be a tree of $F_k$. Then $T$ is obtained from a sequence $T_1, T_2, \ldots, T_p \ (p \geq 1)$ of trees, where $T_1$ is an $N_k$-tree with special vertex $w$ of degree at least $k - 1$, $T = T_p$, and, if $p \geq 2$, $T_{i+1}$ can be obtained recursively from $T_i$ by one of the operations defined above. We will use induction on $p$. If $p = 1$, then $\gamma^k_w(T_1) = \gamma_k(T_1) = n(T_1)$ or $n(T_1) - 1$ depending on whether $w$ has degree $k - 1$ or more, respectively.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in F_k$ that can be constructed from a sequence of length at most $p - 1$, and let $T' = T_{p-1}$. By the inductive hypothesis on $T' \in F_k$ we have $\gamma^k_w(T') = \gamma_k(T')$. Let $T$ be a tree obtained from $T'$ and consider the following cases.

Assume that $T$ is obtained from $T'$ by using Operation $O_1$ or $O_2$. Let $H$ be the added $N_k$-tree. Then by Observations 2.2 and 2.3, $\gamma^k_w(T) = \gamma^k_w(T') + |V(H)| - 1$, $\gamma_k(T) = \gamma_k(T') + |V(H)| - 1$ and hence $\gamma^k_w(T) = \gamma_k(T)$.

Assume now that $T$ is obtained from $T'$ by using operation $O_3$. Let $H$ be the added $N_k$-tree with special vertex $x$ and $H_1, H_2, \ldots, H_q$ the $q$ added new trees attached to $z$ of $T'$. We further assume that $t$ trees among the $q$ new trees are attached to $z$ by vertices of degree exactly $k - 1$, and so such vertices would have degree $k$ in $T$. It can be seen easily that $\gamma^k_w(T) = \gamma^k_w(T') + |V(H)| - 1 + \sum_{i=1}^{q} |V(H_i)| - t$, and $\gamma_k(T) = \gamma_k(T') + |V(H)| - 1 + \sum_{i=1}^{q} |V(H_i)| - t$. Therefore $\gamma^k_w(T) = \gamma_k(T)$. 

We now are ready to give our main result.

**Theorem 2.5.** Let $k \geq 2$ be an integer. A tree $T$ satisfies $\gamma_o^k(T) = \gamma_k(T)$ if and only if either $\Delta(T) \leq k - 2$ or $T \in \mathcal{F}_k$.

*Proof.* If $T$ is a tree with $\Delta(T) \leq k - 2$, then by Observation 2.1, $\gamma_o^k(T) = \gamma_k(T) = n(T)$. If $T \in \mathcal{F}_k$, then by Lemma 2.4, $\gamma_o^k(T) = \gamma_k(T)$.

Let us prove the “only if” part. Let $k \geq 2$ be an integer and $T$ a tree with $\gamma_o^k(T) = \gamma_k(T)$. Suppose that $\Delta(T) \geq k - 1$ and let $B(T) = \{x \in V(T) : \deg_T(x) \geq k\}$. We use an induction on the size of $B(T)$. If $|B(T)| = 0$ or 1, then $T$ is an (exact) $N_k$-tree that belongs to $\mathcal{F}_k$. Let $|B(T)| \geq 2$ and assume that every tree $T'$ with $|B(T')| < |B(T)|$ such that $\gamma_o^k(T') = \gamma_k(T')$ is in $\mathcal{F}_k$. Let $T$ be a tree with $\gamma_o^k(T) = \gamma_k(T)$ and $S$ a $\gamma_o^k(T)$-set.

We now root $T$ at a vertex $r$ of maximum eccentricity. Let $w$ be a vertex of degree at least $k$ at maximum distance from $r$. We further assume that among such vertices $w$ has maximum degree. Clearly since $k \geq 2, w \neq r$ and the subtree induced by $D(w) \cup \{w\}$ is an $N_k$-tree with special vertex $w$ of degree at least $k - 1$. Note that every vertex in $D(w)$ has degree at most $k - 1$ and so $D(w)$ is contained in every $\gamma_o^k(T)$-set and every $\gamma_k(T)$-set. Let $u$ be the parent of $w$ in the rooted tree. We consider the following cases.

**Case 1.** $\deg_T(u) \geq k + 2$. Let $T' = T - T_w$. By Observation 2.3, $\gamma_k(T) = \gamma_k(T') + |V(T_w)| - 1$ and by Observation 2.2, $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(T_w)| + 1$. If $\gamma_o^k(T') < \gamma_o^k(T) - |V(T_w)| + 1$, then using the fact $\gamma_o^k(T) = \gamma_k(T)$ we arrive to $\gamma_o^k(T') < \gamma_k(T')$, a contradiction. Therefore $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Hence we may assume that $w \notin S$ (else replace $w$ by $u$) and so $S' = S \cap V(T')$ is a $\gamma_o^k(T')$-set. Observe that if $u \notin S'$, then since $w \notin S$ the set $S'$ is a $\gamma_o^k(T')$-set for which $w$ satisfies $|N_T(u) \cap S'| > |N_T(u) - S'| + k$. Now it follows by the previous equalities that $\gamma_o^k(T') = \gamma_k(T')$. If $B(T') = \emptyset$, then $\deg_T(u) = k$ and $T'$ is an exact $N_k$-tree with special vertex $u$, that is $T' \in \mathcal{F}_k$. If $B(T') \neq \emptyset$, then clearly $|B(T')| < |B(T)|$ and hence by induction on $T'$, we have $T' \in \mathcal{F}_k$. Therefore in both cases $T \in \mathcal{F}_k$ and is obtained from $T'$ by using Operation $O_1$.

**Case 2.** $\deg_T(u) = k + 1$. Let $T' = T - T_w$. By Observation 2.3, $\gamma_k(T) = \gamma_k(T') + |V(T_w)| - 1$ and by Observation 2.2, $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(T_w)| + 1$. By using the same argument as that used in Case 1, we obtain $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Also $w \notin S$ (else replace $w$ by $u$ in $S$) and hence $u \in S$, implying that $S' = S \cap V(T')$ is a $\gamma_o^k(T')$-set, where $u \in S'$. The previous equalities imply that $\gamma_o^k(T') = \gamma_k(T')$. Clearly $|B(T')| < |B(T)|$ but we note that $B(T') \neq \{u\}$ for otherwise $S' - \{u\}$ would be a global offensive $k$-alliance of $T'$. Now by induction on $T'$ we have $T' \in \mathcal{F}_k$. Hence $T \in \mathcal{F}_k$ and is obtained from $T'$ by using Operation $O_2$.

**Case 3.** $\deg_T(u) = k$. By our choice of $w$ every vertex in $C(u)$ has degree at most $k$. Recall that $|B(T)| \geq 2$. If $\deg_T(u) \leq k$, then let $T' = T - T_w$. It can be seen that $\gamma_k(T) = \gamma_k(T') - |V(T_w)| + 1$ and $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Therefore $\gamma_o^k(T') = \gamma_k(T')$ and by induction on $T'$ we have $T' \in \mathcal{F}_k$. Since $\deg_T(u) \leq k - 1$, $u$ belongs to every $\gamma_o^k(T')$-set. Thus $T \in \mathcal{F}_k$ and is obtained from $T'$ by using Operation $O_2$. Now assume that $\deg_T(u) = q \geq k + 1$, then let $w = w_1, w_2, \ldots, w_{q-k+1}$ be any
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vertices of \( C(u) \), where the first \( t \) \((t \geq 1)\) vertices have degree exactly \( k \) and the remaining vertices have degree at most \( k - 1 \). Let \( T' = T - \bigcup_{j=1}^{q+1-k} T_{w_j} \). Note that \( \deg_{T'}(u) = k - 1 \). By Observations 2.1, 2.2 and 2.3, it can be seen easily that

\[
\gamma^k_o(T) = \gamma^k_o(T') + \left| \bigcup_{j=1}^{q+1-k} D[w_j] \right| - t,
\]

and

\[
\gamma_k(T) = \gamma_k(T') + \left| \bigcup_{j=1}^{q+1-k} D[w_j] \right| - t.
\]

Therefore \( \gamma^k_o(T') = \gamma_k(T') \). Now since \( |B(T')| < |B(T)| \) we obtain by induction \( T' \in \mathcal{F}_k \). Hence \( T \in \mathcal{F}_k \) and is obtained from \( T' \) by using Operation \( O_3 \).

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